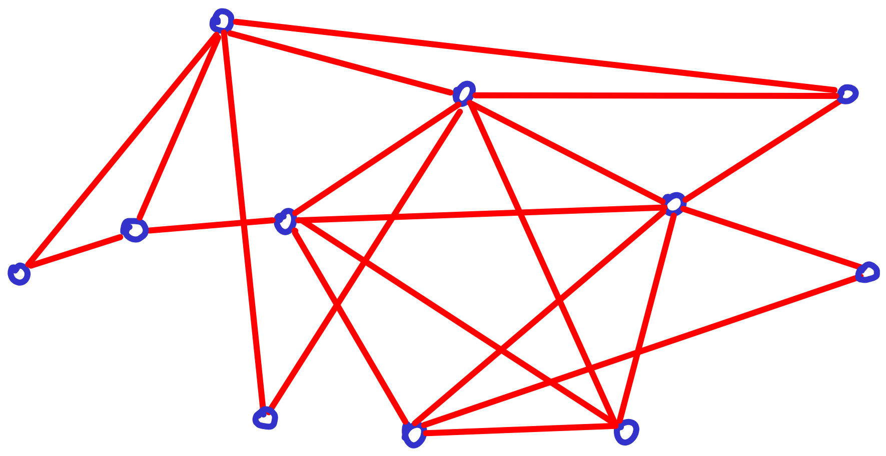
Game for 2 players:

Draw a complete graph,
each player can draw edges using one color,
take turns coloring one edge at a time.

Whoever completes a triangle first wins.

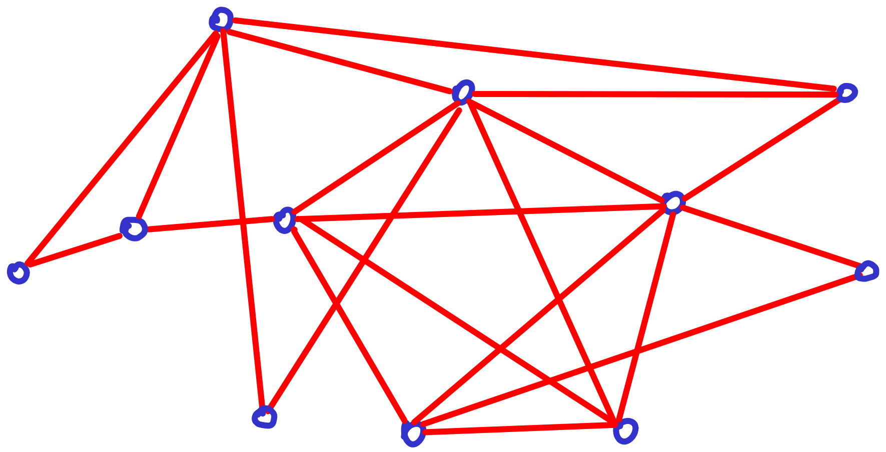
Is there always a winner?

Cliques



Given G, a subset S of $V(G)$ is a clique if every $s_i, s_j \in S$ share an edge in G.

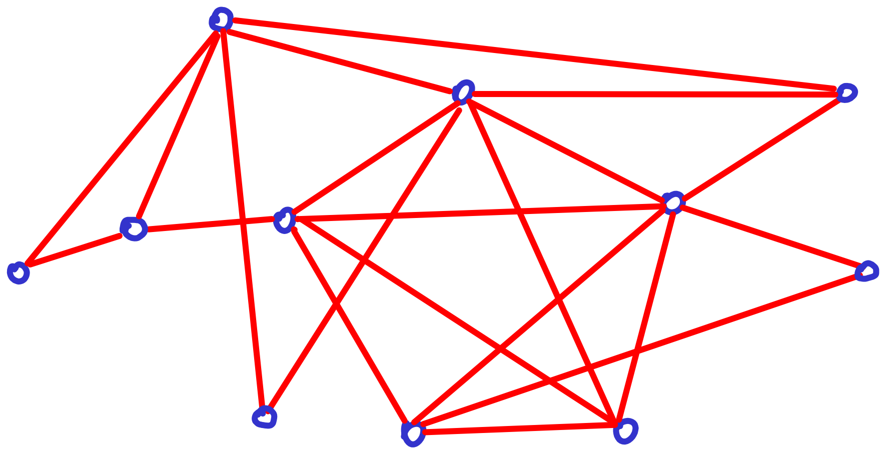
Cliques



Given G, a subset S of V(G) is a clique if every $s_i, s_j \in S$ share an edge in G.

The induced subgraph obtained by removing all but S from V(G) is a complete graph (K_S)

Cliques



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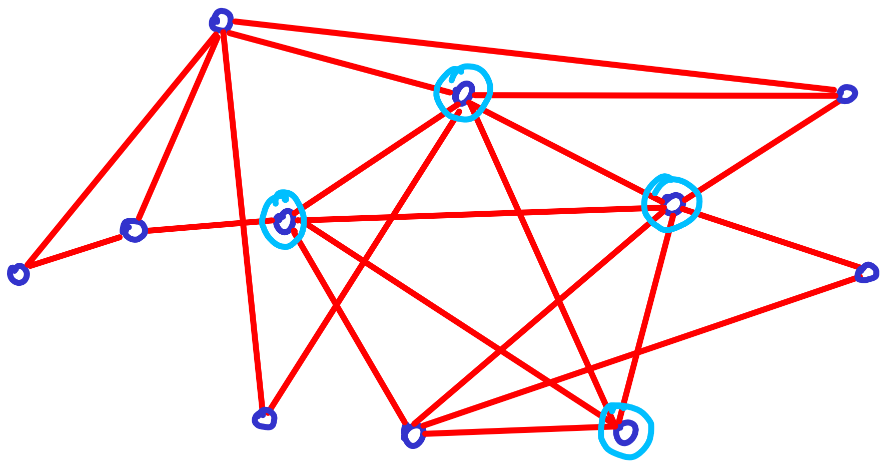
How many cliques?
What is the largest clique?

Cliques



How many cliques?
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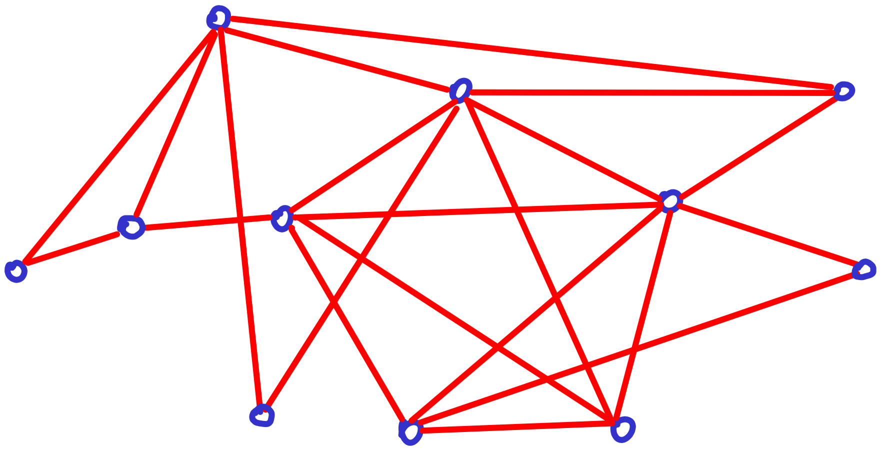
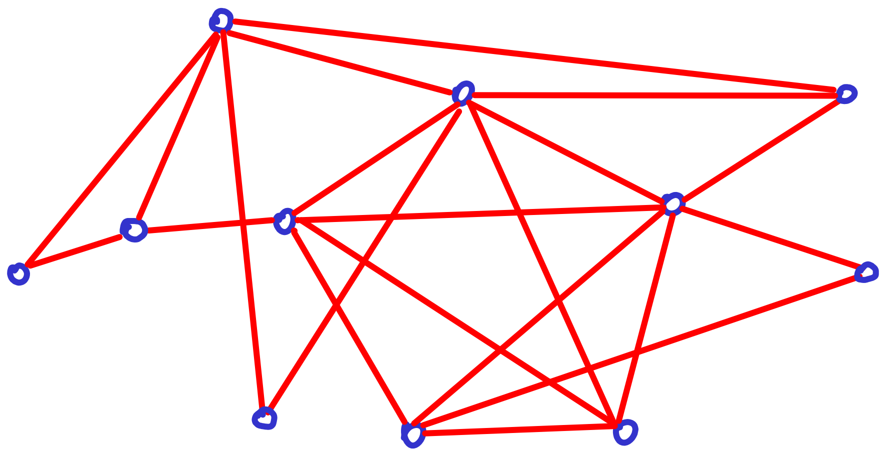
Independent Sets



Given G, a subset S of $V(G)$ is an independent set if no $s_i, s_j \in S$ share an edge in G.

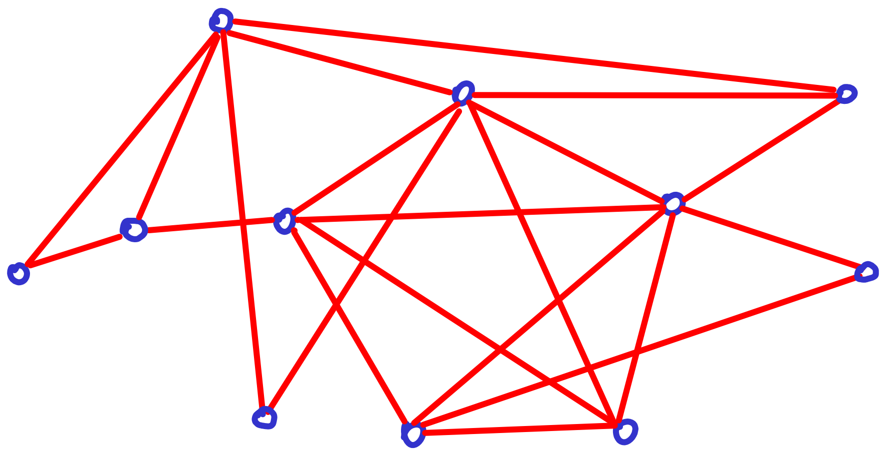
Independent Sets



Given G, a subset S of $V(G)$ is an independent set if no $s_i, s_j \in S$ share an edge in G.

The induced subgraph obtained by removing all but S from $V(G)$ is an edgeless graph.

Independent Sets



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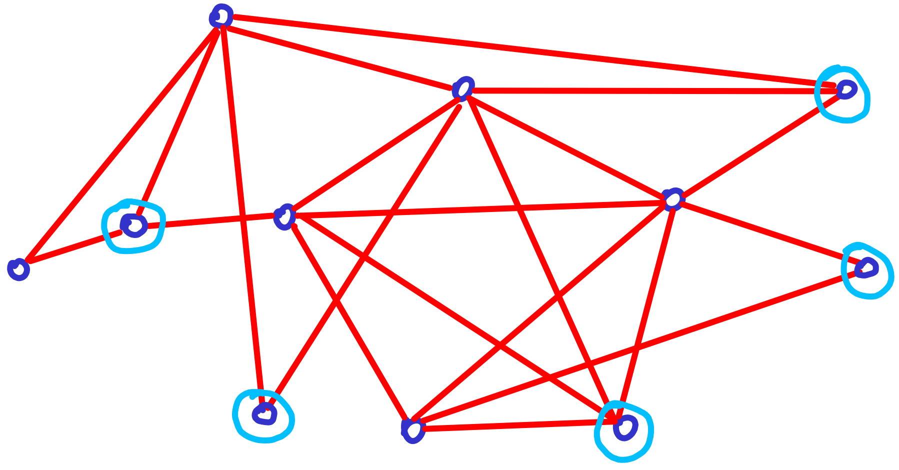

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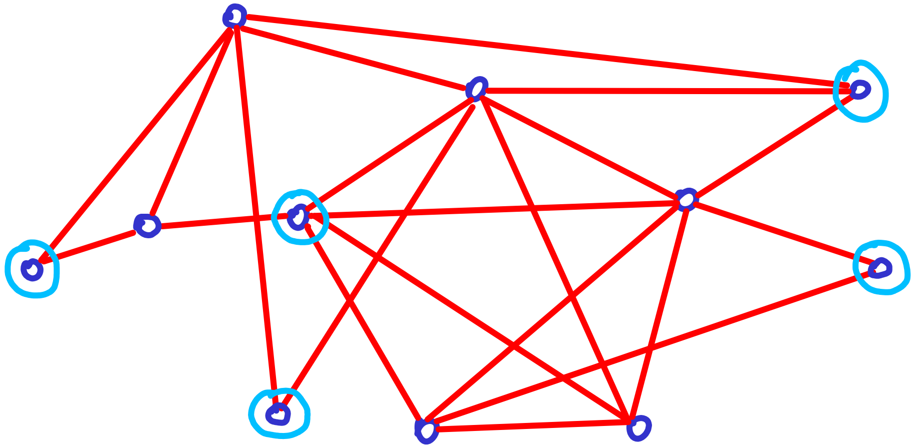

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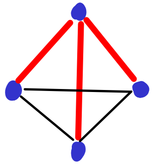
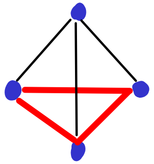
Largest independent set?

The induced subgraph obtained by removing all but S from $V(G)$ is an edgeless graph.

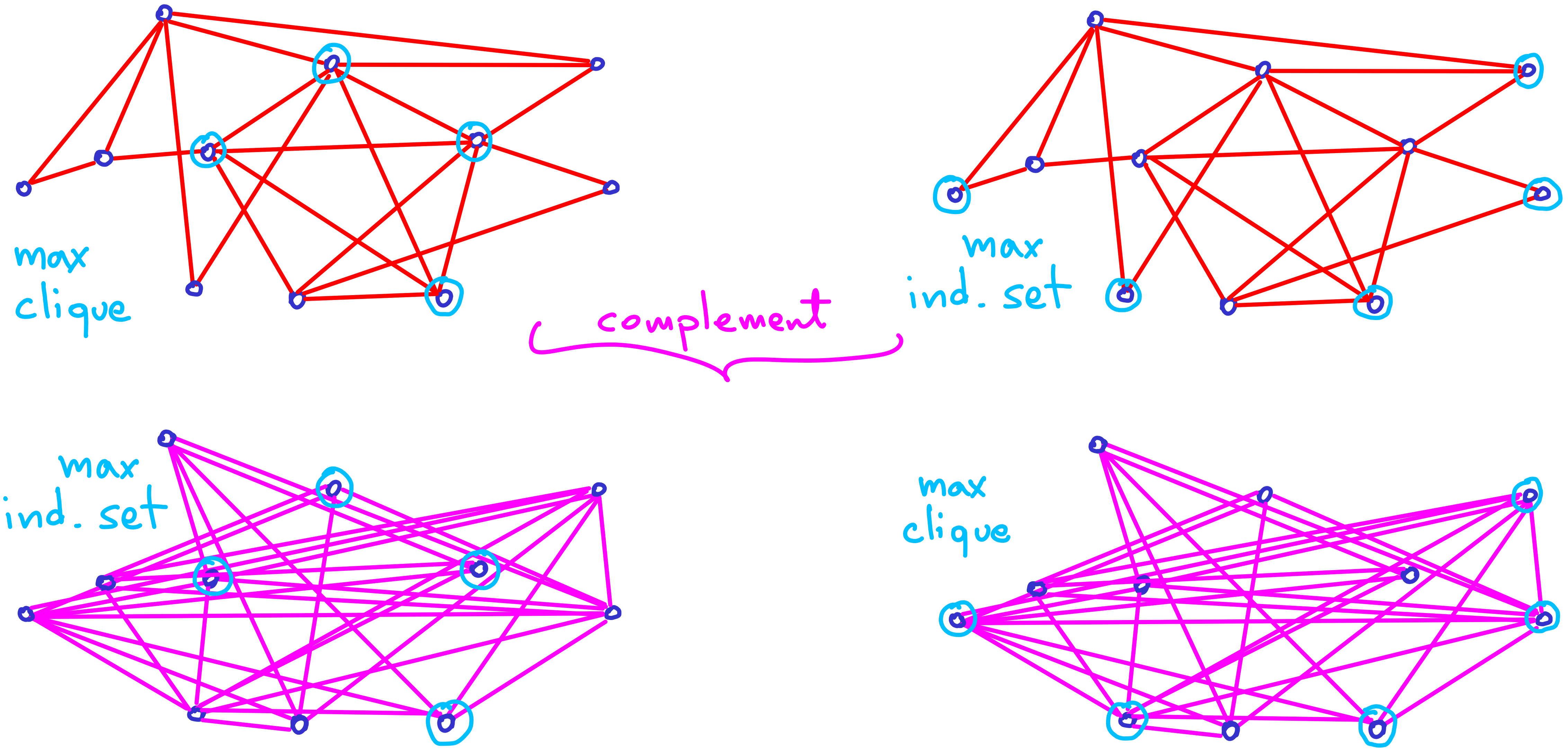
i.e. its complement is a complete graph



complements



cliques vs. independent sets



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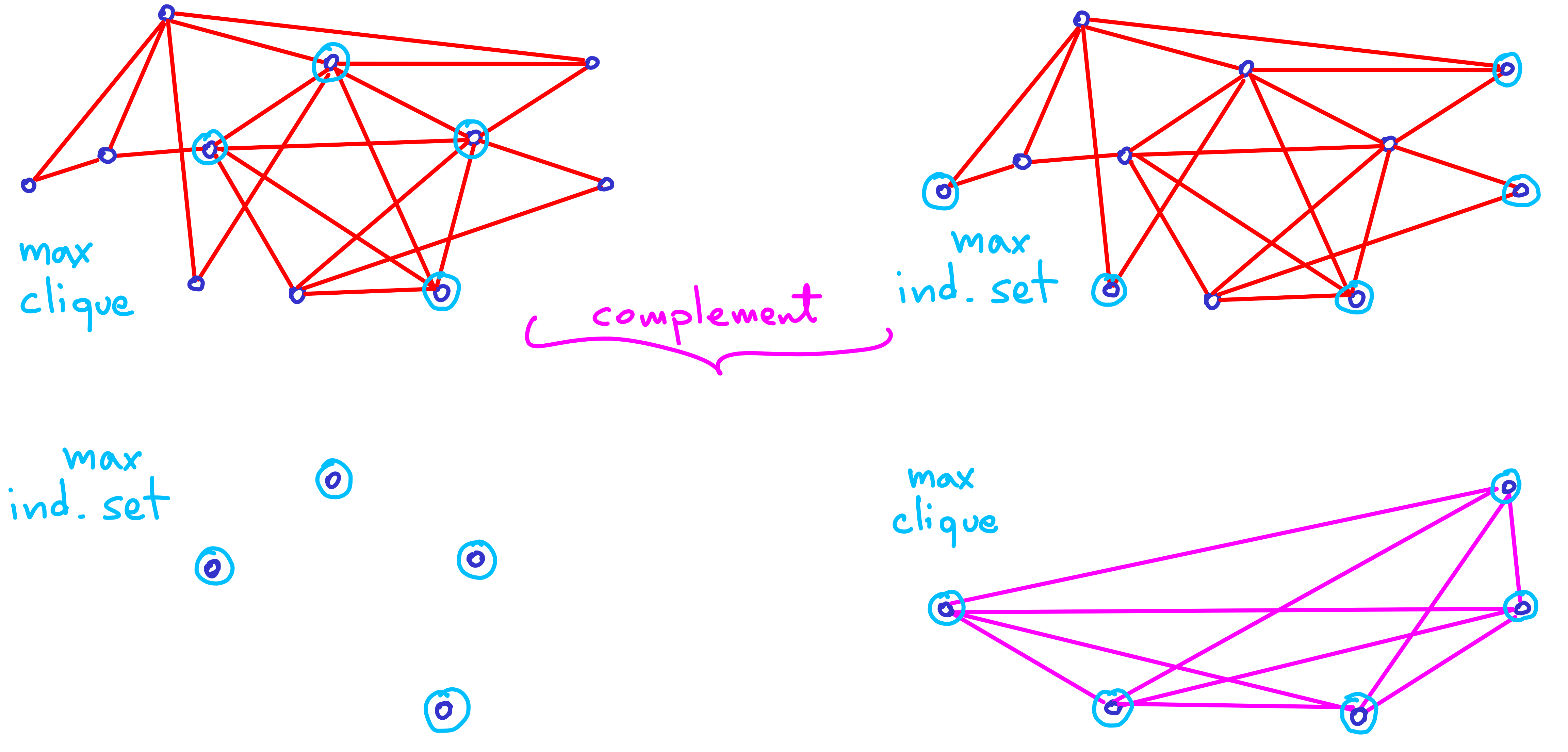

Claim: Every graph with $|V| \geq 6$ contains a triangle (clique of size 3) OR an independent set of size 3

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Rephrase: (for $V \geq 6$) a graph contains a triangle OR its complement does

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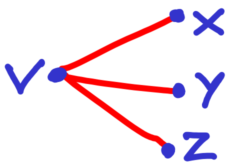
Proof: pick any vertex v.

If $d(v) \geq 3$ we have



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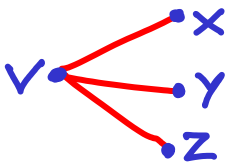

If $\overset{\frown}{xy}$ or $\overset{\frown}{xz}$ or $\overset{\frown}{yz}$: ?

Claim: Every graph with $|V| \geqslant 6$ contains a triangle (clique of size 3) OR an independent set of size 3

Rephrase: (for $V \geqslant 6$) a graph contains a triangle OR its complement does

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Proof: pick any vertex v.

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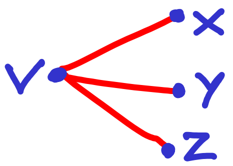

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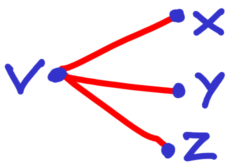

If $\overset{\frown}{xy}$ or $\overset{\frown}{xz}$ or $\overset{\frown}{yz}$: we find a clique Δ

Otherwise ?

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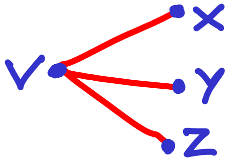

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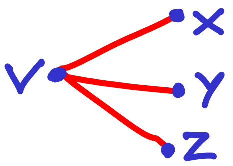

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If $d(v) \leq 2$ ... ?

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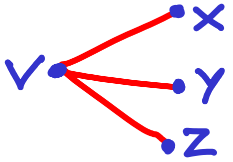
<u>If $d(v) \geq 3$</u> we have 

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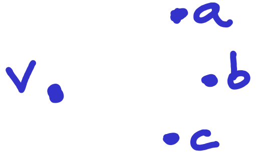
<u>If $d(v) \leq 2$</u>, there are ≥ 3 vertices not neighboring v. $\rightarrow$ 

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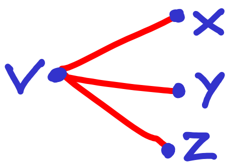
If $d(v) \geq 3$ we have 

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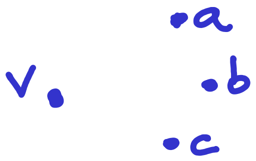
If $d(v) \leq 2$, there are ≥ 3 vertices not neighboring v. $\rightarrow$ 

If $\widehat{ab}$, $\widehat{bc}$, $\widehat{ac}$ are edges ... ?

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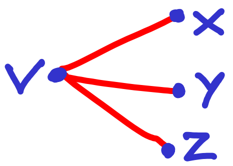
If $d(v) \geq 3$ we have 

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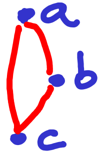
If $d(v) \leq 2$, there are ≥ 3 vertices not neighboring v. $\rightarrow$ v. 

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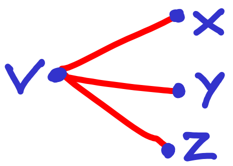
If $d(v) \geq 3$ we have 

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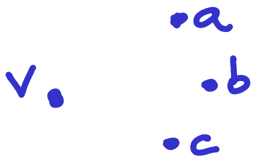
If $d(v) \leq 2$, there are ≥ 3 vertices not neighboring v. $\rightarrow$ 

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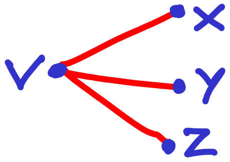
If $d(v) \geq 3$ we have 

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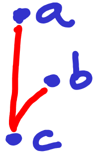
If $d(v) \leq 2$, there are ≥ 3 vertices not neighboring v. $\rightarrow$ v. 

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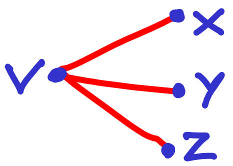
If $d(v) \geq 3$ we have 

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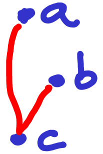
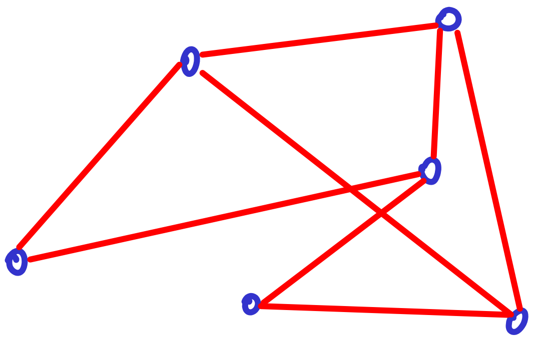
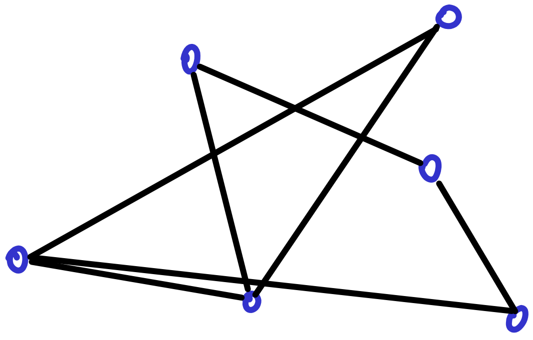
If $d(v) \leq 2$, there are ≥ 3 vertices not neighboring v. $\rightarrow$ v. 

If $\overline{ab}, \overline{bc}, \overline{ac}$ are edges, they are a clique Δ.

Otherwise one edge is missing (w.l.o.g. ab) ... so vab is an ind. set. $\square$

(for $V \geq 6$) a graph contains a triangle OR its complement does





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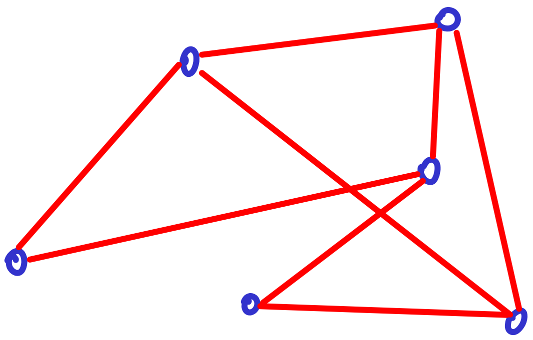

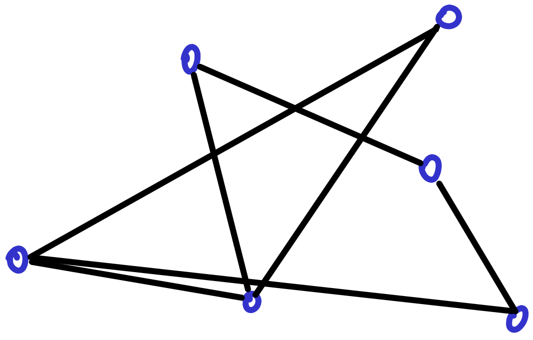

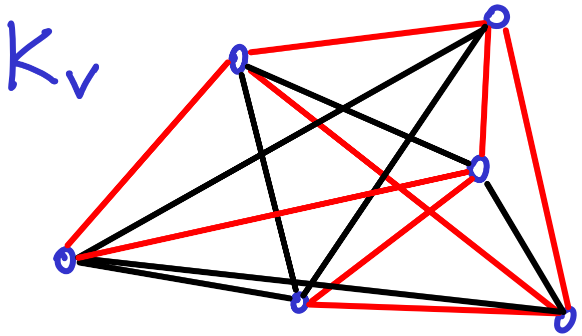

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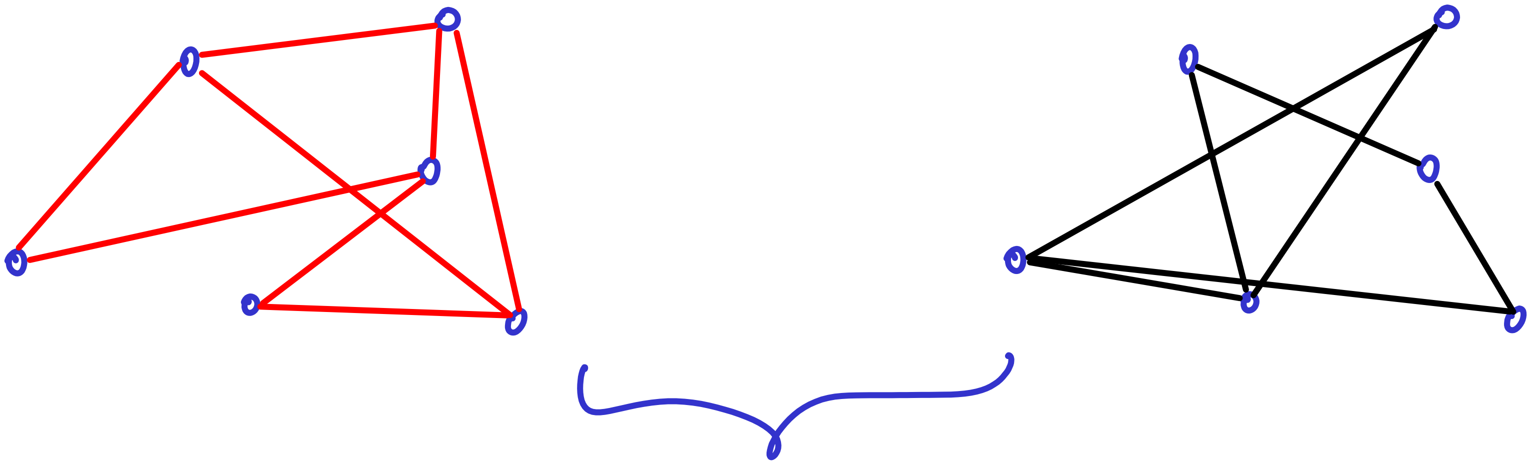
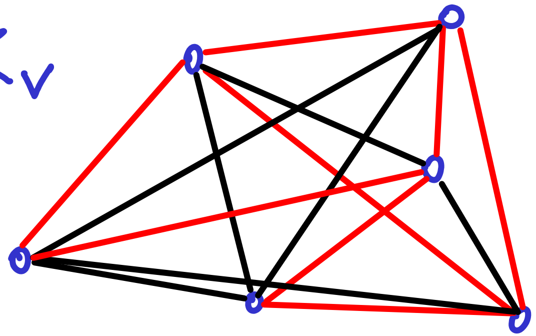


K_v



Equivalent statement

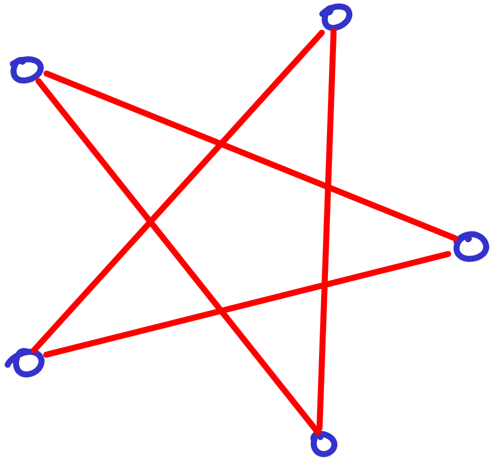
If we color each edge of K_v red or black, then we must get a red triangle or a black triangle

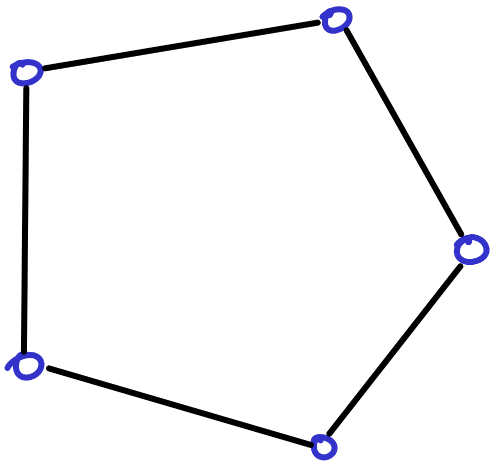
Recap: if you want a clique or an independent set of size $\underline{3}$ then you'll be happy as long as $|V| \geq \underline{6}$

$|V| < 6$?

Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as $|V| \geq 6$

$\hookrightarrow R(3,3) = 6$





Recap: if you want a clique or an independent set of size $\underline{3}$

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$\hookrightarrow R(3,3) = 6$

$R(4,4)$

$\downarrow$

18 (one direction to be shown)

Recap: if you want a clique or an independent set of size 3

then you'll be happy as long as $|V| \geq 6$

$\hookrightarrow R(3,3) = 6$

$R(4,4)$	$R(5,5)$
$\downarrow$	$\downarrow$
18	we don't know!
	$[43 \ldots 49]$

Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as $|V| \geq 6$

$\hookrightarrow R(3,3) = 6$

$R(4,4)$

$\downarrow$

18

$R(5,5)$

$\downarrow$

we don't know!

[43...49]

$R(n,n)$

~ exponential

$\downarrow$

even for small values we will probably never know the exact answer

Recap: if you want a clique or an independent set of size $\underline{3}$

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↳ $R(3,3) = 6$

$R(4,4)$ → 18

$R(5,5)$ → we don't know! [43...49]

$R(n,n)$ ~ exponential → even for small values we will probably never know the exact answer

For more, see Ramsey's Thm.

$R(x,y)$: smallest number N such that any graph with $\geq N$ vertices has a clique of size x or an independent set of size y

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$$R(4,2) = ?$$

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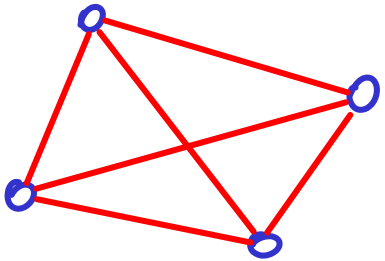
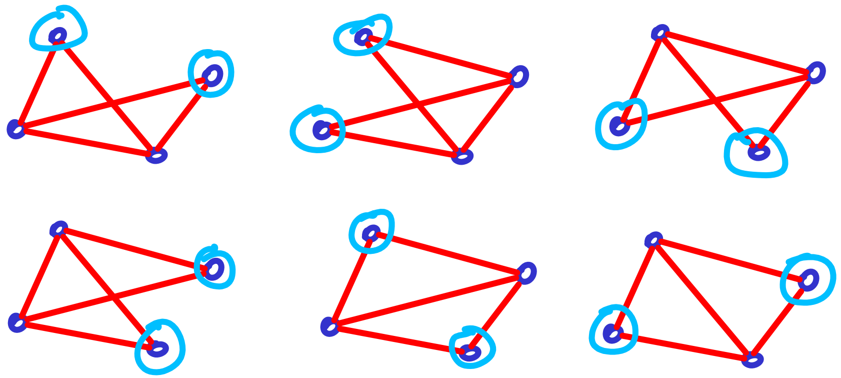
$R(4,2) = 4$



OR



$R(4,3)$

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Suppose $|V| \geq 10$

Pick any vertex, A. ≥ 9 vertices remain.

A

$R(4,3)$

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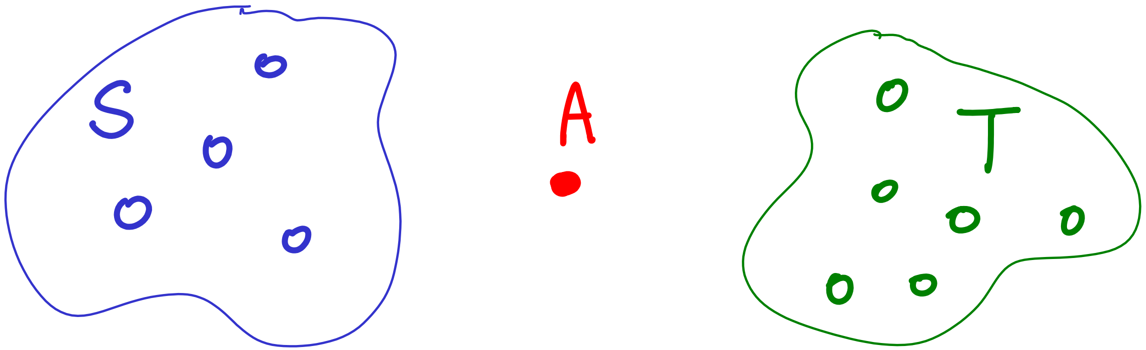
Form 2 groups:

S & T



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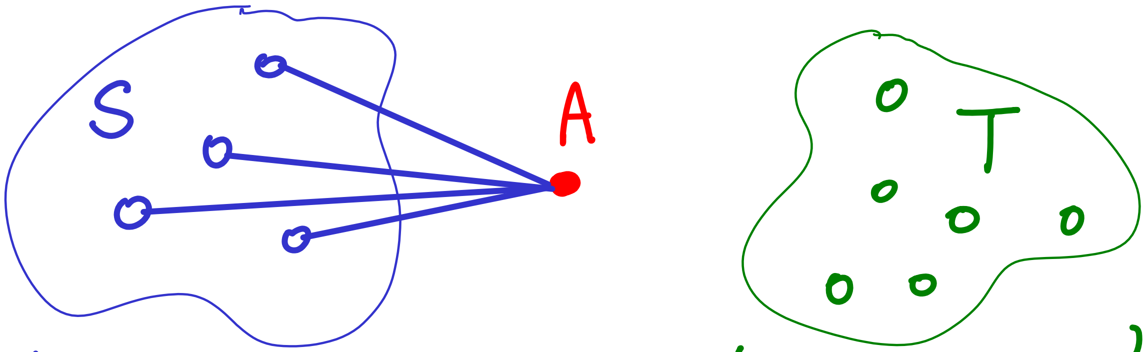

edges to A

no edges to A

$R(4,3)$

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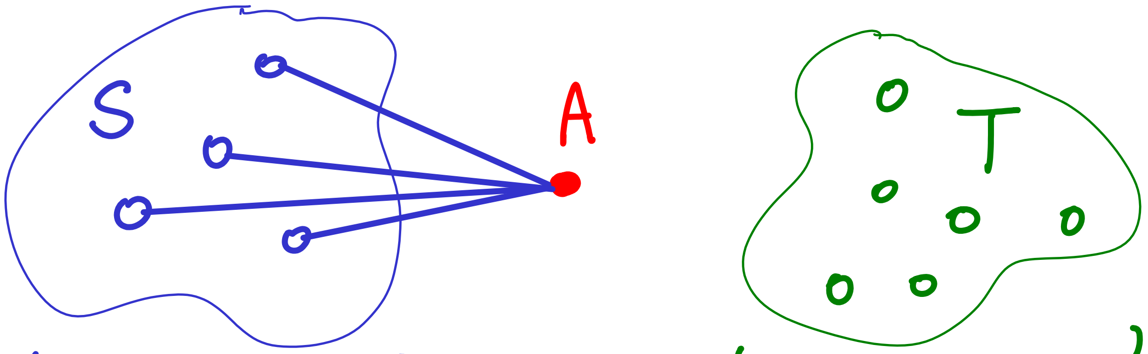


edges to A

no edges to A

either $|S| \geq 6$ or $|T| \geq 4$

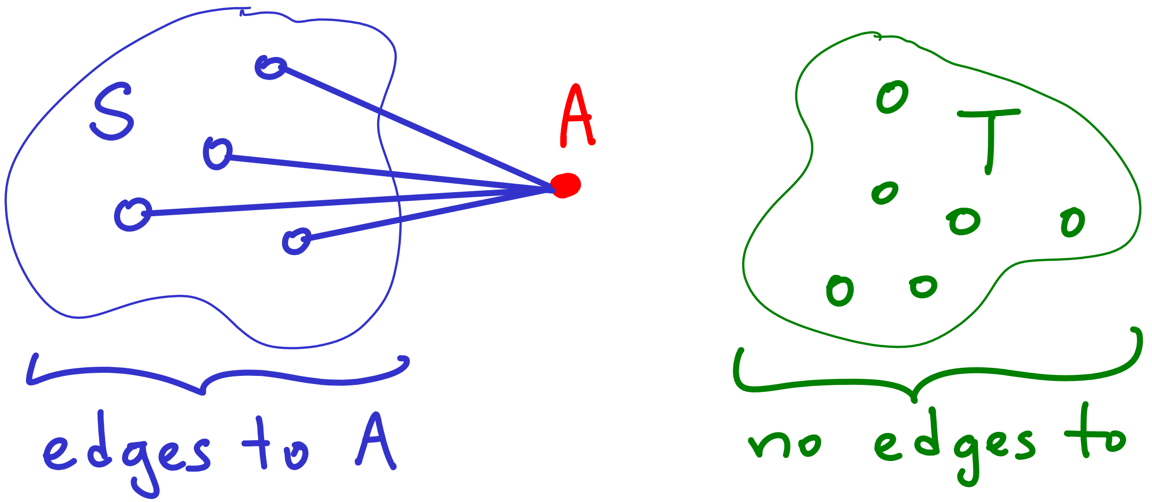
$R(4,3)$

Suppose $|V| \geq 10$

Pick any vertex, A. ≥ 9 vertices remain.

Form 2 groups:

S & T



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If $|S| \geq 6$, ...?

$R(4,3)$

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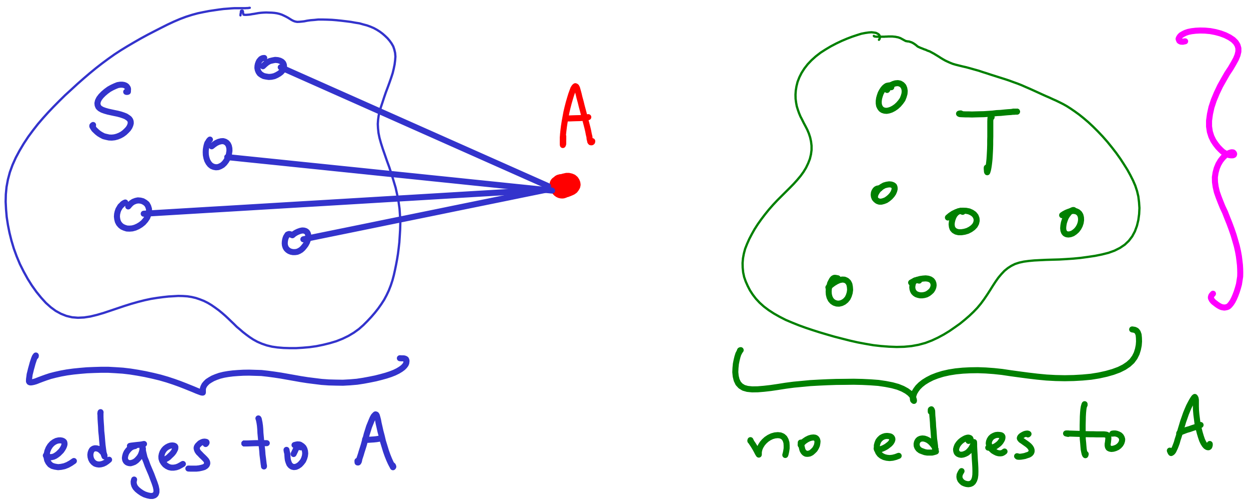

either $|S| \geq 6$ or $|T| \geq 4$

If $|S| \geq 6$, use $R(3,3) = 6$ ... ?

$R(4,3)$

Suppose $|V| \geq 10$

Pick any vertex, A. ≥ 9 vertices remain.

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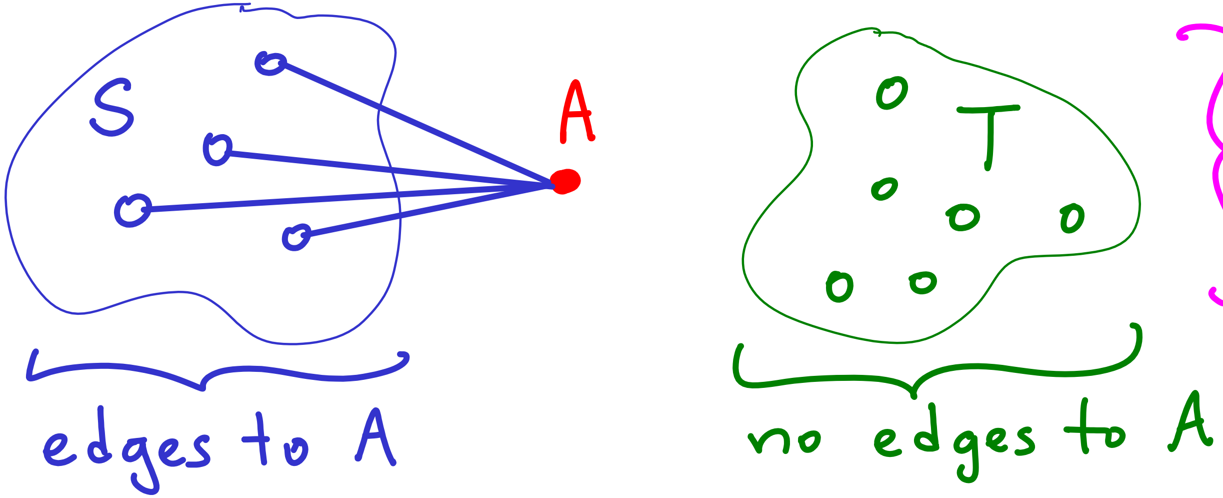

either $|S| \geq 6$ or $|T| \geq 4$

If $|S| \geq 6$, use $R(3,3)=6$: S has 3 independent vertices

OR S has a 3-clique

...?

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Suppose $|V| \geq 10$

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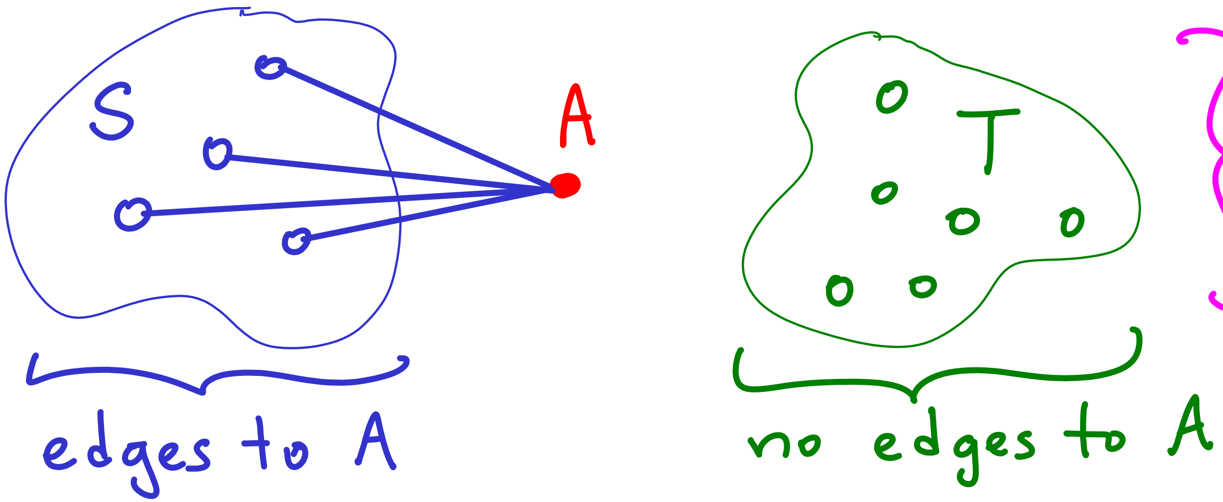

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If $|S| \geq 6$, use $R(3,3) = 6$: S has 3 independent vertices (done),
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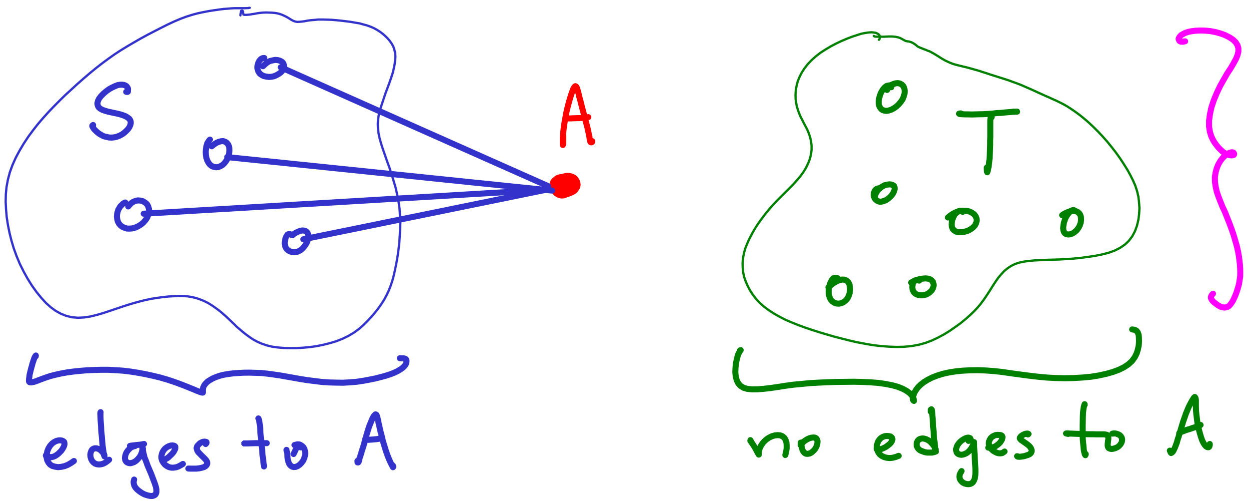

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$R(4,3)$

Suppose $|V| \geq 10$

Pick any vertex, A. ≥ 9 vertices remain.

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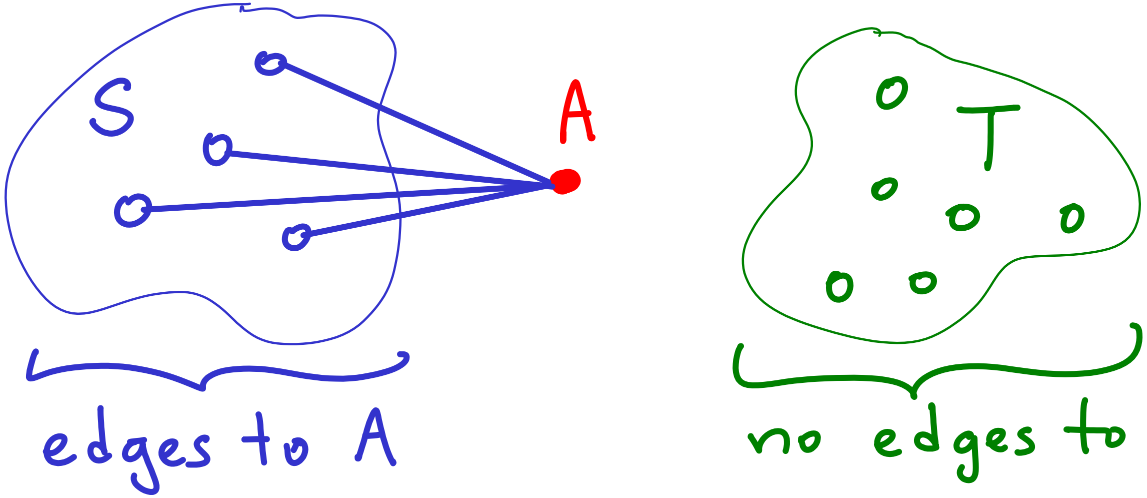

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R(4,3)

Suppose $|V| \geq 10$

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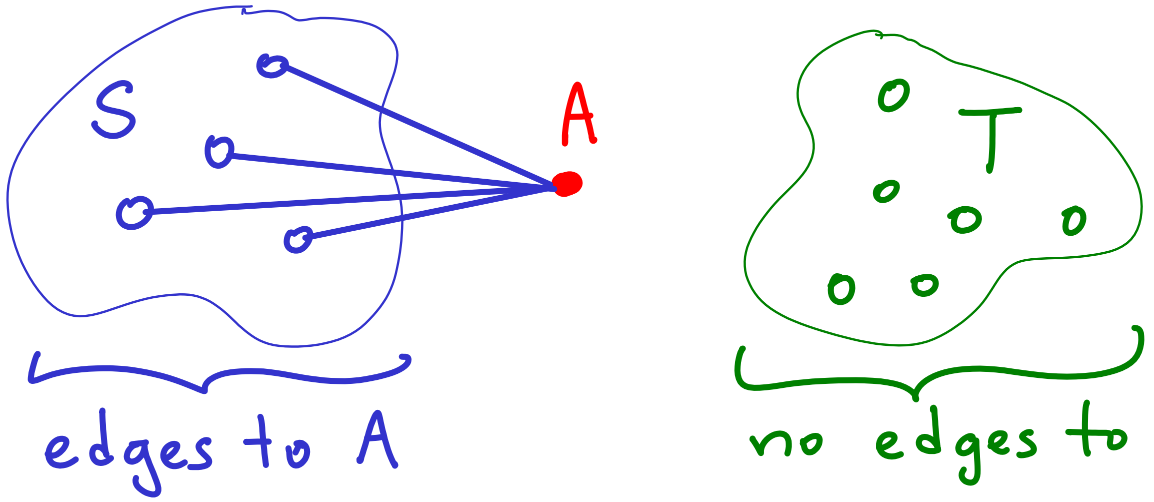

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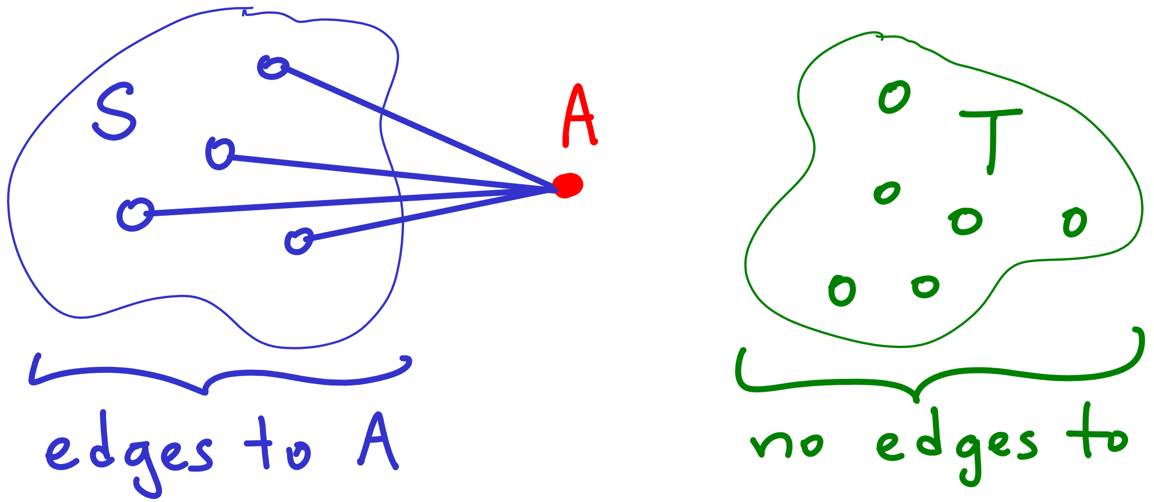
Otherwise $\exists$ a,b in T w/ no edge. Combine w/ A.

$R(4,3) \leq 10$

Suppose $|V| \geq 10$

Pick any vertex, A. ≥ 9 vertices remain.

Form 2 groups: S & T



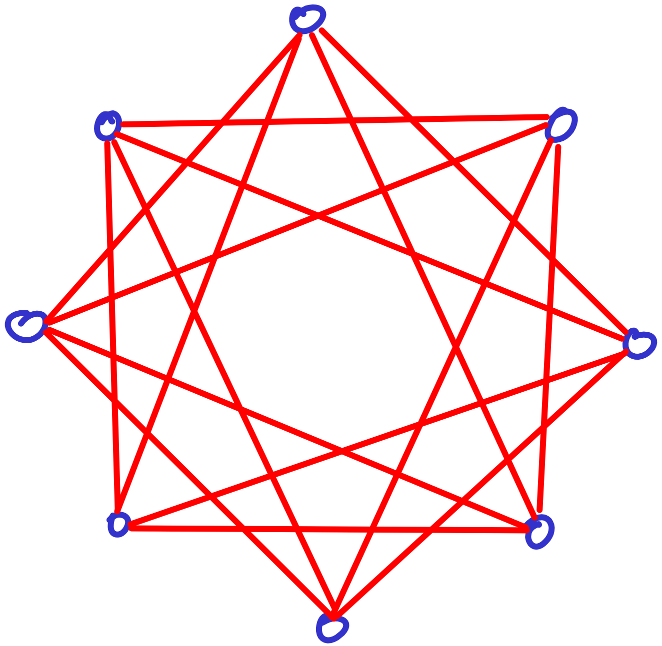
either $|S| \geq 6$ or $|T| \geq 4$

If $|S| \geq 6$, use $R(3,3)=6$: S has 3 independent vertices (done), OR S has a 3-clique, so with A we get a 4-clique.

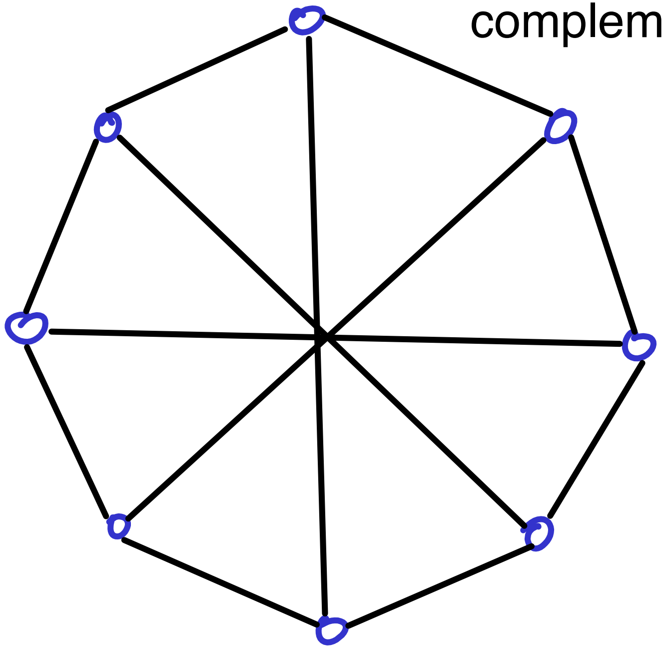
If $|T| \geq 4$, if T is a clique: done. Otherwise $\exists\, a,b$ in T w/ no edge. Combine w/ A. $\square$

$R(4,3) \leq 10$

$R(4,3) > 8$



no 4-clique



complement(G)

no 3-clique = no 3-independent in G

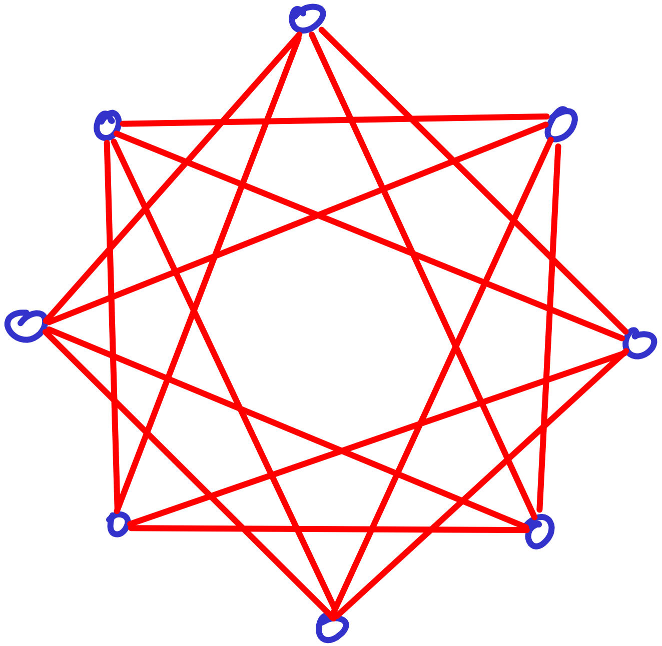
$R(4,3) \leq 10$

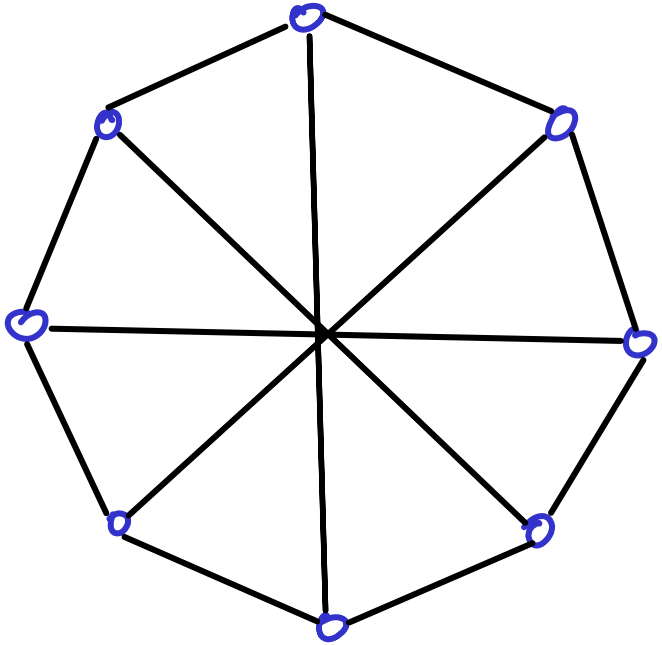
$R(4,3) > 8$

... turns out $R(4,3) = 9$

↳ not terribly hard

↳ notice $R(x,y) = R(y,x)$





$R(4,4)$

$R(4,4)$

Suppose $|V| \geq 18$

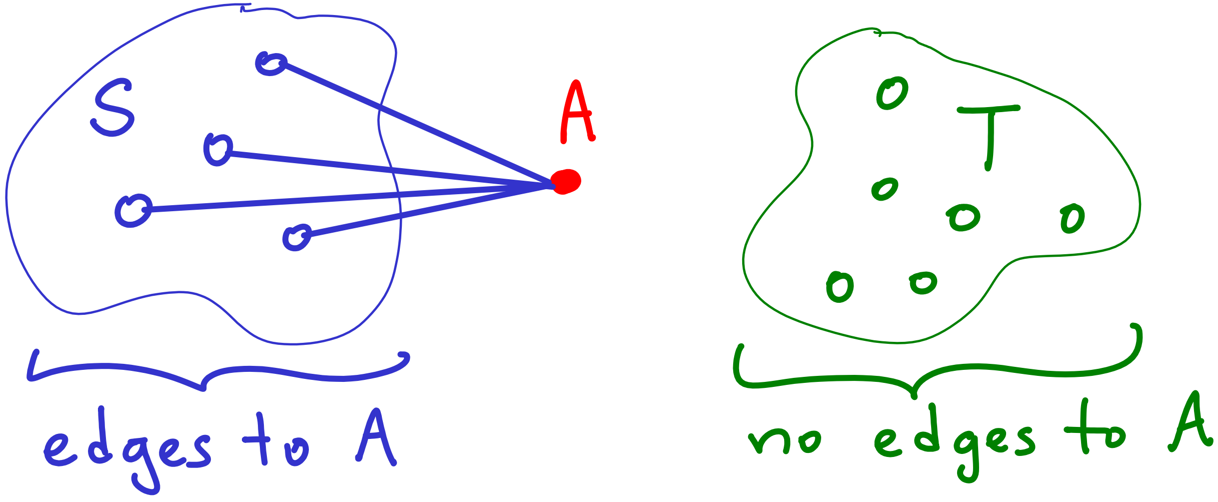
Pick any vertex, A. ≥ 17 vertices remain.

R(4,4)

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:
S & T



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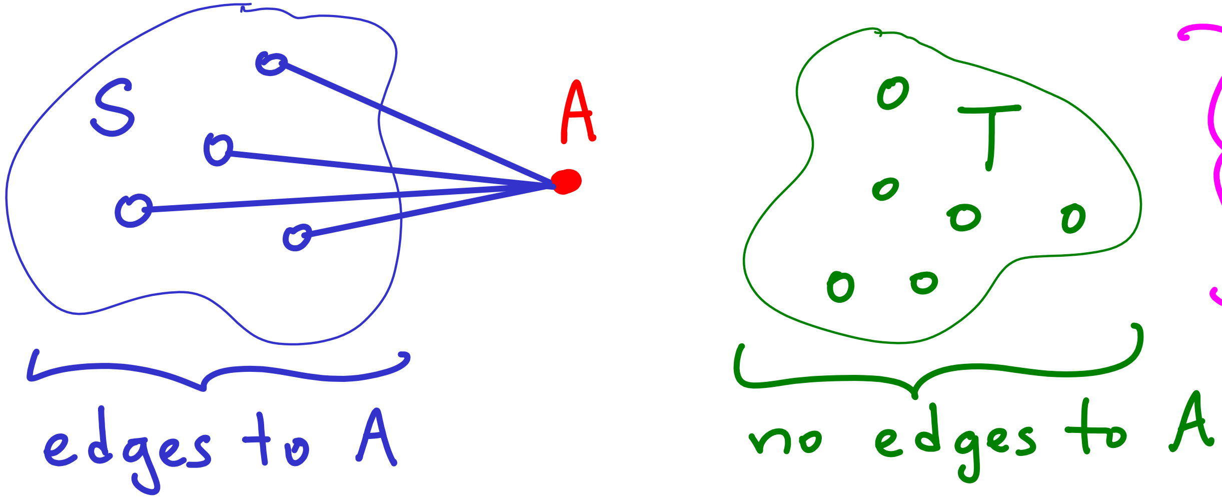


either $|S| \geq 9$ or $|T| \geq 9$

If $|S| \geq 9$?

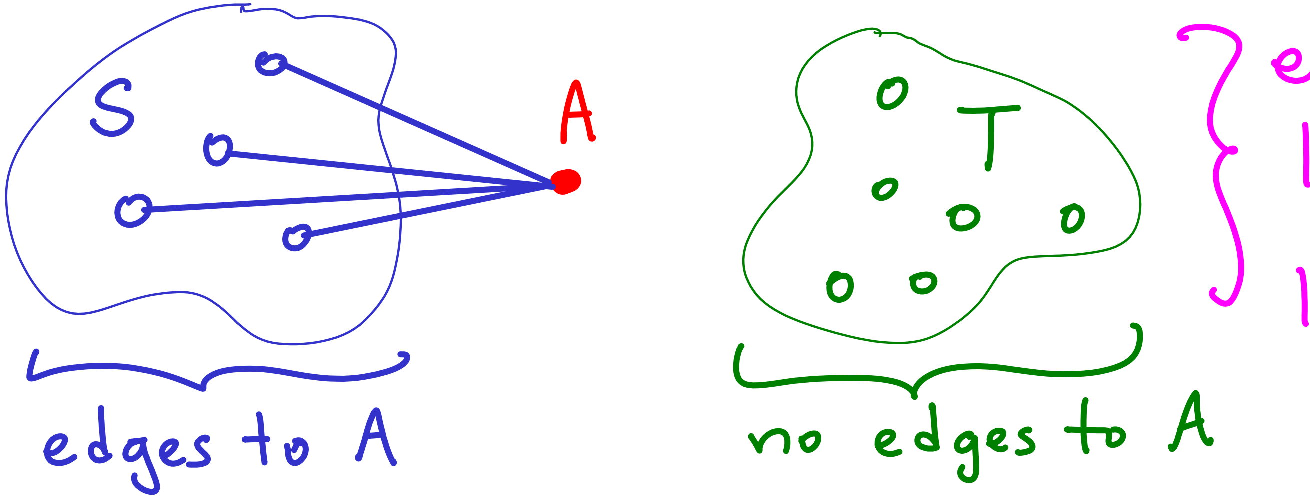
$R(4,4)$

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:

S & T



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If $|S| \geq 9$, use $R(3,4) = 9$ ... ?

$R(4,4)$

Suppose $|V| \geq 18$

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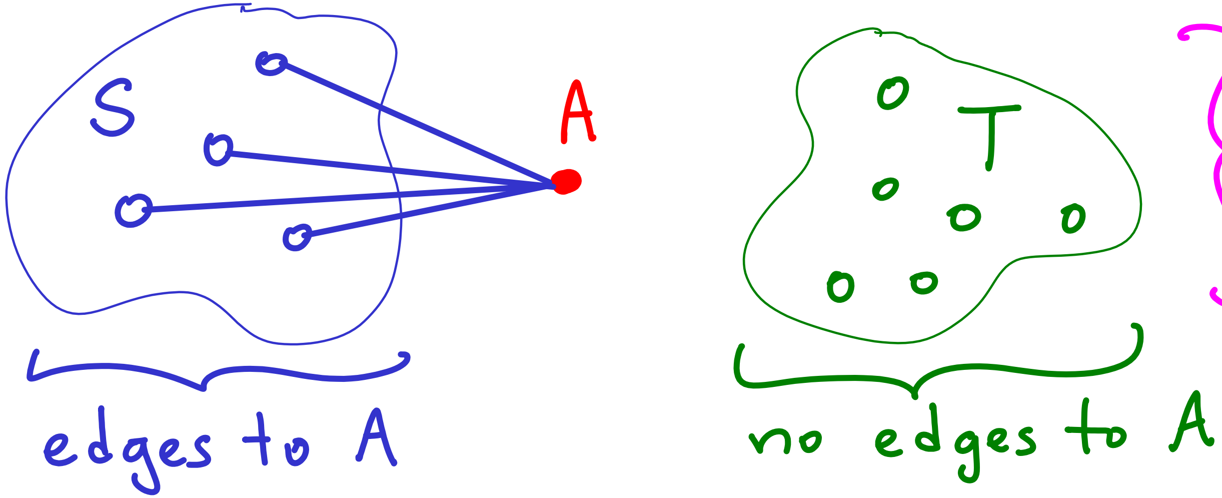
Form 2 groups:
S & T



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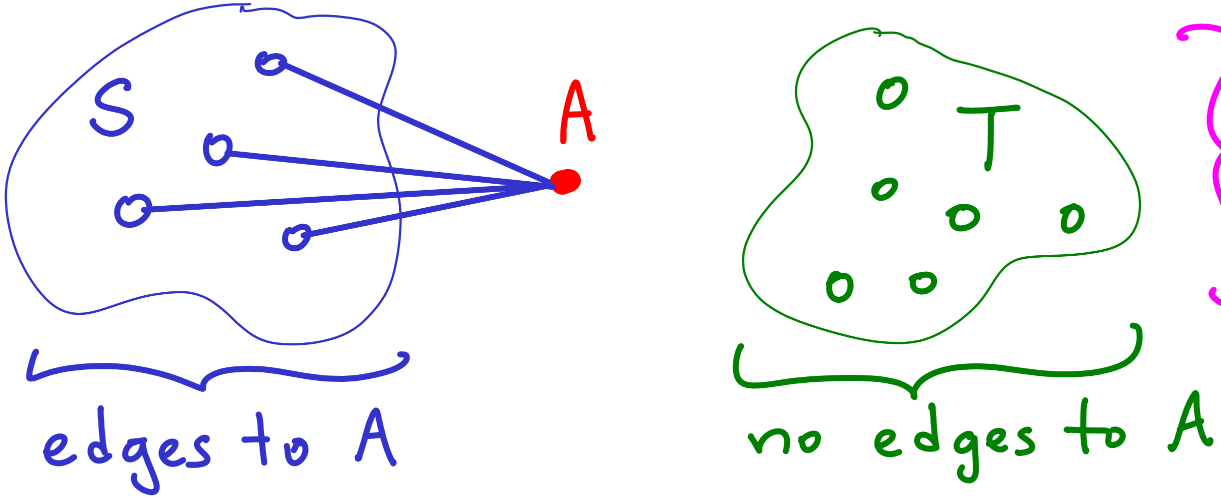

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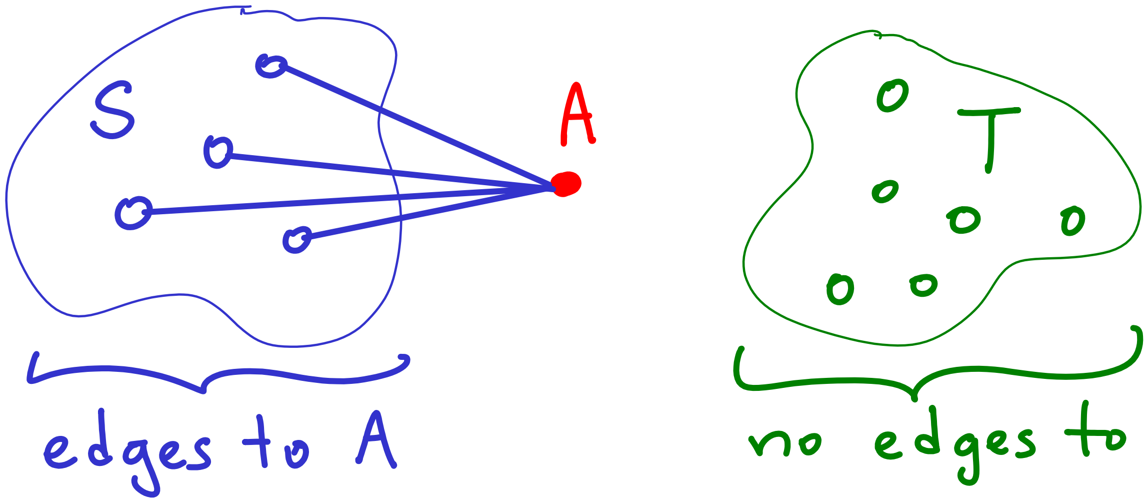
$R(4,4)$

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:

S & T



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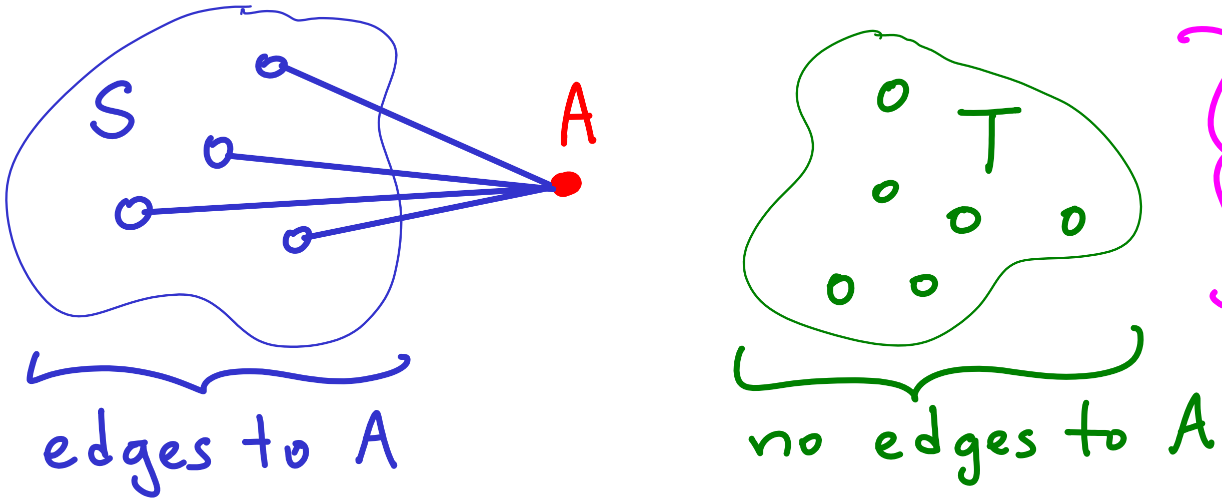
$R(4,4)$

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:
S & T



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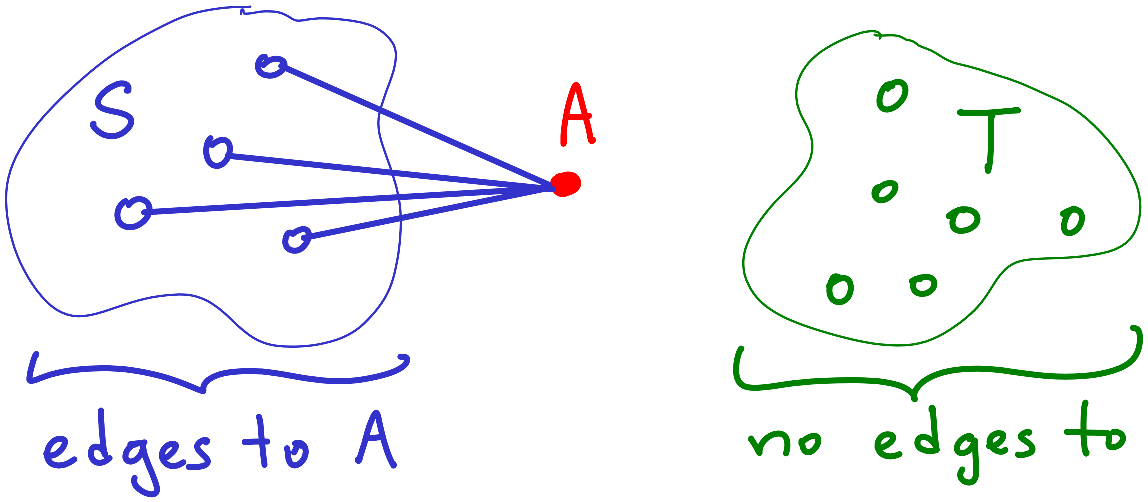
If $|T| \geq 9$, use $R(4,3) = 9$ ...?

$R(4,4)$

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups:
S & T



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<u>OR</u> S has a 3-clique, so with A we get a 4-clique.

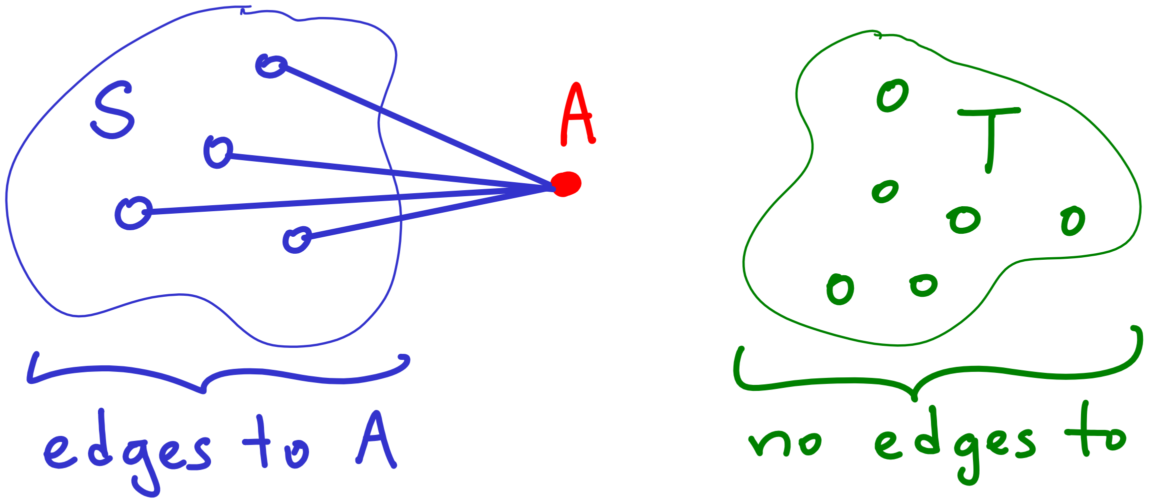
If $|T| \geq 9$, use $R(4,3) = 9$: T has a 4-clique, <u>OR</u>
T has 3 independent vertices ... ?

R(4,4)

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups: S & T



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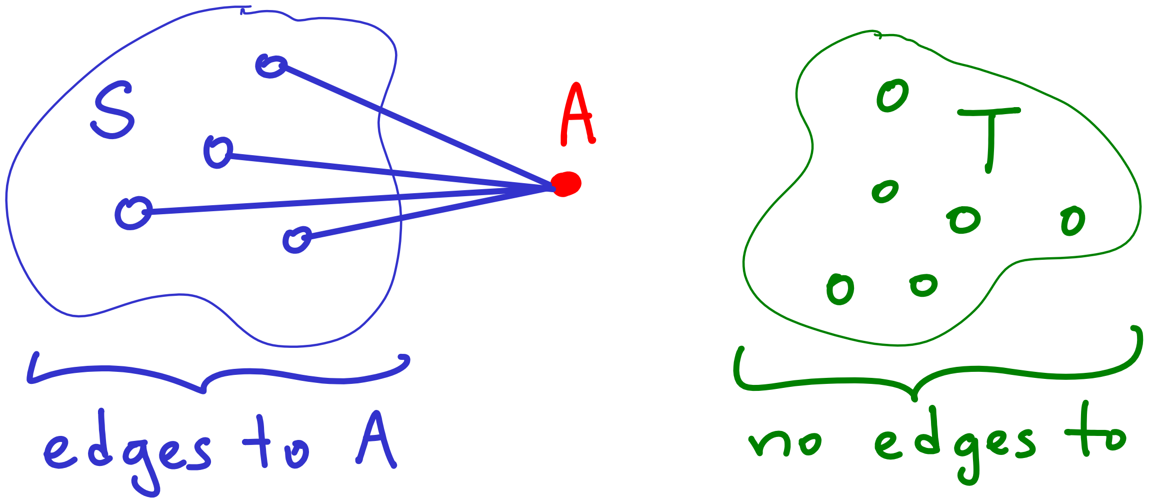
If $|T| \geq 9$, use $R(4,3) = 9$: T has a 4-clique, (done) OR T has 3 independent vertices, so with A we have 4.

$R(4,4) \leq 18$

Suppose $|V| \geq 18$

Pick any vertex, A. ≥ 17 vertices remain.

Form 2 groups: S & T



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Notes:

(1) if we only knew that $R(4,3) \leq 10$ (instead of $=9$)

then ... ?

Notes:

(1) if we only knew that $R(4,3) \leq 10$ (instead of $= 9$)

we could have used $|V| \geq 20$ for $R(4,4)$

As you bound smaller R( ) values, you can get (loose) bounds for larger ones

Notes:

(1) if we only knew that $R(4,3) \leq 10$ (instead of $= 9$)

we could have used $|V| \geq 20$ for $R(4,4)$

(2) there is a graph w/ 17 vertices with no clique or independent set of size 4

↳ $R(4,4) = 18$