

INDICATOR RANDOM VARIABLES

(IRV)

(taking value 0 or 1)

We already saw this : Y : parity of rolling one die.

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Another example: flip a coin 10 times.

X = #times we see pattern HT

$$E[X] = ?$$

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HT could appear at flips 1 & 2, or 2 & 3, ..., or 9 & 10

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← linearity of expectation

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Notice X_1 & X_2 are not independent. $P(X_i=1) = \frac{1}{4}$
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$$\begin{aligned} E[X_i] &= 0 \cdot P(X_i=0) + 1 \cdot P(X_i=1) = \frac{1}{4} \\ &= 9 \cdot \frac{1}{4} \end{aligned}$$

INDICATOR RANDOM VARIABLES

The hat-check problem (a.k.a. coat-check)

- ◆ n people at a party leave their hats with an attendant
- ◆ The attendant gives hats back randomly.

How many people do we expect to get their own hats back?

The hat-check problem

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$$E[X] = E\left[\sum_{k=1}^n X_k\right]$$

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$$E[X_k] = \frac{1}{n} \quad \begin{array}{l} \text{(random)} \\ P(X_k = 1) \end{array}$$

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The hiring problem: you need one assistant.

◆ n candidates, interviewed in random order.

◆ No 2 equally skilled.

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◆ any time you interview someone better than all previous, you hire the new person & fire the current assistant.

How many people do you expect to hire?

The hiring problem

$X =$ # people you will hire

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iff better than
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$$= \sum_{k=1}^n \frac{1}{k} \leq 1 + \ln n$$

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The birthday problem

How many people do we need in a room so that we expect to have (at least) one birthday match?

The birthday problem

X = # birthday matches among n people

For what n do we get $E[X] \geq 1$?

↳ Set up $E[X]$ as function of n

The birthday problem

$X = \#$ birthday matches among n people

What should our I.R.V. be? $X?$

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all $\binom{n}{2}$ pairs

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$$X_{ij} = \begin{cases} 1 & \text{if persons } i \text{ \& } j \text{ match} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_{ij}] = \frac{1}{365}$$

$$= \underbrace{\sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij}}_{\text{all } \binom{n}{2} \text{ pairs}}$$

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$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{1}{365} = \frac{n \cdot (n-1)}{2} \cdot \frac{1}{365}$$

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we said we want $E[X]=1$

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$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{1}{365} = \frac{n \cdot (n-1)}{2} \cdot \frac{1}{365} = 1 \Rightarrow n \approx 28$$

□

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

(new problem)

Not about
expected value or IRV

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

$$= 1 - P(\text{nobody has the same birthday})$$

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$= 1 - P(\text{nobody has the same birthday})$

$= 1 - \frac{365}{365}$
person #1

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

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person #1
person #2

$$= 1 - \frac{365}{365} \cdot \frac{364}{365}$$

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$$= 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$$

person #1 person #2 person #3

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$$= 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365 - (k-1)}{365}$$

Diagram illustrating the probability calculation for the birthday problem. The expression is $1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365 - (k-1)}{365}$. The terms are annotated with green brackets and labels: "person #1" above the first fraction, "person #2" above the second, "person #3" above the third, and "person #k" above the final fraction.

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

$= 1 - P(\text{nobody has the same birthday})$

$$= 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365 - (k-1)}{365} = 1 - \frac{365!}{(365-k)! 365^k}$$

The diagram shows the probability calculation for the birthday problem. It starts with the probability of no two people sharing a birthday, which is the product of probabilities for each person having a unique birthday. The first person has 365 choices, the second has 364, the third has 363, and so on, until the k-th person has 365 - (k-1) choices. This is written as a product of fractions: $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365 - (k-1)}{365}$. The terms are labeled with green brackets as "person #1", "person #2", "person #3", and "person #k". The final result is simplified to $1 - \frac{365!}{(365-k)! 365^k}$.

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$$k=2 \rightarrow P \sim 0.27\% \left(\frac{1}{365} \right)$$

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$$k \sim 116 \rightarrow P \sim 1 - \frac{1}{10^9}$$

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$$k=300 \rightarrow P \sim 1 - \frac{1}{10^{80}}$$

$(10^{80} \sim \# \text{ atoms in universe})$

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$(10^{80} \sim \# \text{ atoms in universe})$

$$(k > 365 \rightarrow P=1)$$

