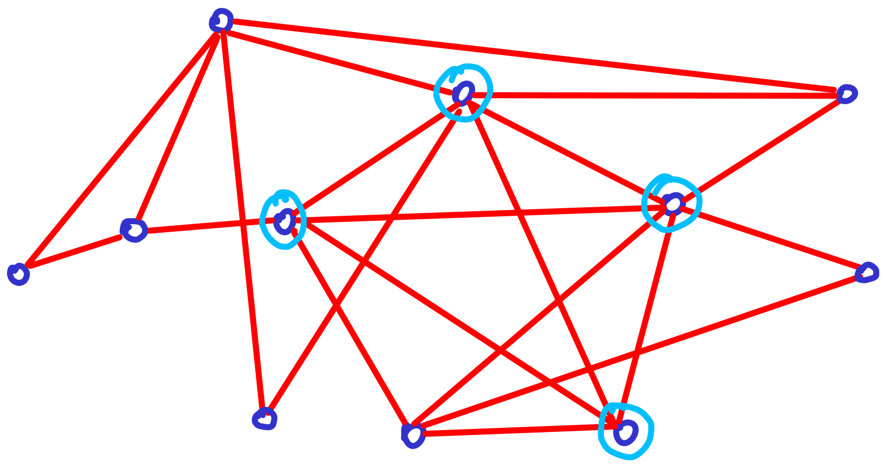
Game for 2 players:

Draw a complete graph,
each player can draw edges using one color,
take turns coloring one edge at a time.

Whoever completes a triangle first wins.

Is there always a winner?

Cliques

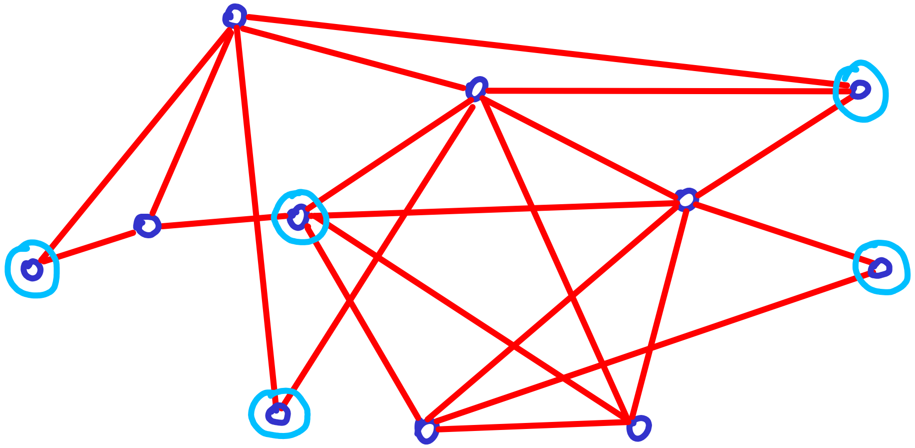


Given G, a subset S of $V(G)$ is a clique if every $s_i, s_j \in S$ share an edge in G.

How many cliques?
What is the largest clique?

The induced subgraph obtained by removing all but S from $V(G)$ is a complete graph (K_S)

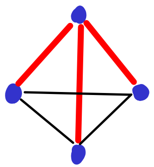
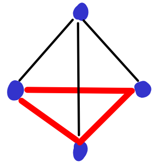
Independent Sets



Given G, a subset S of $V(G)$ is an independent set if <u>no</u> $s_i, s_j \in S$ share an edge in G.

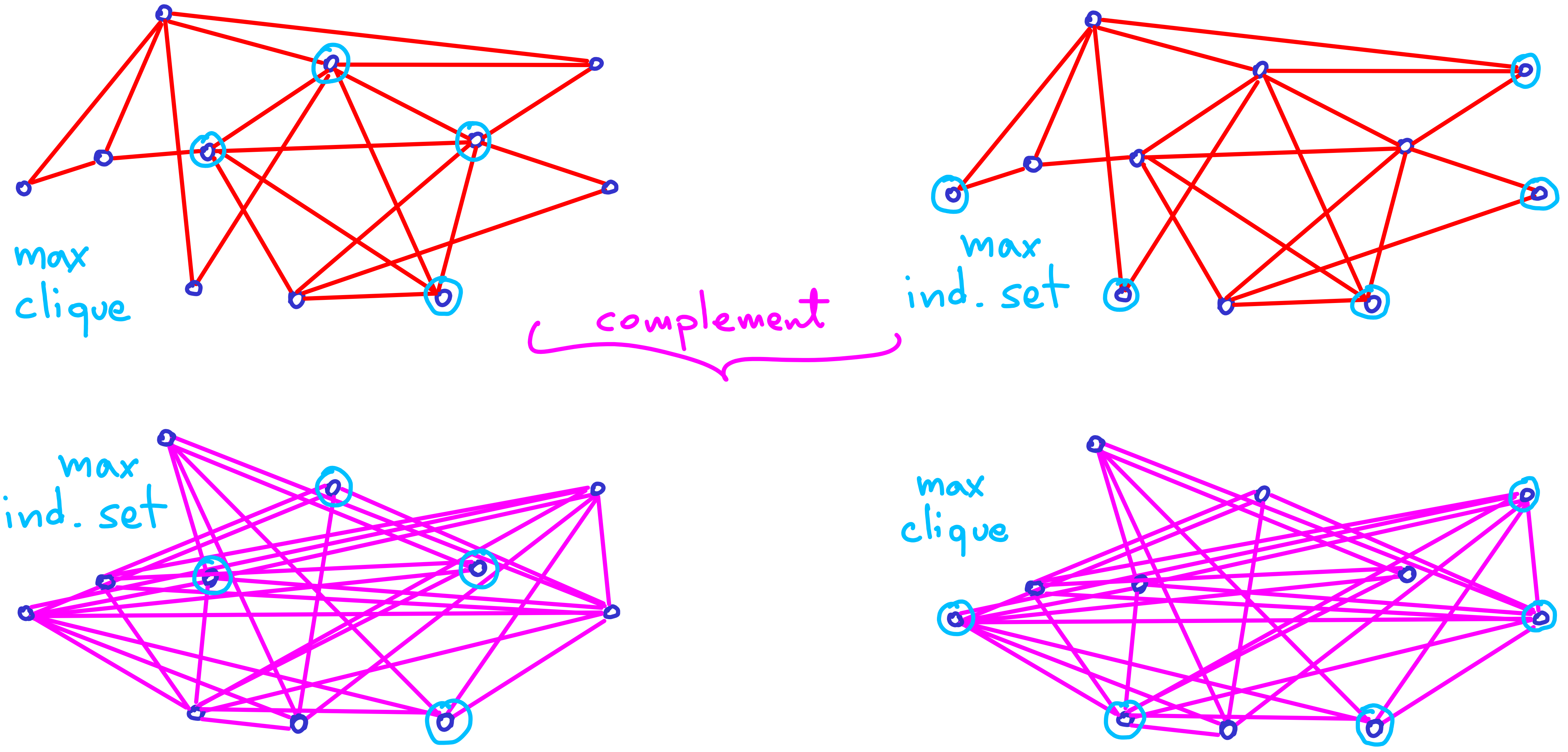
Largest independent set?

The induced subgraph obtained by removing all but S from $V(G)$ is an edgeless graph.

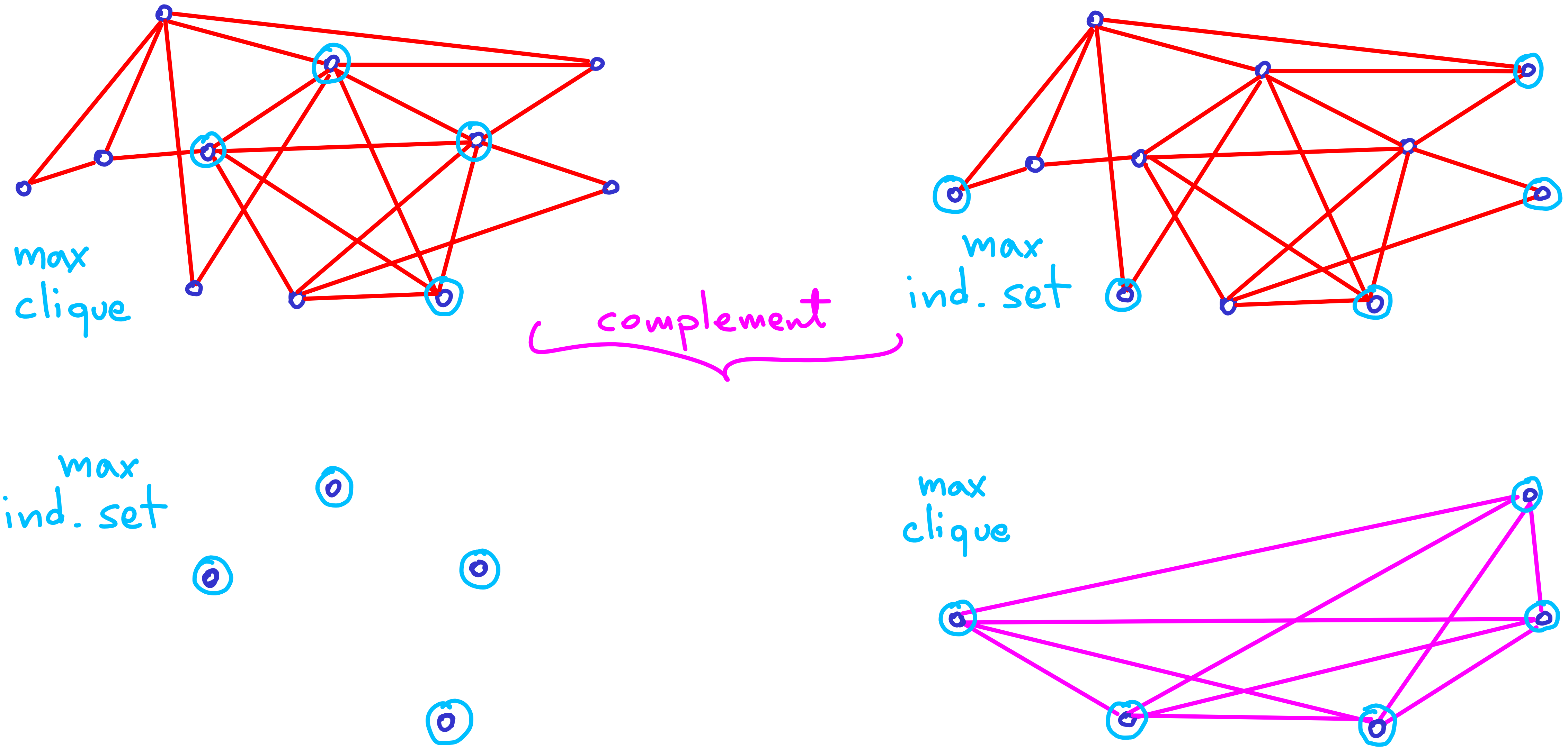
 complements 

i.e. its complement is a complete graph

cliques vs. independent sets



cliques vs. independent sets



Claim: Every graph with $|V| \geqslant 6$ contains a triangle (clique of size 3) OR an independent set of size 3

Rephrase: (for $V \geqslant 6$) a graph contains a triangle OR its complement does

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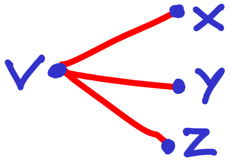
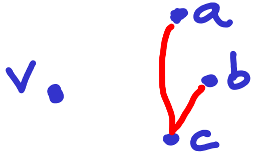
Proof: pick any vertex v.

If $d(v) \geqslant 3$ we have 

If $\overset{\frown}{xy}$ or $\overset{\frown}{xz}$ or $\overset{\frown}{yz}$: we find a clique $\triangle$

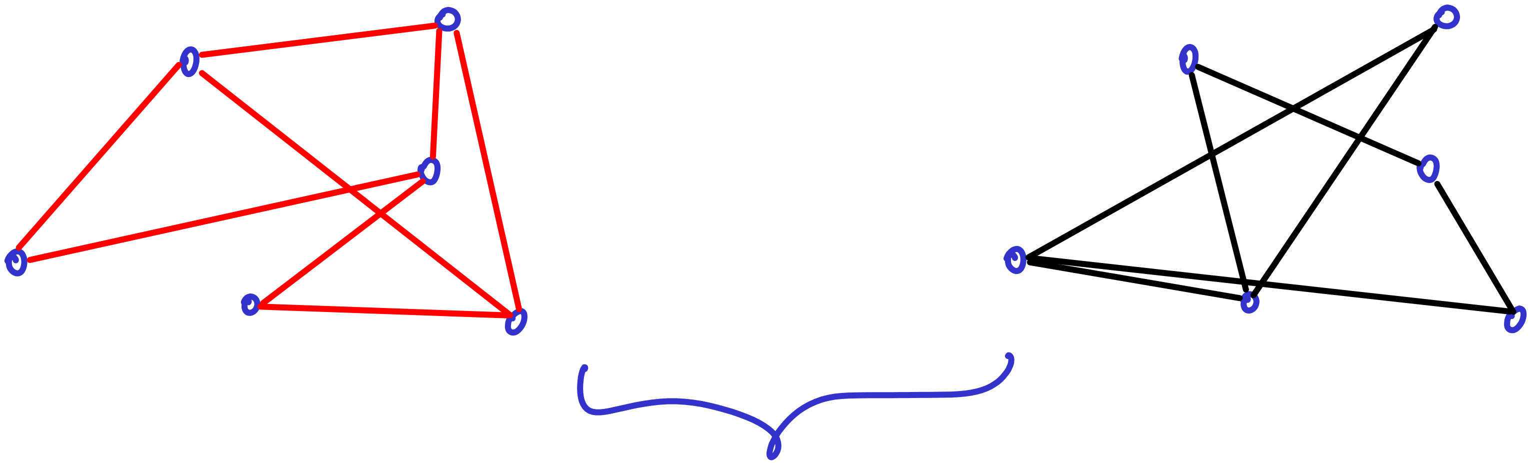
Otherwise, x, y, z are an independent set.

If $d(v) \leqslant 2$, there are $\geqslant 3$ vertices not neighboring v. $\rightarrow$ 

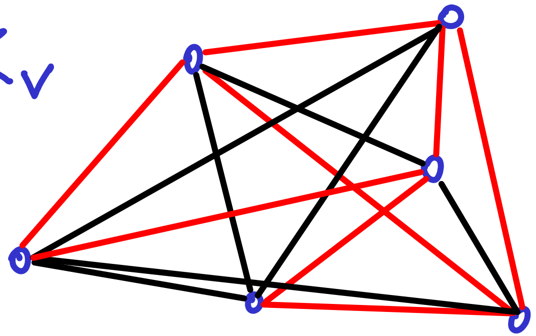
If $\overset{\frown}{ab}$, $\overset{\frown}{bc}$, $\overset{\frown}{ac}$ are edges, they are a clique $\triangle$.

Otherwise one edge is missing (w.l.o.g. ab) ... so vab is an ind. set. $\square$

(for $V \geq 6$) a graph contains a triangle OR its complement does



K_V

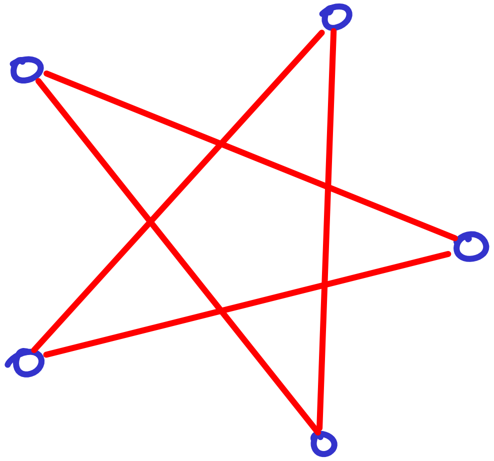


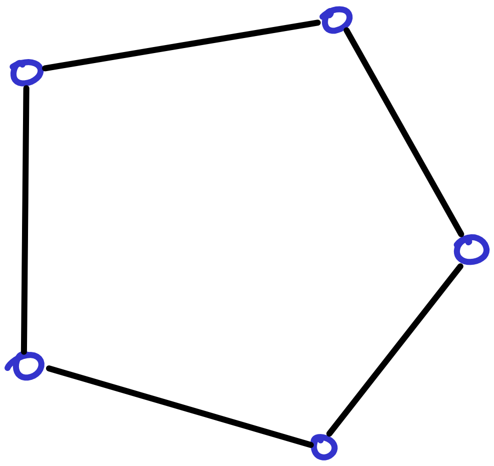
Equivalent statement

If we color each edge of K_V red or black, then we must get a red triangle or a black triangle

Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as $|V| \geq 6$

$\hookrightarrow R(3,3) = 6$





Recap: if you want a clique or an independent set of size 3

then you'll be happy as long as $|V| \geq 6$

$\hookrightarrow R(3,3) = 6$

$R(4,4)$ $\downarrow$ 18

(one direction to be shown)

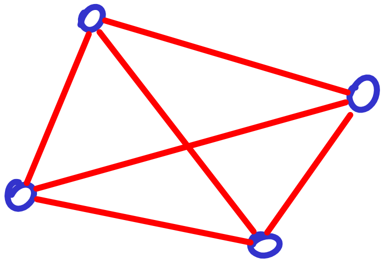
$R(5,5)$ $\downarrow$ we don't know! [43...49]

$R(n,n)$ ~ exponential $\downarrow$ even for small values we will probably never know the exact answer

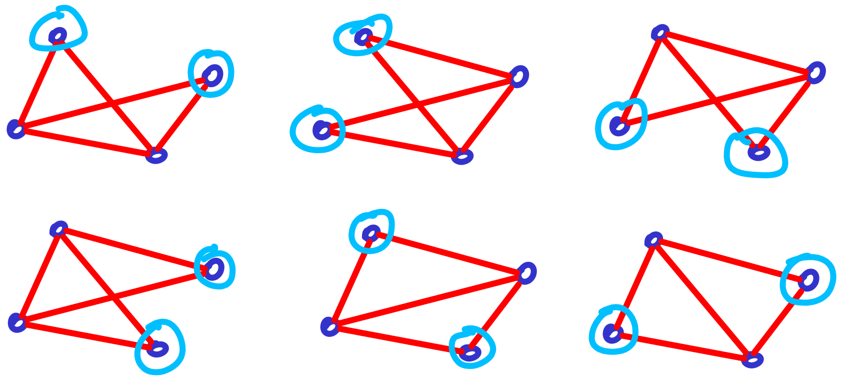
For more, see Ramsey's Thm.

$R(x,y)$: smallest number N such that any graph with $\geq N$ vertices has a clique of size x or an independent set of size y

$R(4,2) = 4$



OR

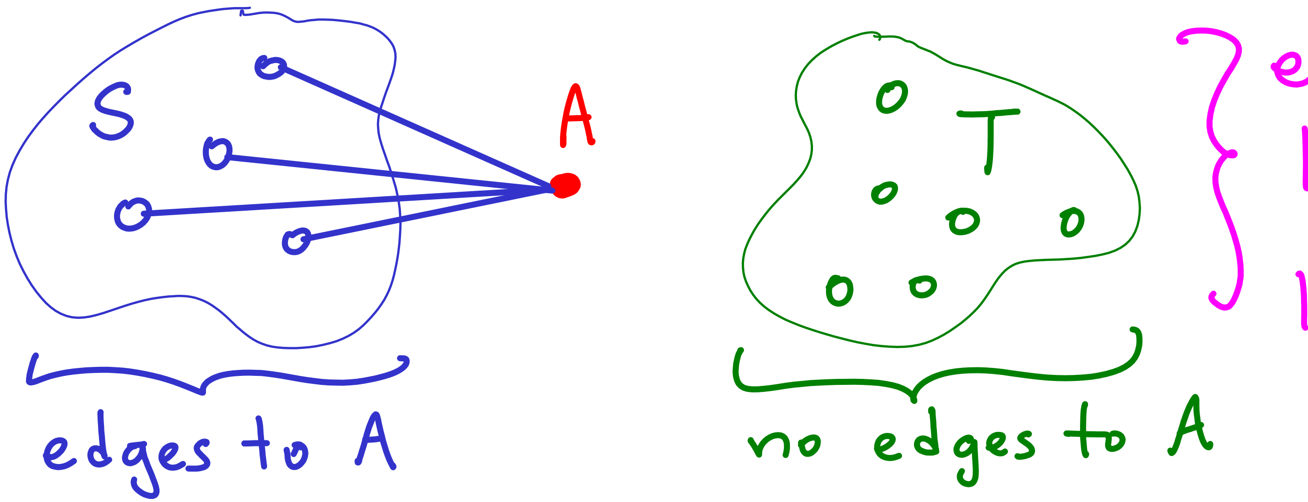


$R(4,3) \leq 10$

Suppose $|V| \geq 10$

Pick any vertex, A. ≥ 9 vertices remain.

Form 2 groups:
S & T



either $|S| \geq 6$ or $|T| \geq 4$

If $|S| \geq 6$, use $R(3,3) = 6$: S has 3 independent vertices (done),
or S has a 3-clique, so with A we get a 4-clique.

If $|T| \geq 4$, if T is a clique: done.
Otherwise $\exists$ a,b in T w/ no edge. Combine w/ A. $\square$

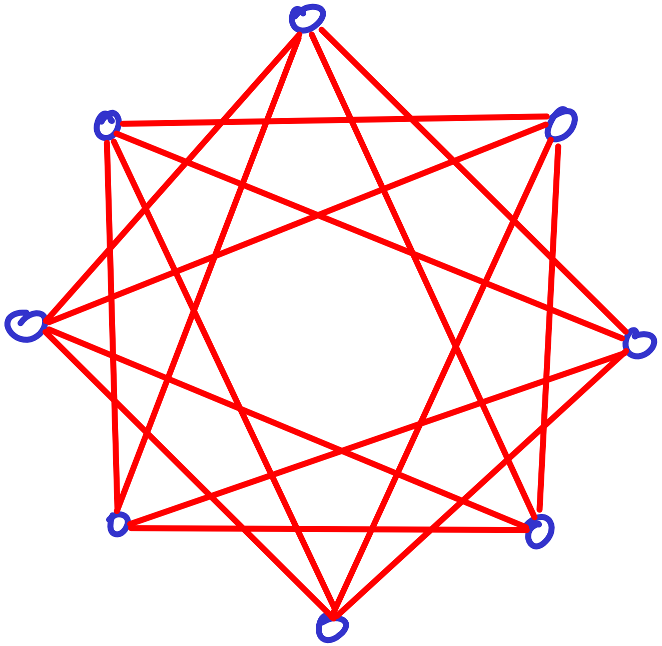
$R(4,3) \leq 10$

... turns out $R(4,3) = 9$

$R(4,3) > 8$

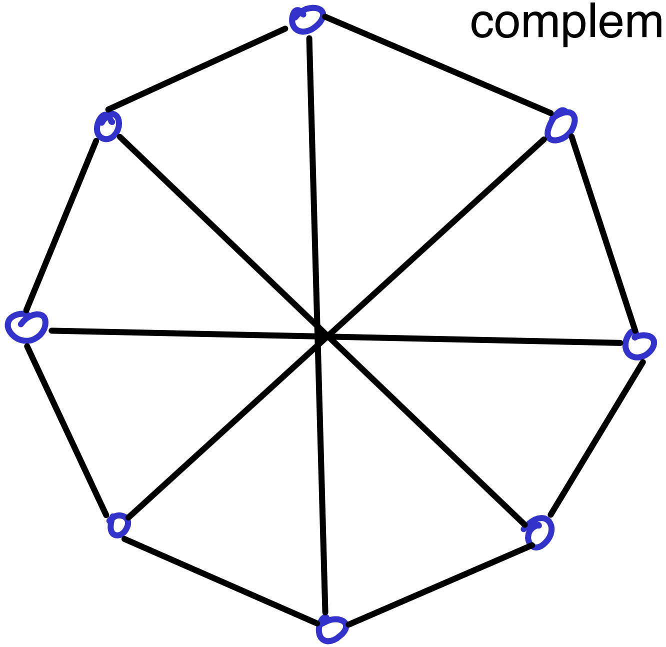
↳ not terribly hard

↳ notice $R(x,y) = R(y,x)$



no 4-clique

complement(G)



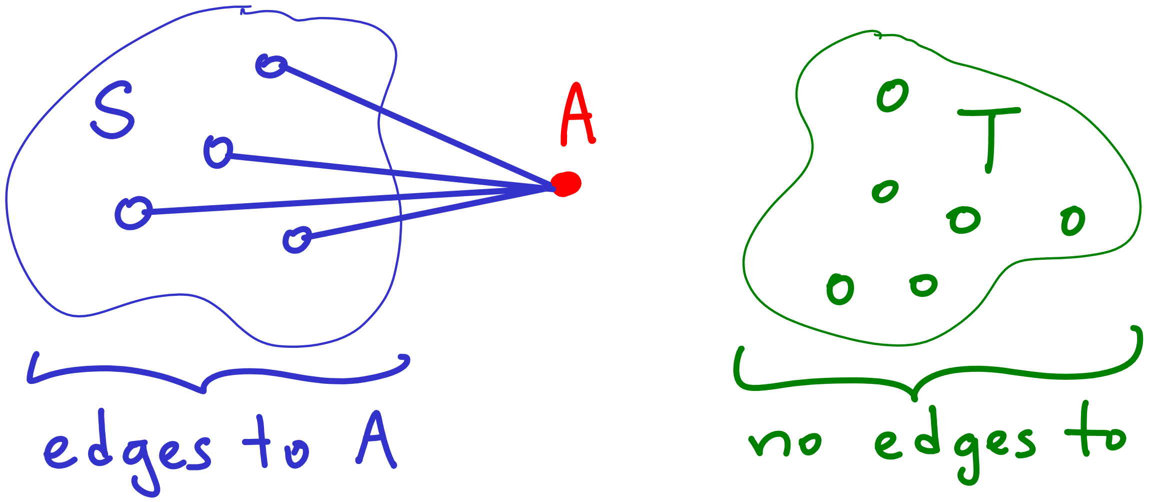
no 3-clique = no 3-independent in G

$R(4,4) \leq 18$

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:
S & T



either $|S| \geq 9$ or $|T| \geq 9$

If $|S| \geq 9$, use $R(3,4) = 9$: S has 4 independent vertices (done)
OR S has a 3-clique, so with A we get a 4-clique.

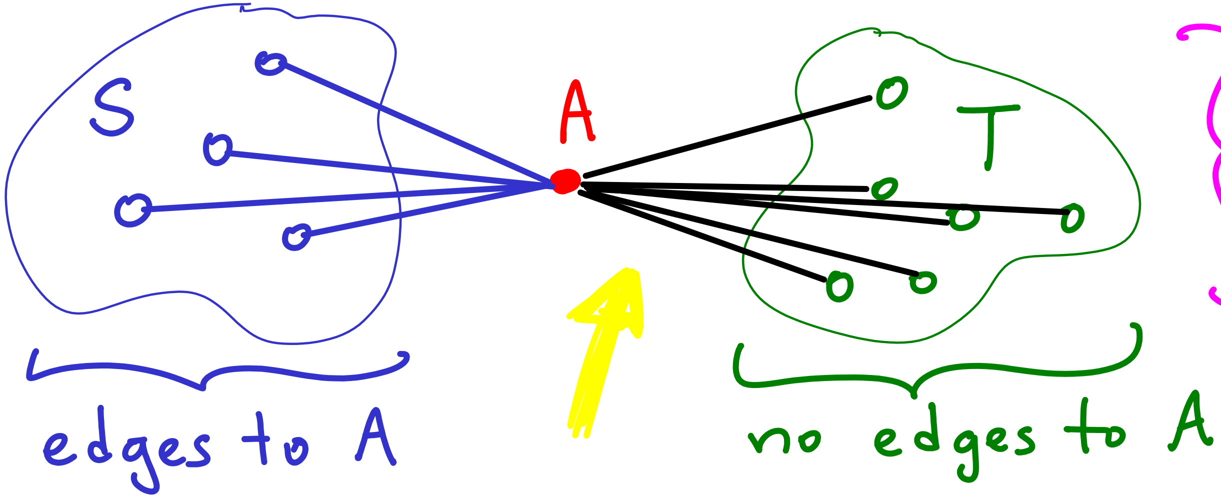
If $|T| \geq 9$, use $R(4,3) = 9$: T has a 4-clique, (done) OR
T has 3 independent vertices, so with A we have 4. $\square$

$R(4,4) \le 18$

Suppose $|V| \ge 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:
S & T



either
$|S| \ge 9$
or
$|T| \ge 9$

If $|S| \ge 9$, use $R(3,4) = 9$: S has 4 independent vertices (done)
<u>OR</u> S has a 3-clique, so with A we get a 4-clique.

If $|T| \ge 9$, use $R(3,4) = 9$ on the <u>complement</u> graph.
(SYMMETRIC)

Notes:

(1) if we only knew that $R(4,3) \le 10$ (instead of $= 9$)

we could have used $|V| \geq 20$ for $R(4,4)$

As you bound smaller R( ) values, you can get (loose) bounds for larger ones

(2) there is a graph w/ 17 vertices with no clique or independent set of size 4

↳ $R(4,4) = 18$