The game of 20 questions

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Player 1 thinks of someone that player 2 could name.

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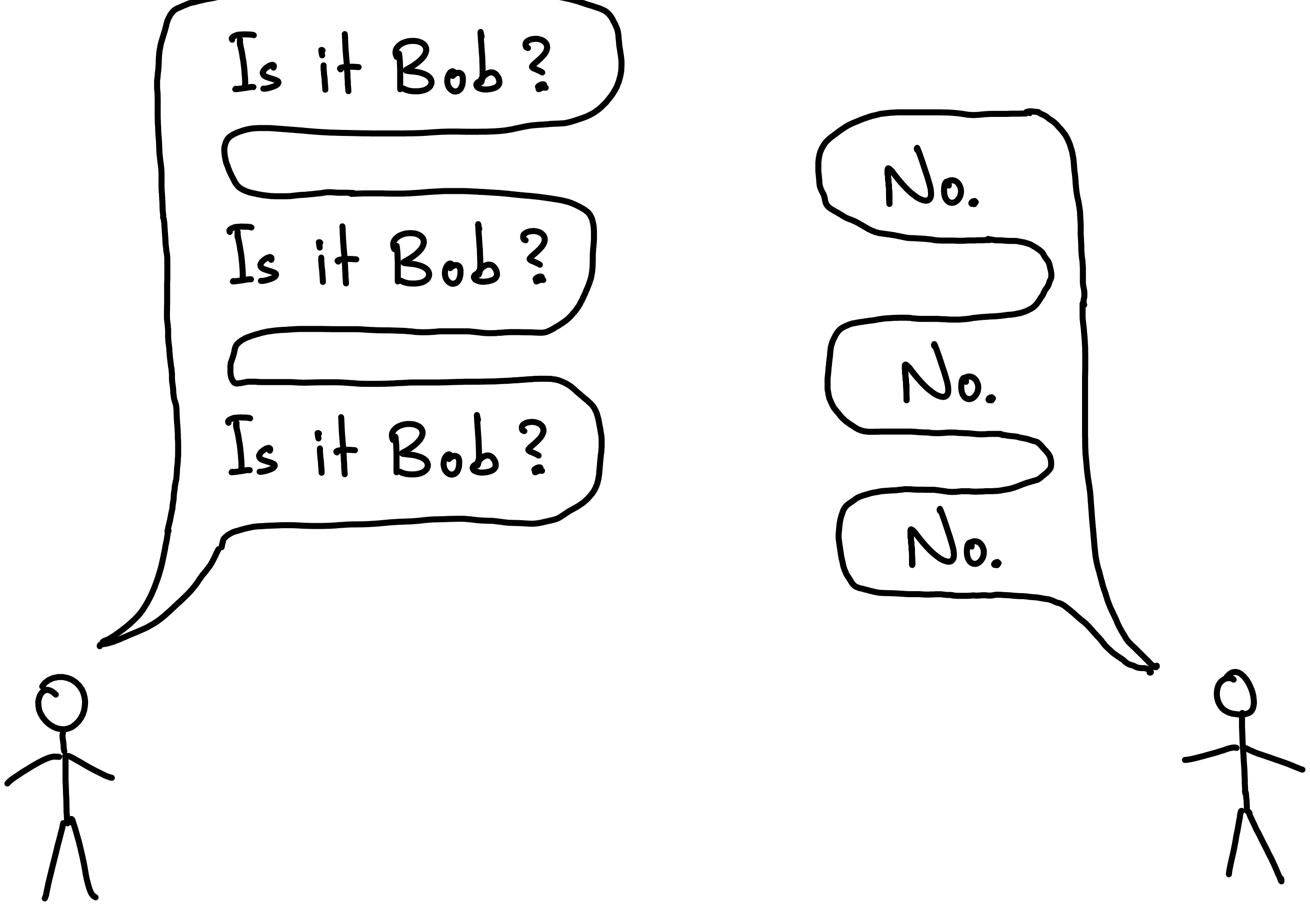
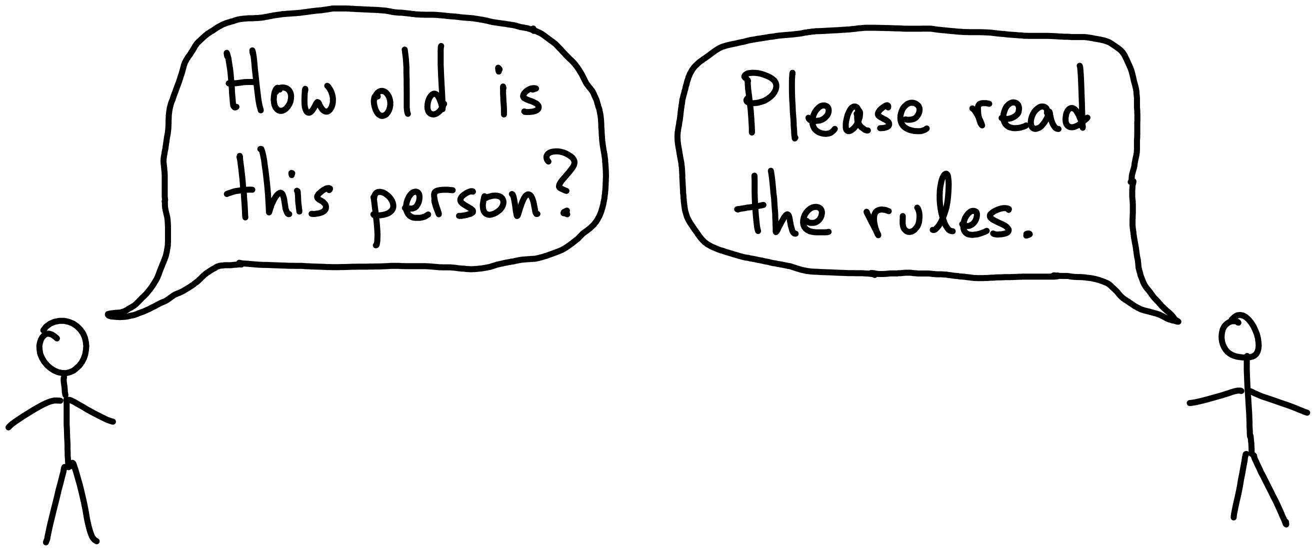
Player 1 thinks of someone that player 2 could name.

Player 2 can ask 20 questions that have a YES or NO answer

& must determine who player 1 is thinking of.


Is it Bob?
Is it Bob?
Is it Bob?
No.
No.
No.


How old is this person?
Please read the rules.

- Suppose there are k possible answers (candidates)

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Bob vs not Bob

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  ↳ Player 2 should ask questions that split the set of candidates.

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For k candidates, how many questions does player 2 <u>need</u>?

$$\lceil \log_2 k \rceil$$

COMPARISON-BASED ALGORITHMS represented as DECISION TREES

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example: an algorithm to sort 3 numbers: a_1, a_2, a_3 (no duplicates)

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$a_1 : a_2$ compare 2 numbers

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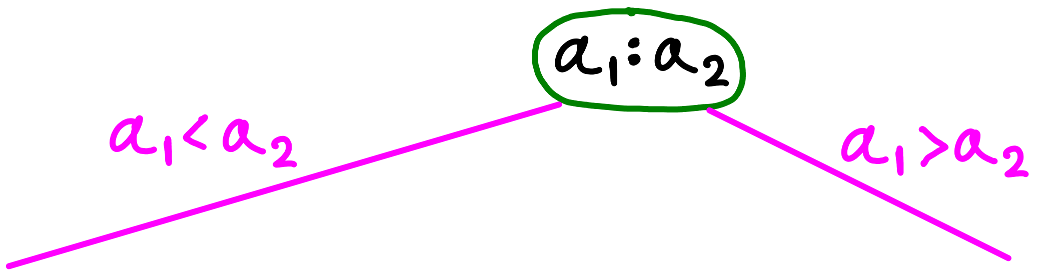

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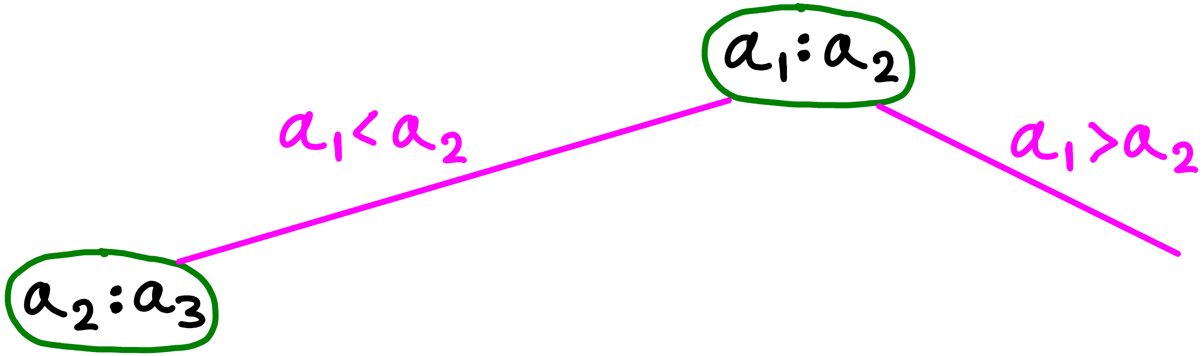

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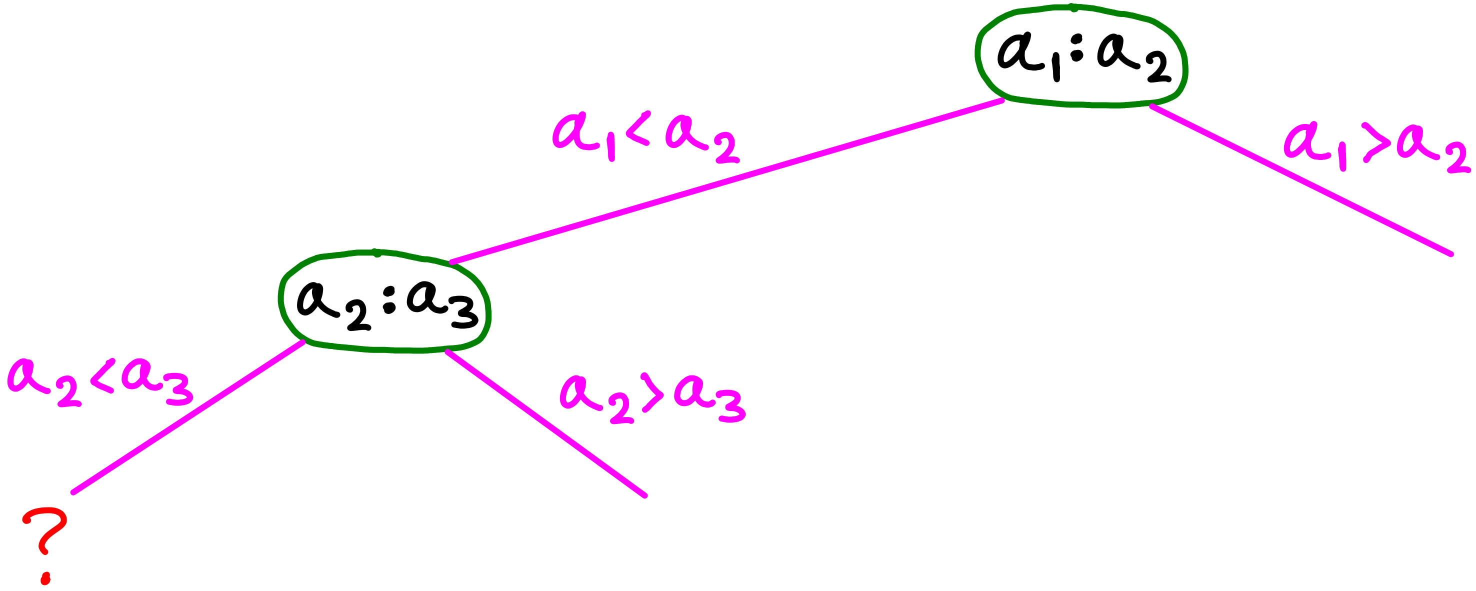

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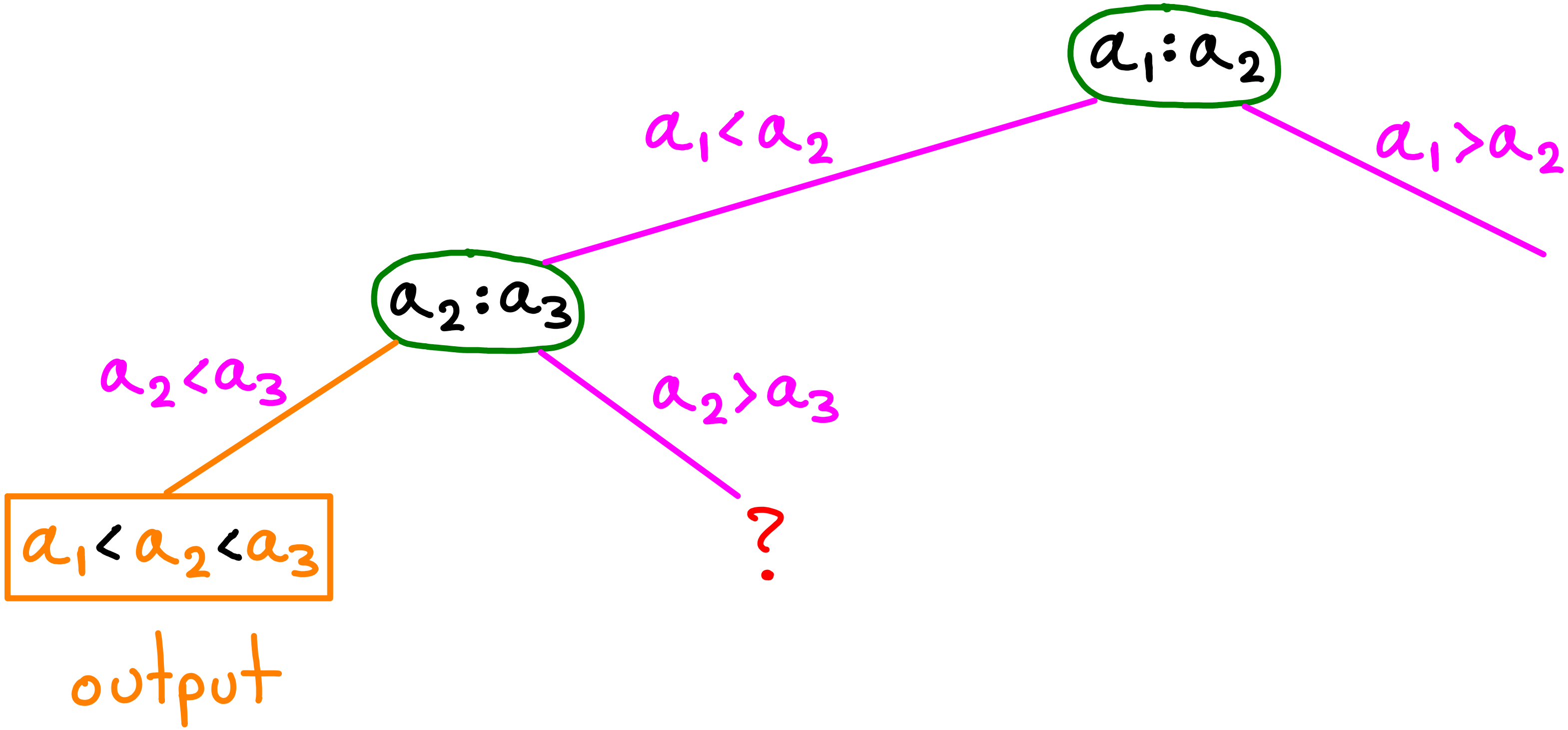

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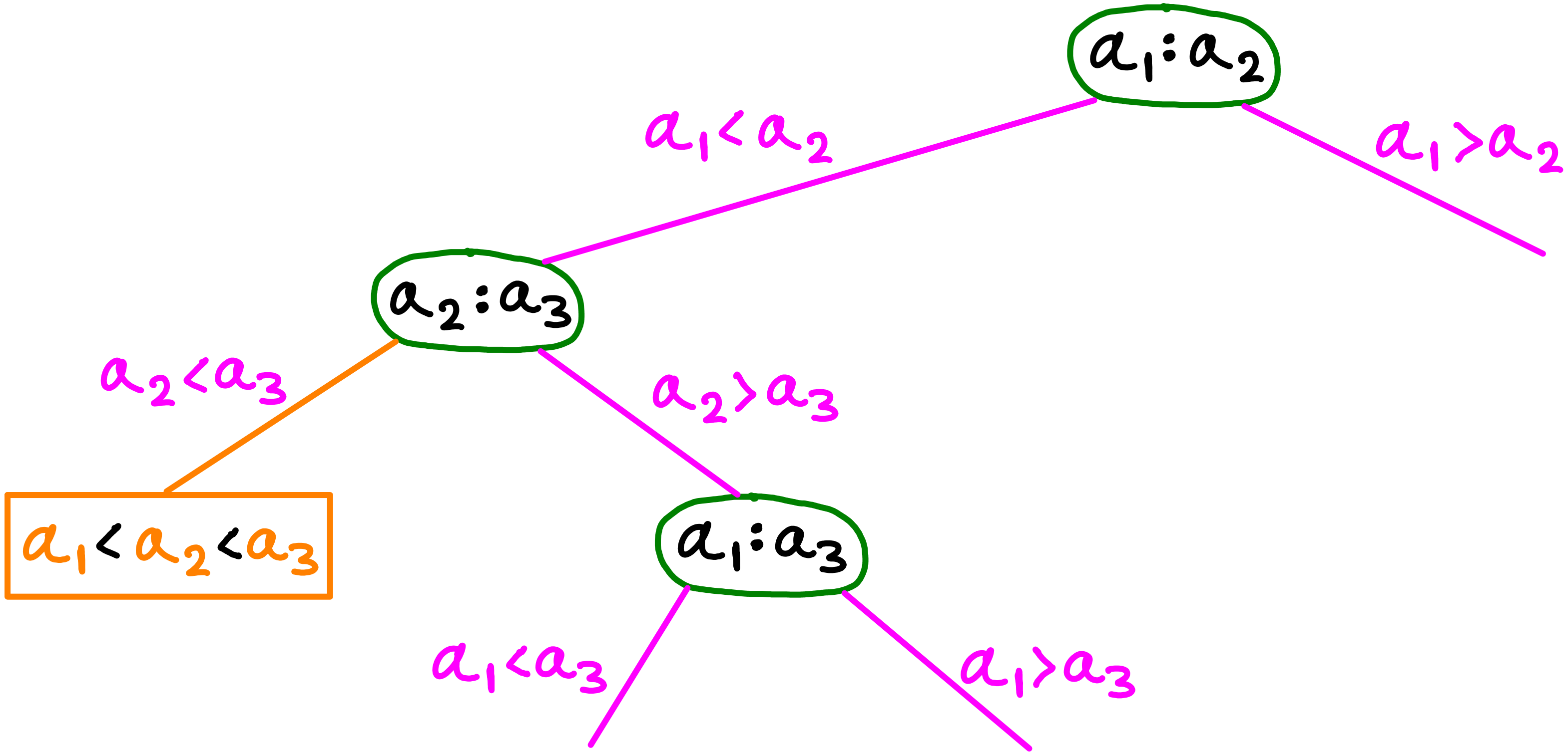

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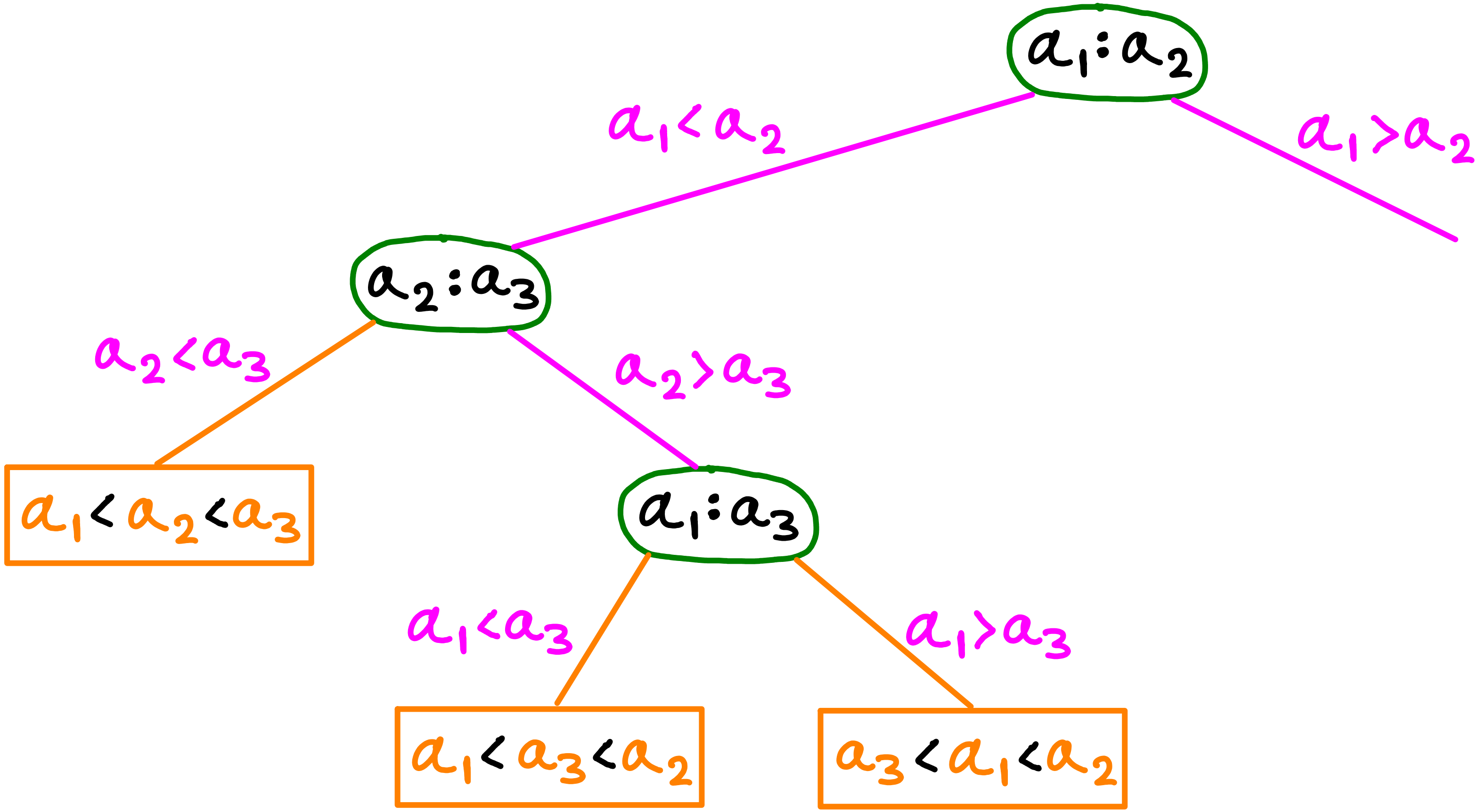

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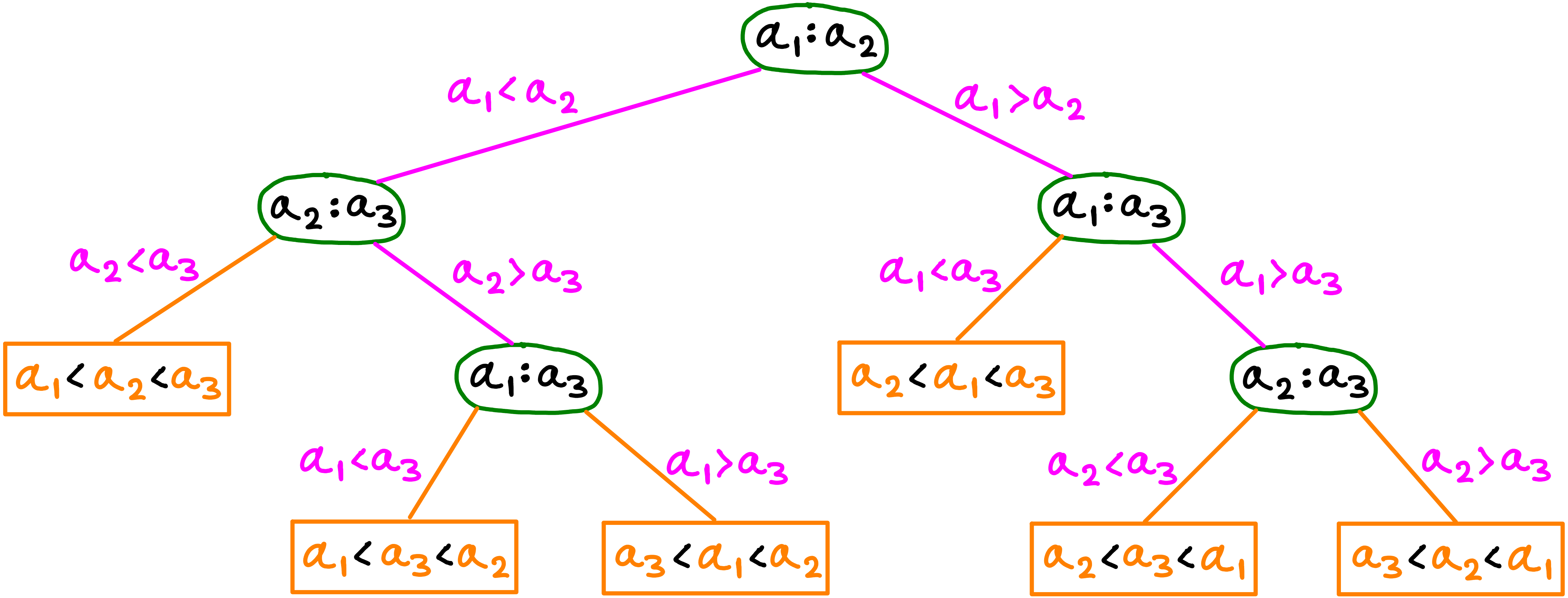
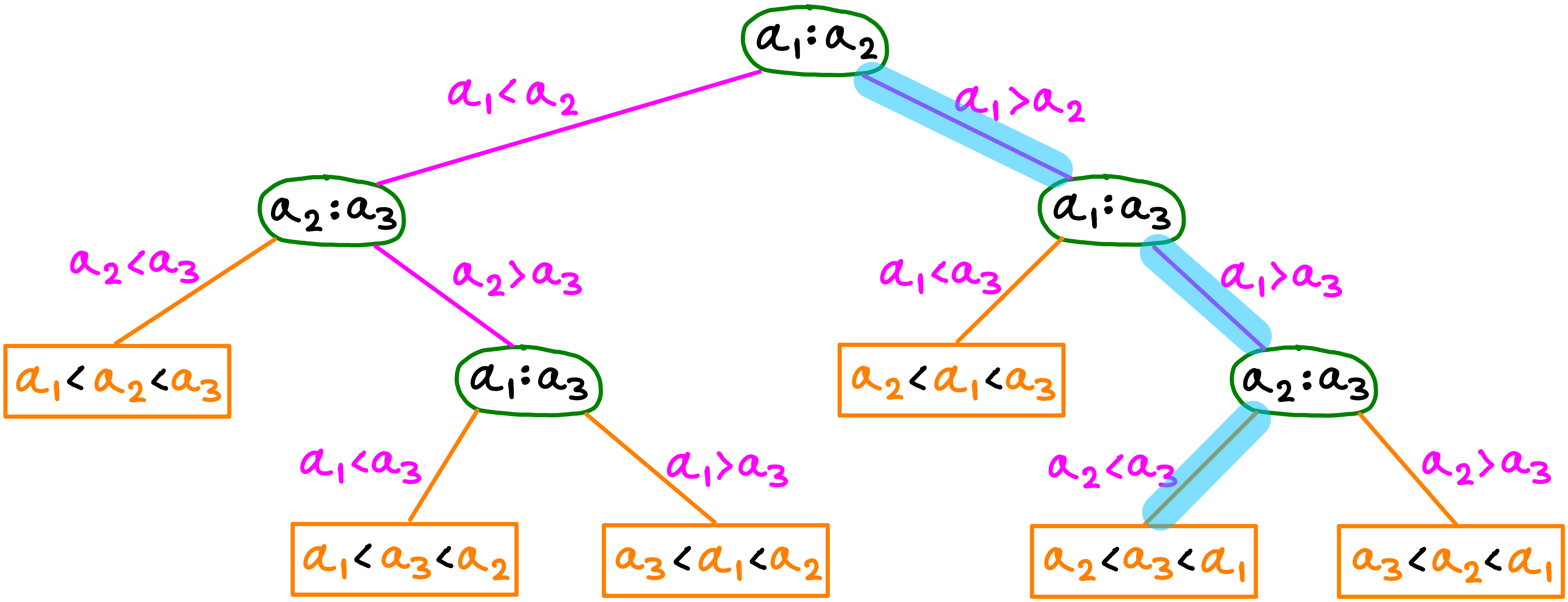


COMPARISON-BASED ALGORITHMS represented as DECISION TREES

example: an algorithm to sort 3 numbers: 9, 4, 6

a_1 a_2 a_3



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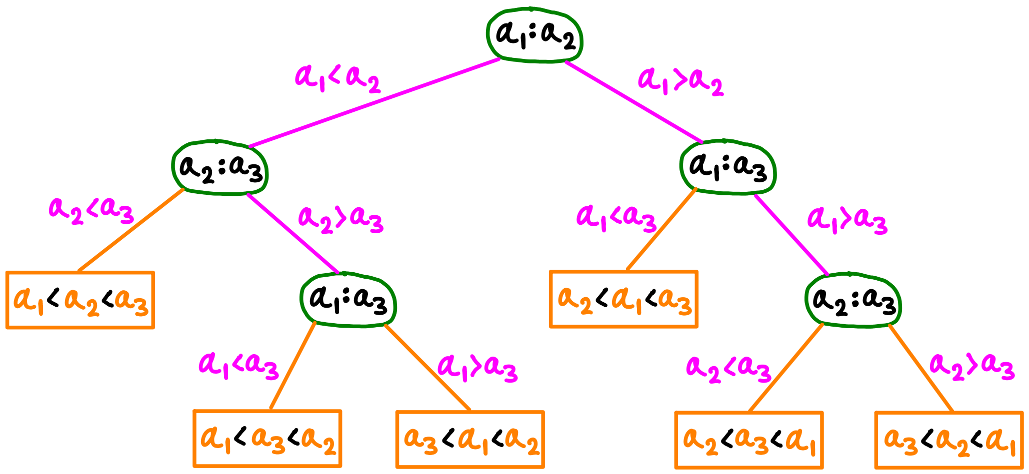

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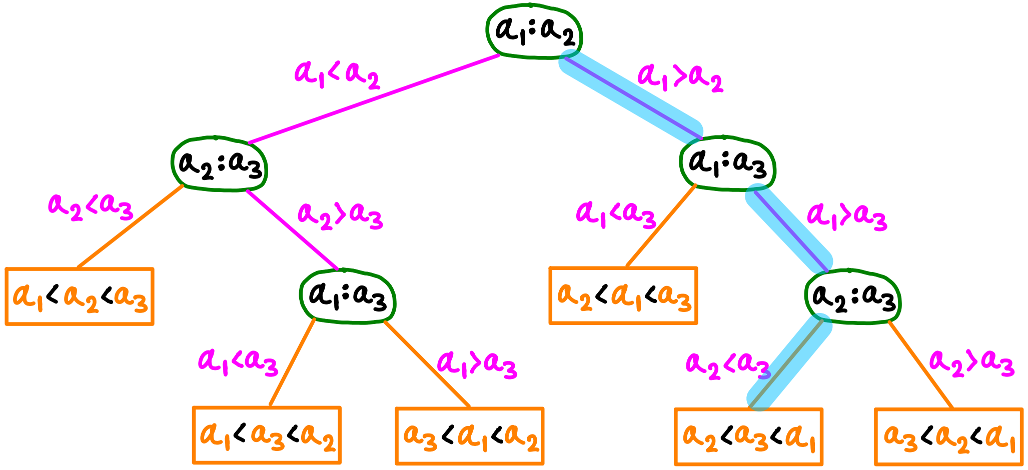

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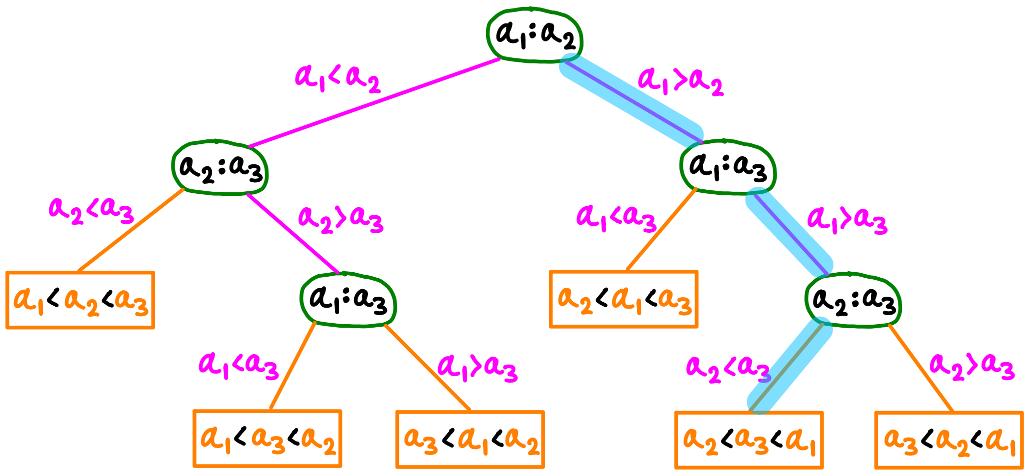

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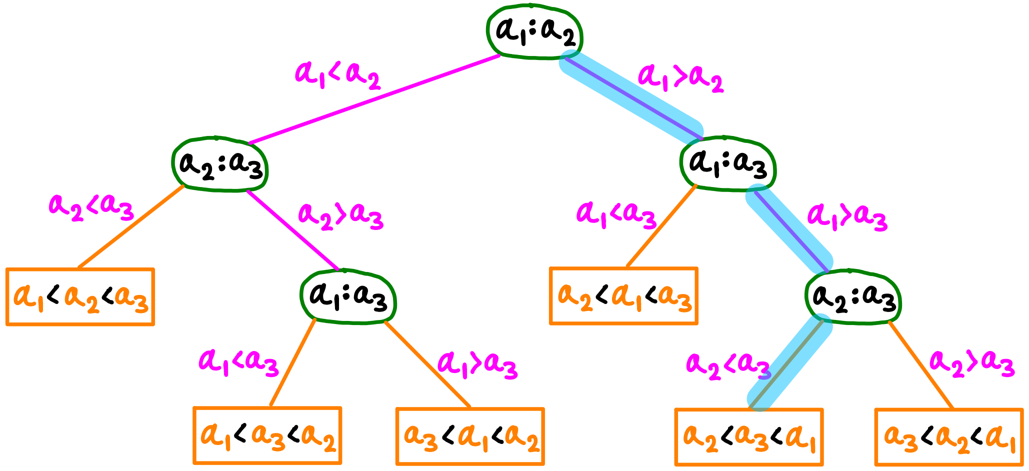

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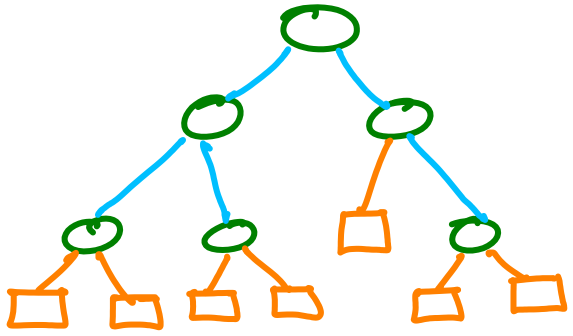
Longest path → worst-case time complexity

the SORTING LOWER BOUND

a lower bound for the worst-case time complexity of all comparison-based sorting algorithms

Consider the decision tree corresponding to any algorithm that can sort n items.

↳ e.g., the best algo EVER.



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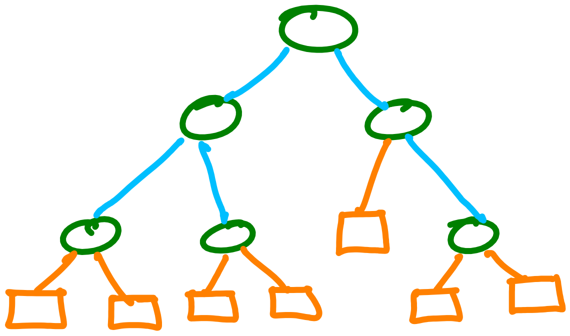

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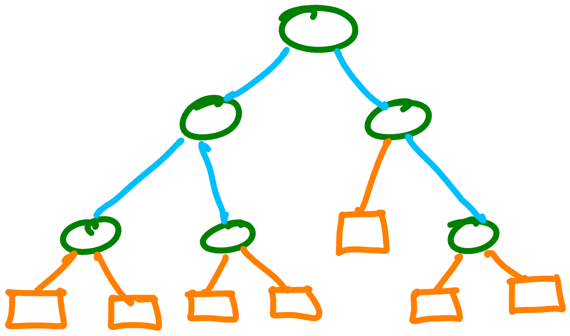

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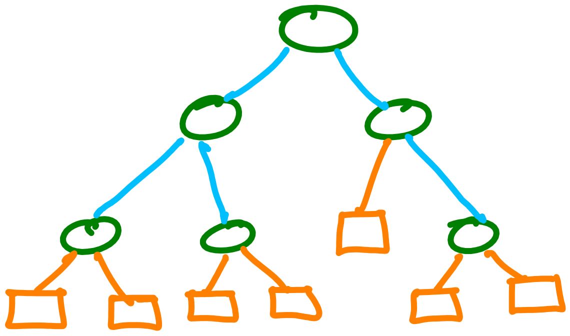

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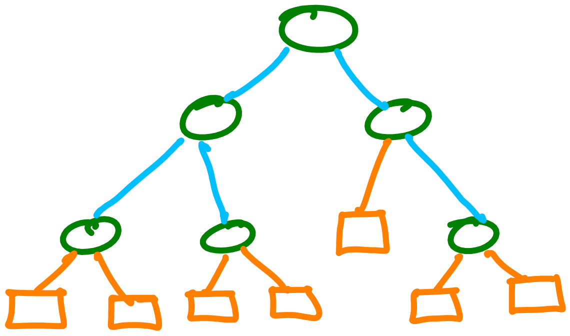

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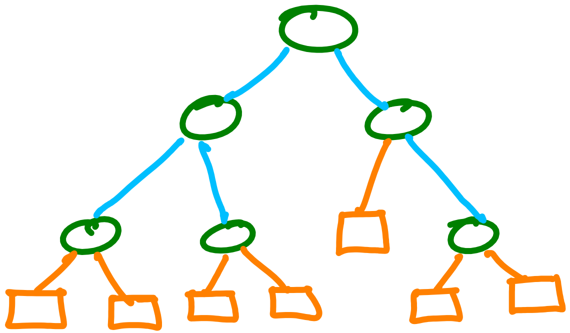

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Conclusion: For every comparison-sort algorithm,

worst-case time complexity $\geq \lceil \log_2 n! \rceil$

Approximating $\log_2 n!$

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Stirling's formula: $n! \geqslant \left(\frac{n}{e}\right)^n$

(method 1)

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$$= \Omega(n\log n)$$

Approximating $\log_2 n!$

(method 2)

Let’s try without using a formula

$$\log(n!) = O(?)$$

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$$\log(n!) \leq \log(n^n)$$

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$\log(n!) \leq \log(n^n) = n \log n$

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$$\log(n!) = \Omega(?)$$

$$\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots \cdots 3 \cdot 2 \cdot 1)$$

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$$\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots \cdots 3 \cdot 2 \cdot 1)$$

$$= \log(n \cdot 1 \cdot (n-1) \cdot 2 \cdot (n-2) \cdot 3 \cdot (n-3) \cdot 4 \cdots \cdots \sim (n-\tfrac{n}{2}) \cdot (n-\tfrac{n}{2}))$$

exactly if n: even

$$\log(n!) = O(n \log n)$$

$$\log(n!) \leq \log(n^n) = n \log n$$

$$\log(n!) = \Omega(?)$$

$$\begin{aligned}
\log(n!) &= \log\big(n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots\ \cdots 3\cdot 2\cdot 1\big)\\
&= \log\big(\underbrace{n\cdot 1}\cdot\underbrace{(n-1)\cdot 2}\cdot\underbrace{(n-2)\cdot 3}\cdot\underbrace{(n-3)\cdot 4}\cdots\ \cdots \sim \underbrace{(n-\tfrac{n}{2})\cdot(n-\tfrac{n}{2})}\big)\\
&\geq \log\big(n\ \cdot\ n\ \cdot\ n\ \cdot\ n\ \cdot\ \cdots\ n\big)
\end{aligned}$$

(~ n/2 terms)

$$\log(n!) = O(n\log n)$$

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$$\geq \log(n\cdot n\cdot n\cdot n\cdot\cdots n)$$

$$= \log(n^{n/2})$$ (assume n: even otherwise $\lfloor\frac{n}{2}\rfloor$)

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(assume n: even
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$\Rightarrow \log(n!) \geq \frac{n}{2} \log n$

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---

so $\frac{1}{2} n \log n \leq \log(n!) \leq n \log n$