# SELECTION

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Given n values, find the k-th smallest

in an array or linked list

rank = k

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(n values → in an array or linked list)

(k-th smallest → rank = k)

- easy if we sort first → $O(n\log n)$

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Given n values, find the k-th smallest

in an array or linked list (Given n values)

rank = k (k-th smallest)

- easy if we sort first → $O(n\log n)$
- easy and fast if $k = O(1)$ or $k = n - O(1)$ → $O(n)$

# SELECTION

Given $n$ values, find the k-th smallest

(Given $n$ values → in an array or linked list)

(k-th smallest → rank = $k$)

- easy if we sort first → $O(n \log n)$
- easy and fast if $k = O(1)$ or $k = n - O(1)$ → $O(n)$

harder as $k \to \frac{n}{2}$ (median)

# SELECTION - Given n values, find the k-th smallest

Simple algorithm:

# SELECTION - Given $n$ values, find the $k$-th smallest

Simple algorithm:

- pick a random input value: $x$

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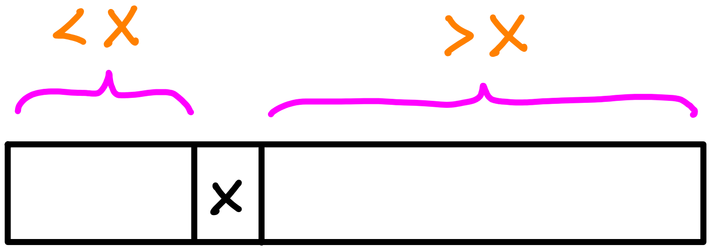

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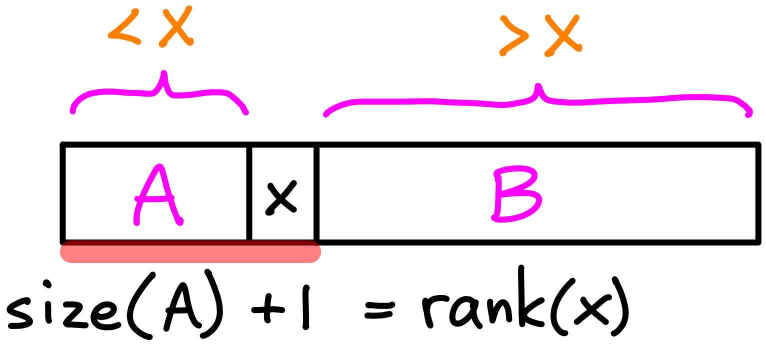

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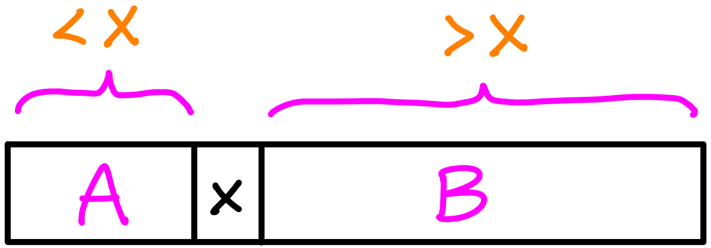


size(A) + 1 = rank(x)

- if rank(x) = k, return x

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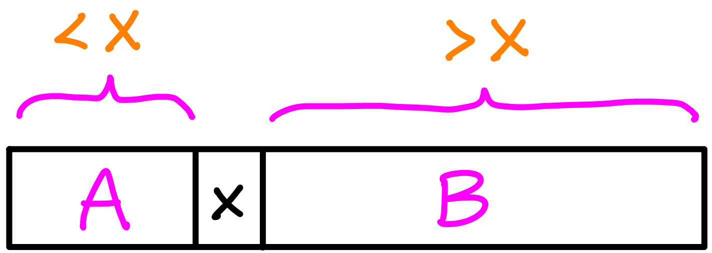


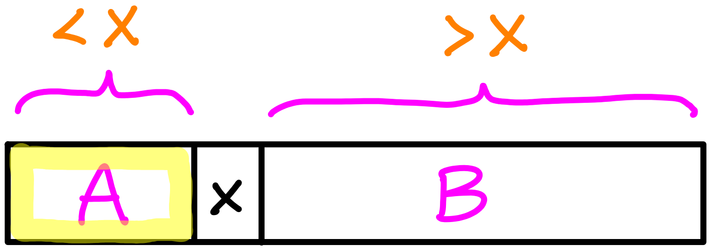

$$\text{size}(A) + 1 = \text{rank}(x)$$

- if $\text{rank}(x) = k$, return $x$
- if $k < \text{rank}(x)$ ...?

# SELECTION - Given n values, find the k-th smallest

## Simple algorithm:

- pick a random input value: x
- use x as pivot, partition input →
- ┌ if rank(x) = k, return x
  └ if k < rank(x), recurse on A

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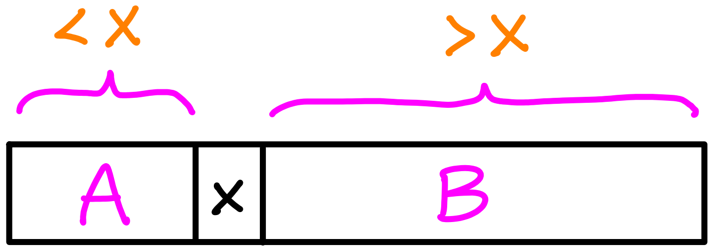


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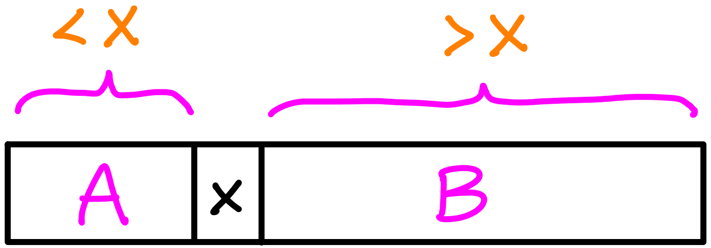


$size(A) + 1 = rank(x)$

- if $rank(x) = k$, return $x$
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  - if $rank(x) < k$ ... ?

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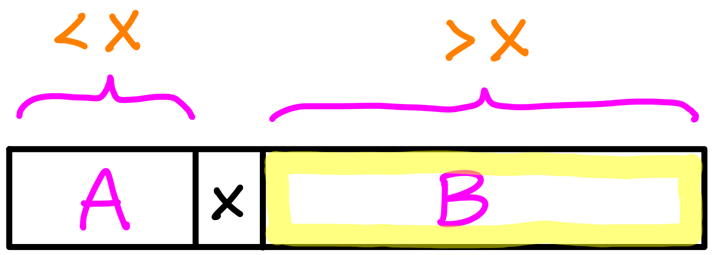


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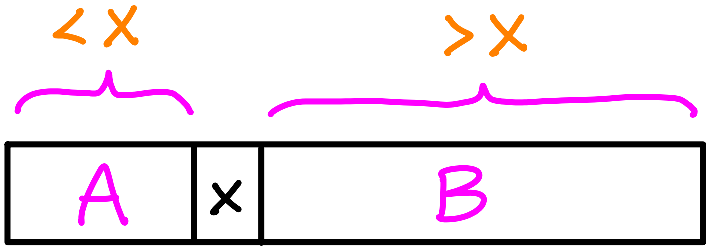


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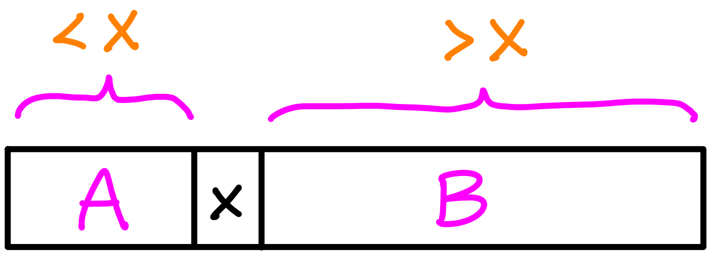


$\text{size}(A) + 1 = \text{rank}(x) = r$

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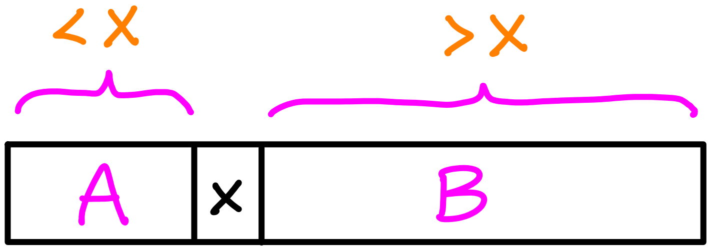


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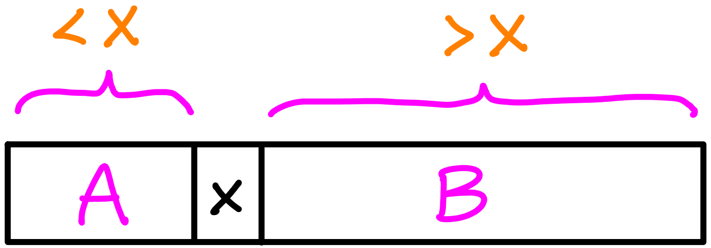
Worst case : $T(n) = \Theta(\,?\,) + T(\,?\,)$

# SELECTION – Given n values, find the k-th smallest

## Simple algorithm:

- pick a random input value: $x$
- use $x$ as pivot, partition input → $\Theta(n)$

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# SELECTION – Given $n$ values, find the $k$-th smallest

**Simple algorithm:**

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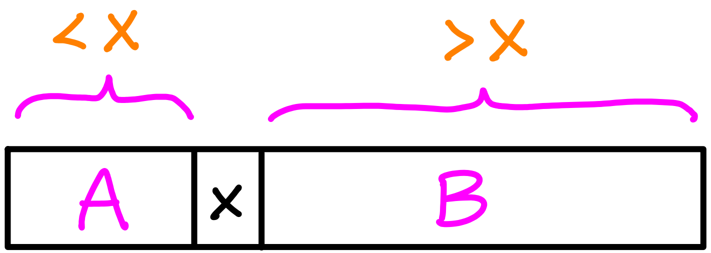


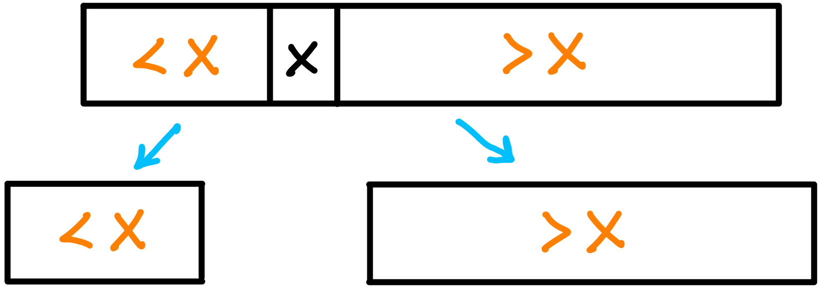

$\text{size}(A) + 1 = \text{rank}(x) = r$

**Worst case:** $T(n) = \Theta(n) + T(n-1) = O(n^2)$

# SELECTION

## summary

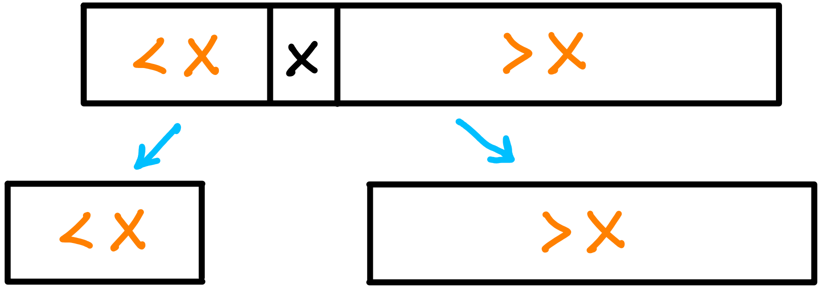
- pick a random input value: x
- use x as pivot, partition input →
- recurse left or right if necessary



# Deterministic SELECTION algorithm (1973)

↳ Blum, Floyd, Pratt, Rivest, Tarjan

- pick a ~~random~~ input value: x
- use x as pivot, partition input →
- recurse left or right if necessary

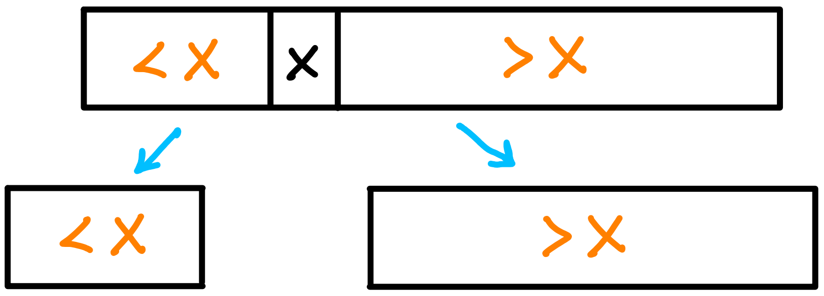


# Deterministic SELECTION algorithm (1973)

↳ Blum, Floyd, Pratt, Rivest, Tarjan

- do something more clever: use some other x
- get O(n) time

- pick a ~~random~~ input value: x
- use x as pivot, partition input →
- recurse left or right if necessary



# Deterministic SELECTION algorithm

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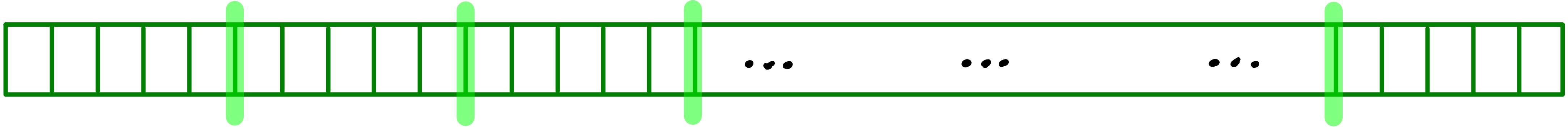
For now, assume n = multiple of 5

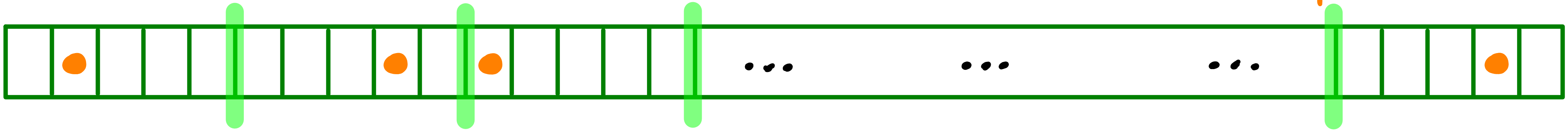
blocks of size 5



# Deterministic SELECTION algorithm

For now, assume n = multiple of 5

- for each block of size 5: find 3rd smallest value = representative



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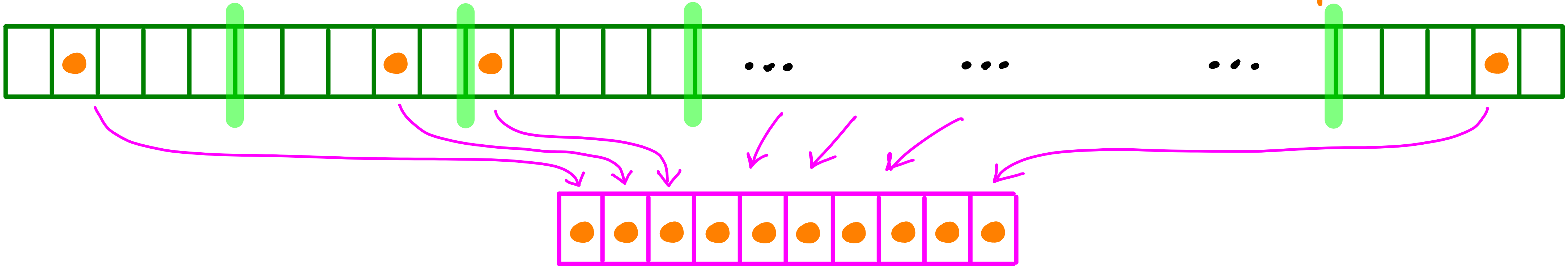

- copy representatives to new list

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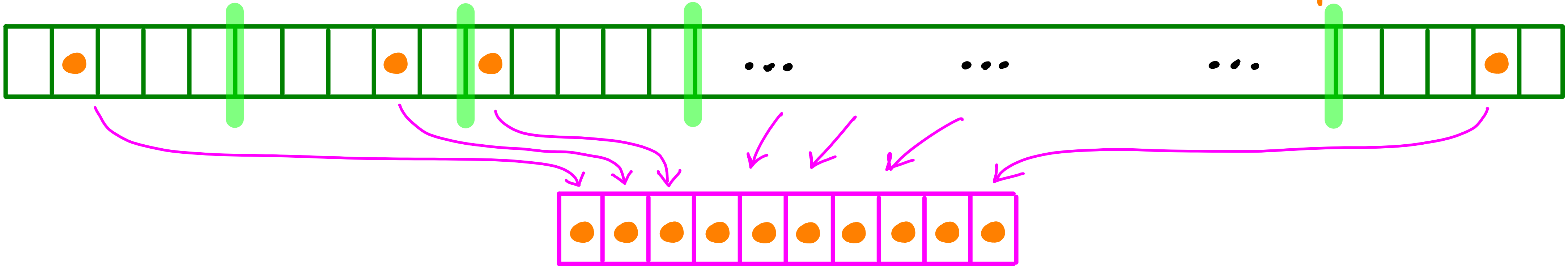


- copy representatives to new list
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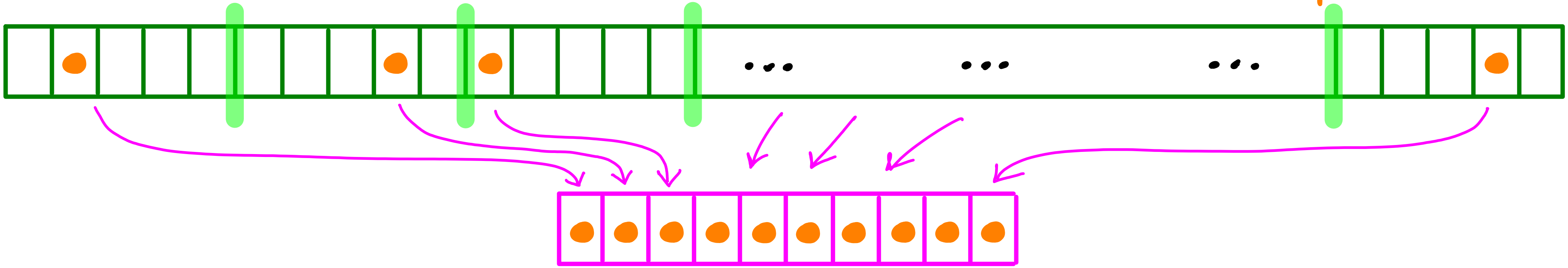


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How?

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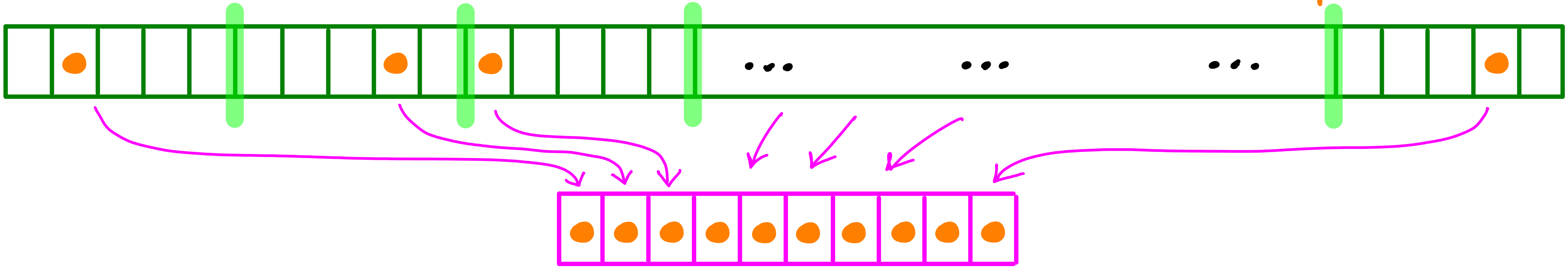

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How? Use the algorithm (recurse)

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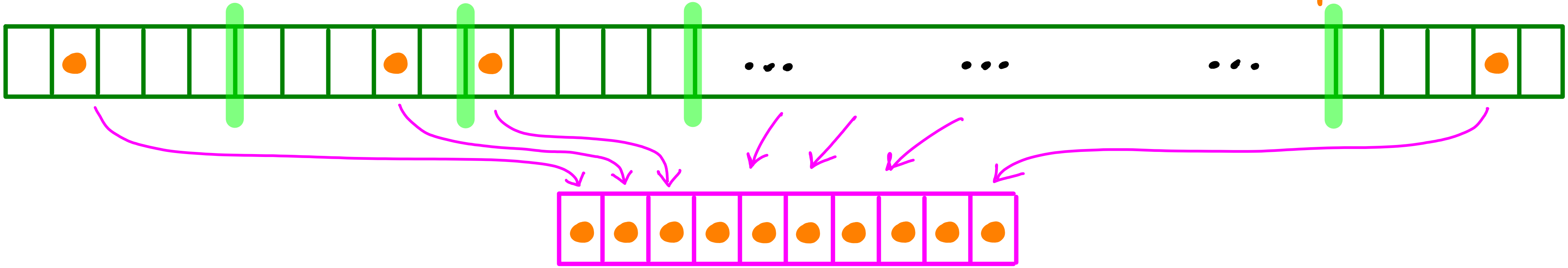

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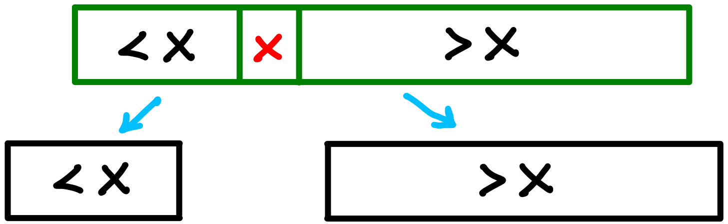
How? Use the algorithm (recurse)

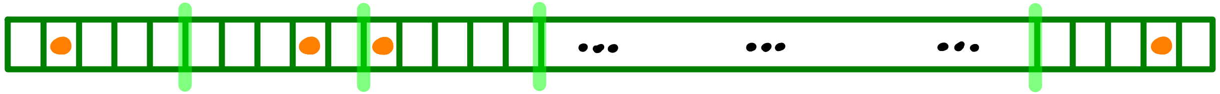
- use x as pivot, partition input




- recurse left or right if necessary

$T(n) = ?$

Time to run Selection on n elements

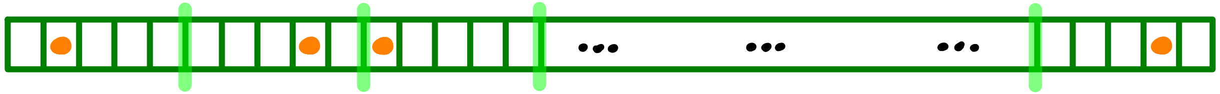



$T(n) = ?$

Time to run Selection on n elements

- for each block of size 5: find representative = ?

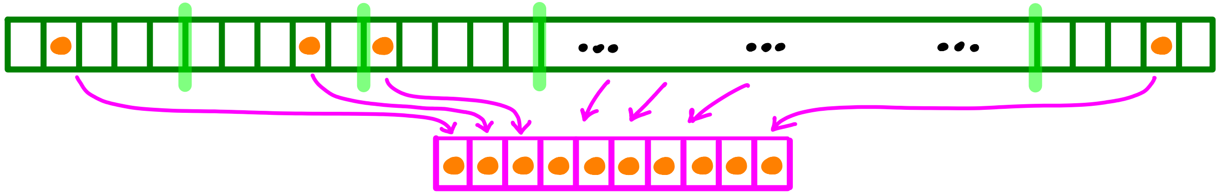
$T(n) = ?$



Time to run Selection on n elements

- for each block of size 5: find representative $= \frac{n}{5} \cdot \Theta(1) = \Theta(n)$

$T(n) = ?$

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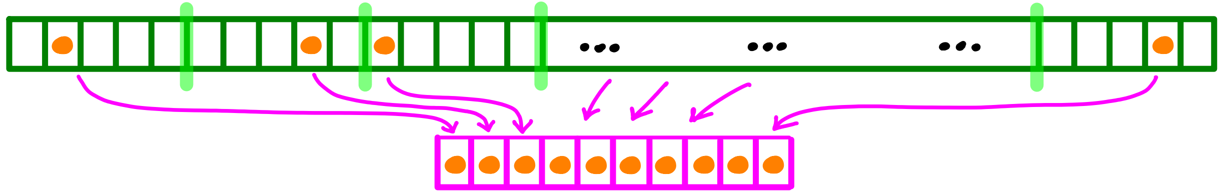

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- find median of representatives $\rightarrow x$ ... ?

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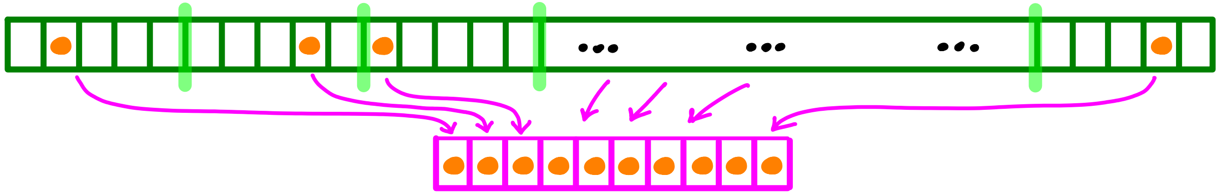
Time to run Selection on n elements



- for each block of size 5: find representative $= \frac{n}{5} \cdot \Theta(1) = \Theta(n)$
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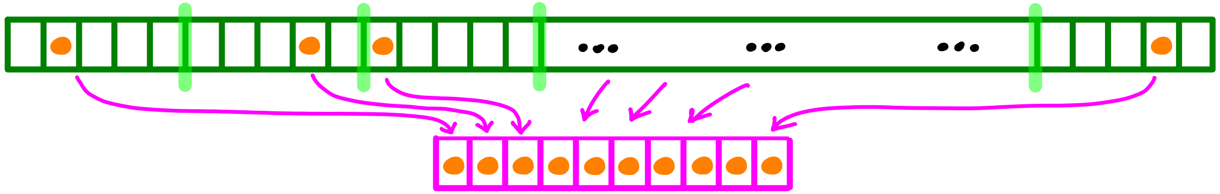

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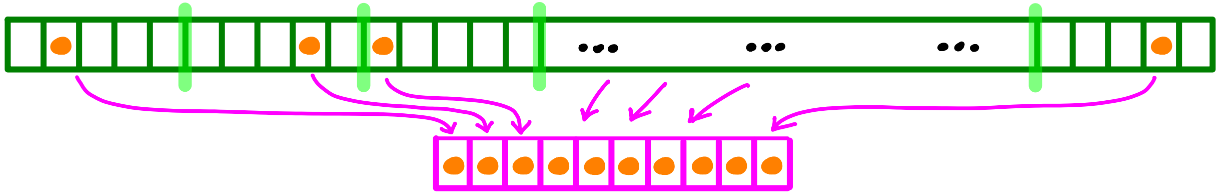

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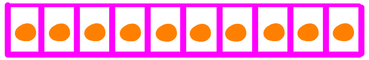
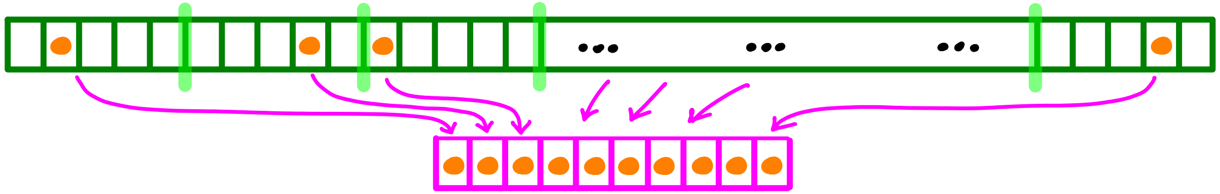


- find median of representatives $\rightarrow x$ (recurse) $= T(\frac{n}{5})$

$T(n) = ?$

Time to run Selection on $n$ elements



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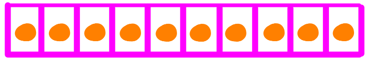
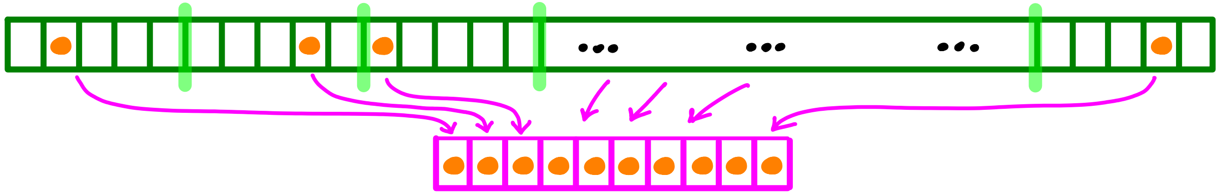


- find median of representatives $\rightarrow x$ (recurse) $= T(\frac{n}{5})$
- use $x$ as pivot, partition input $= \Theta(n)$

$T(n) = ?$



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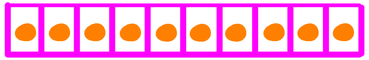

- find median of representatives $\rightarrow x$ (recurse) $= T\left(\frac{n}{5}\right)$
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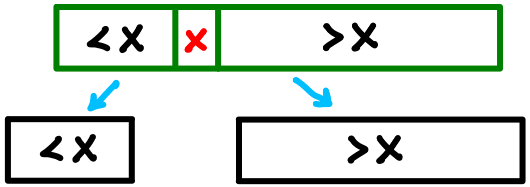
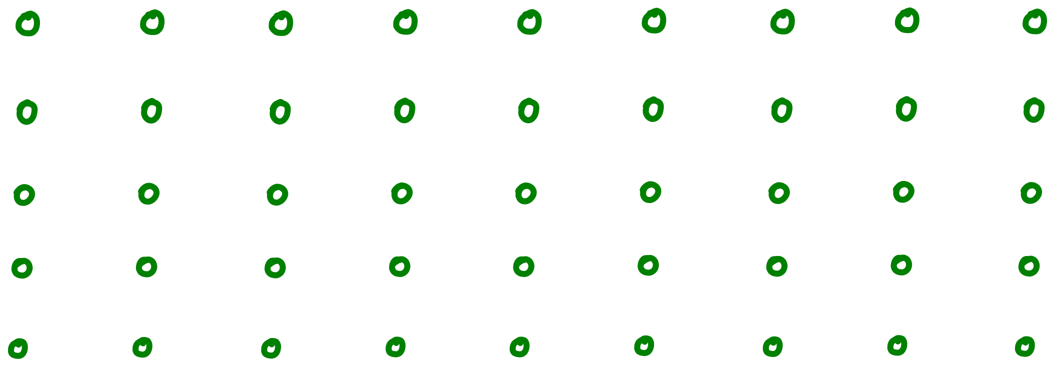



- recurse left or right if necessary $= T(?)$

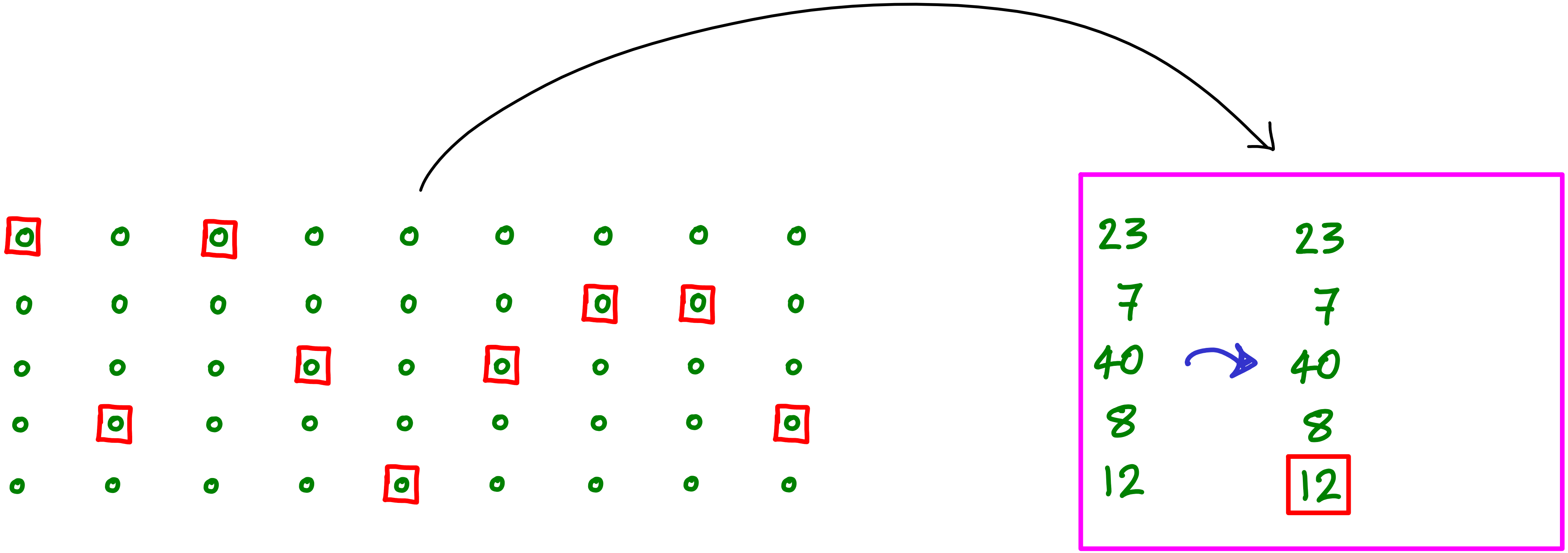
# VISUALIZATION - not part of algorithm

- blocks of size 5



# VISUALIZATION – not part of algorithm

- blocks of size 5
- find representatives



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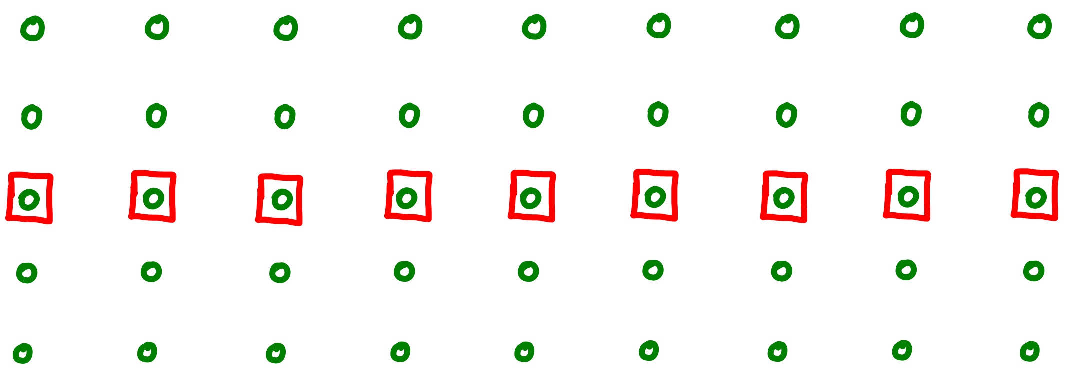

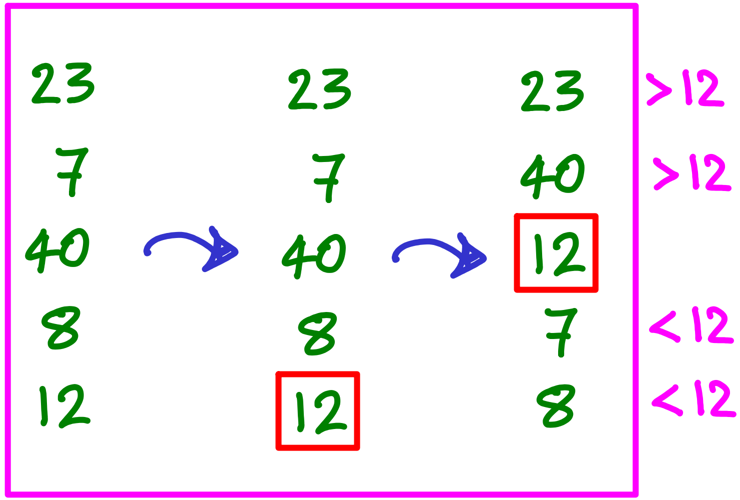

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- blocks of size 5
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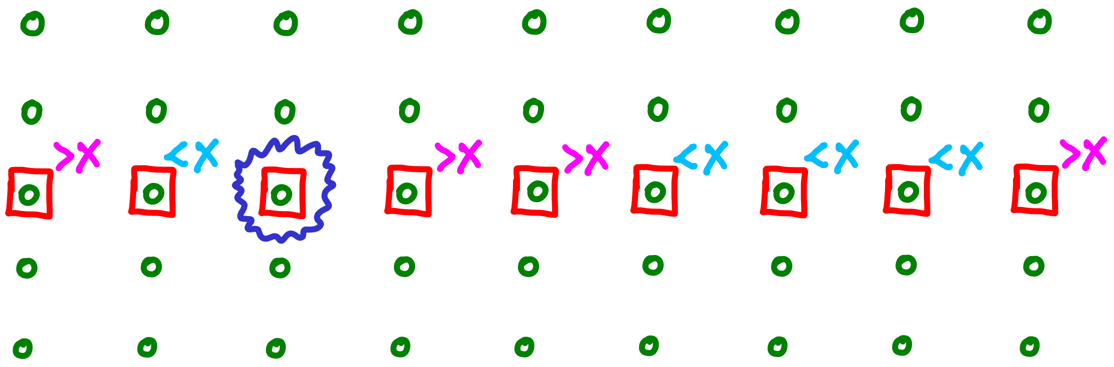
- find median of representatives → x



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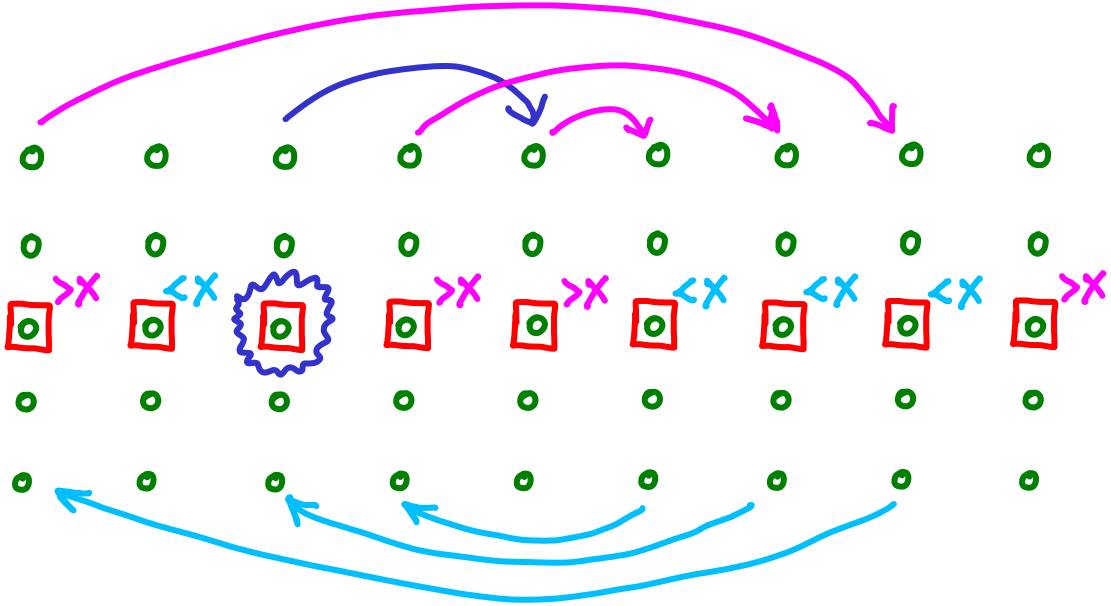

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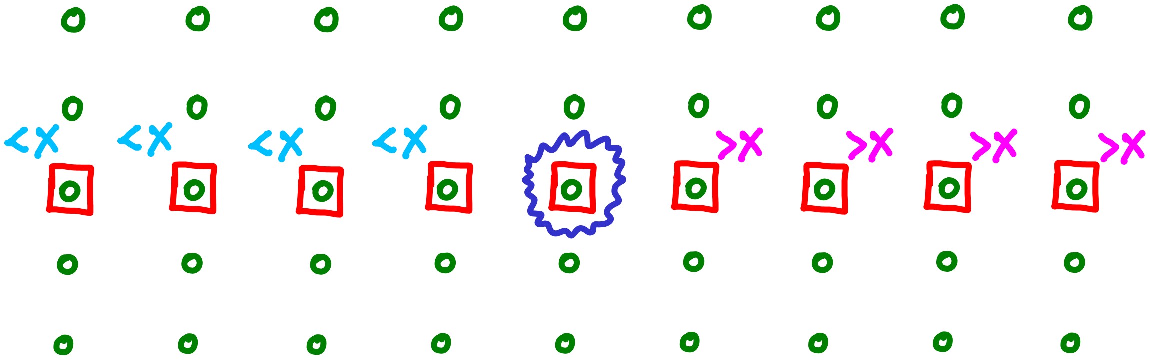

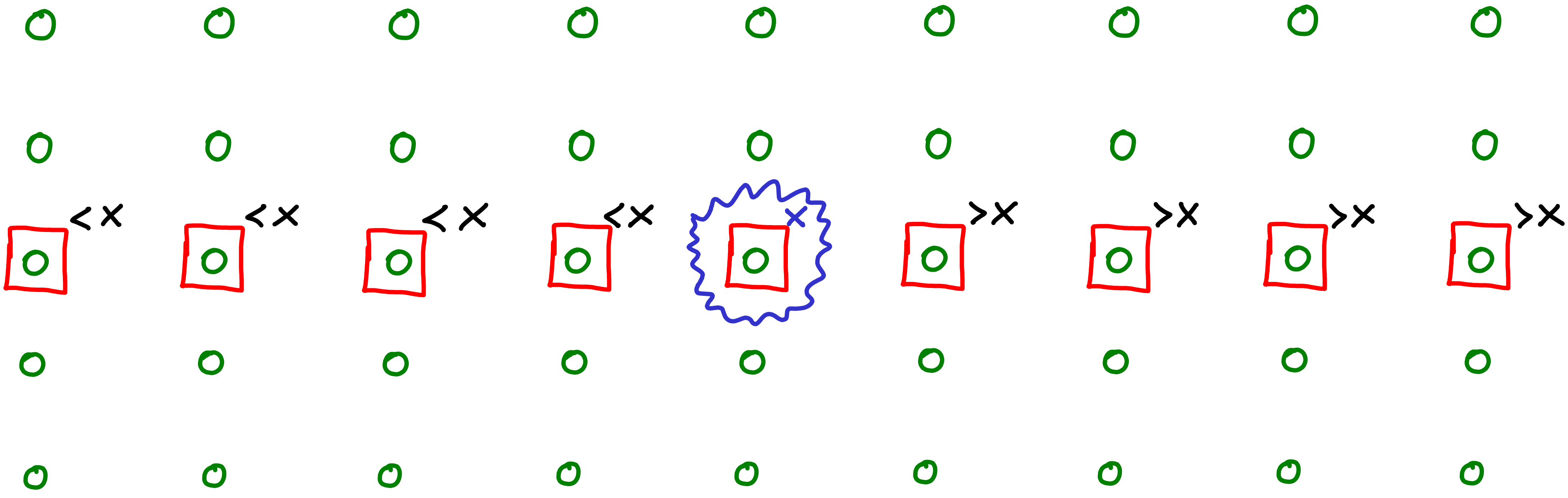
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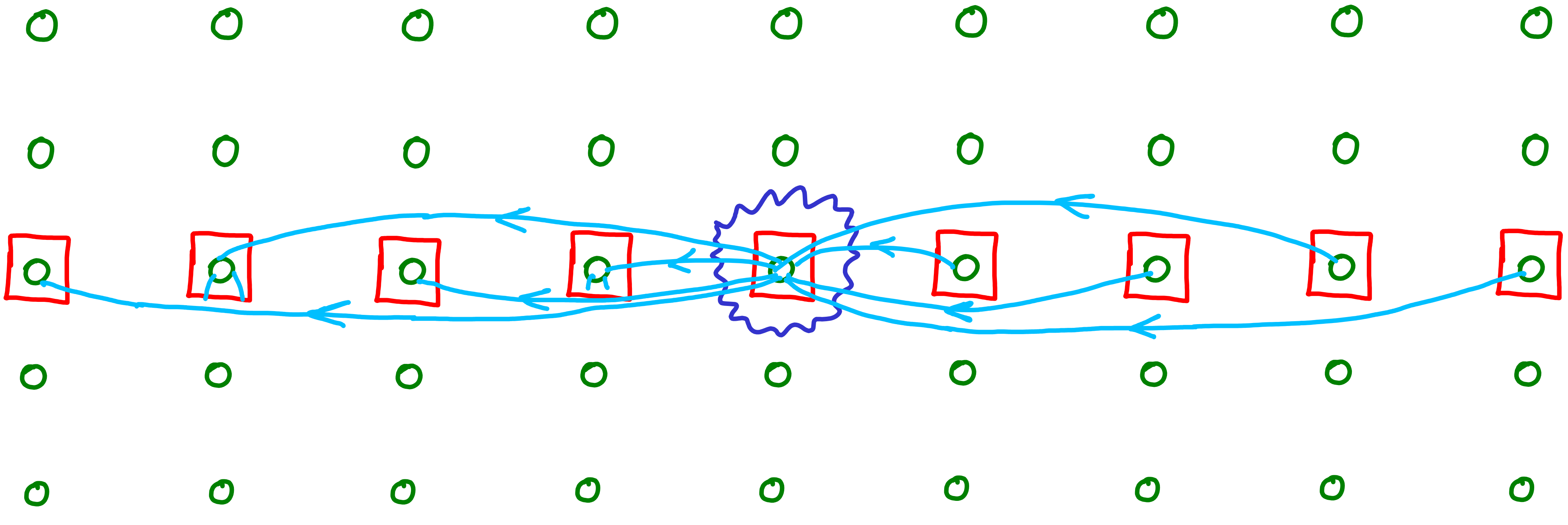
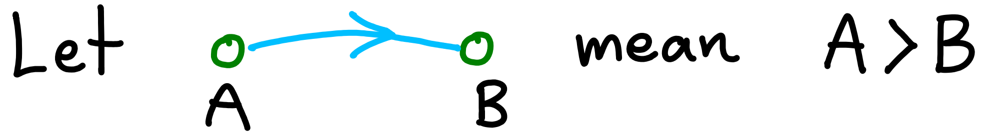
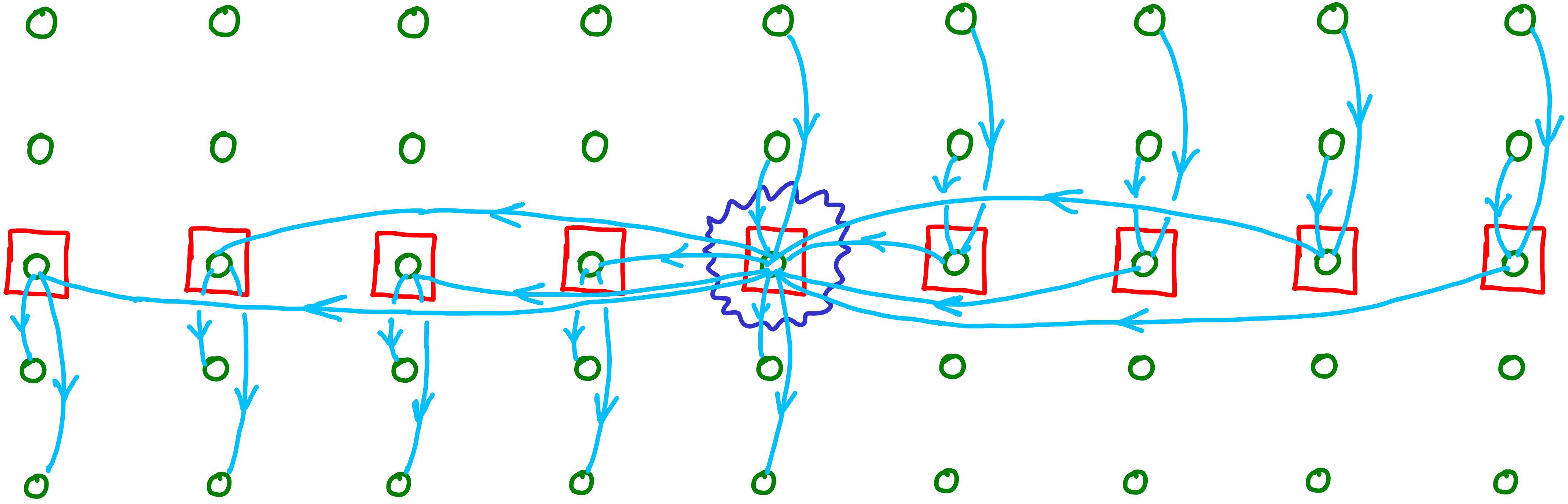
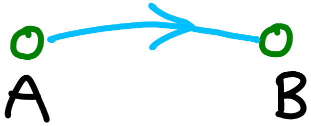
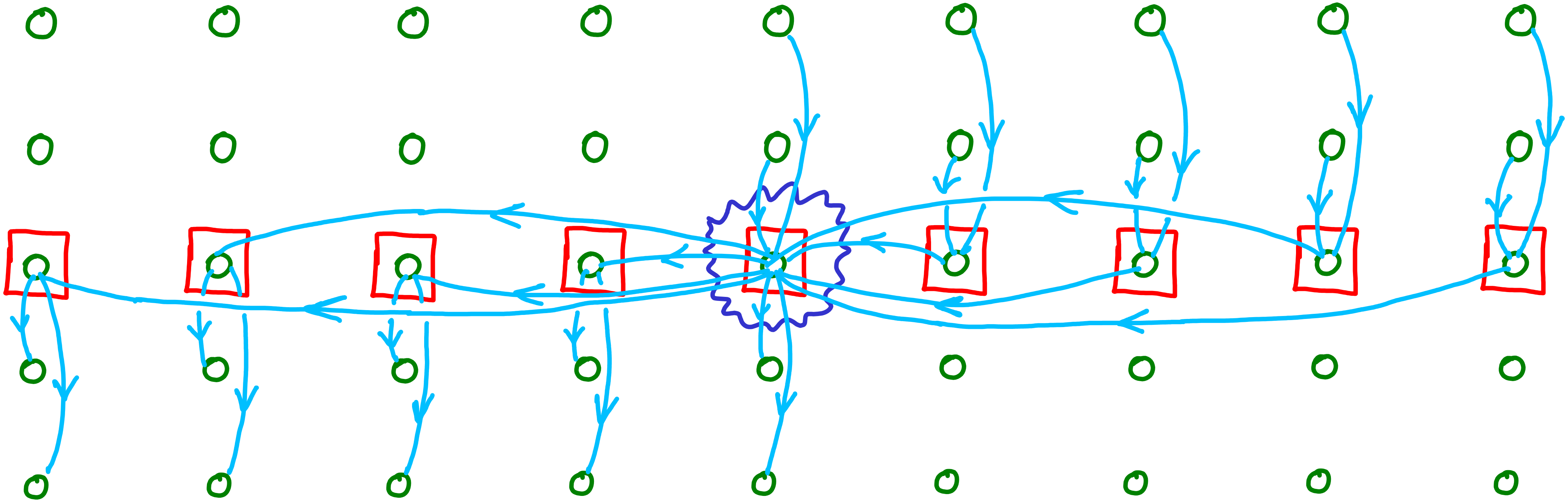


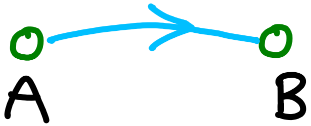

Let
A
B
mean A > B



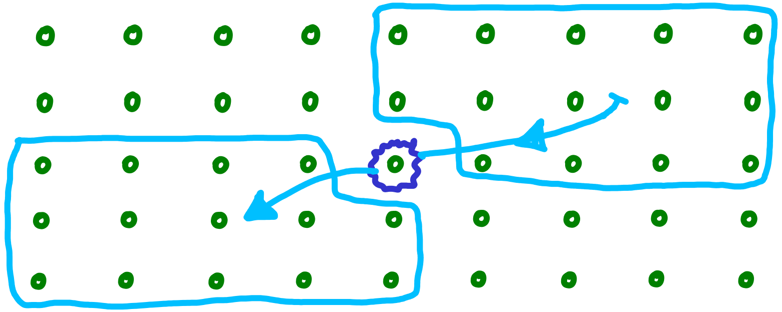
Let 

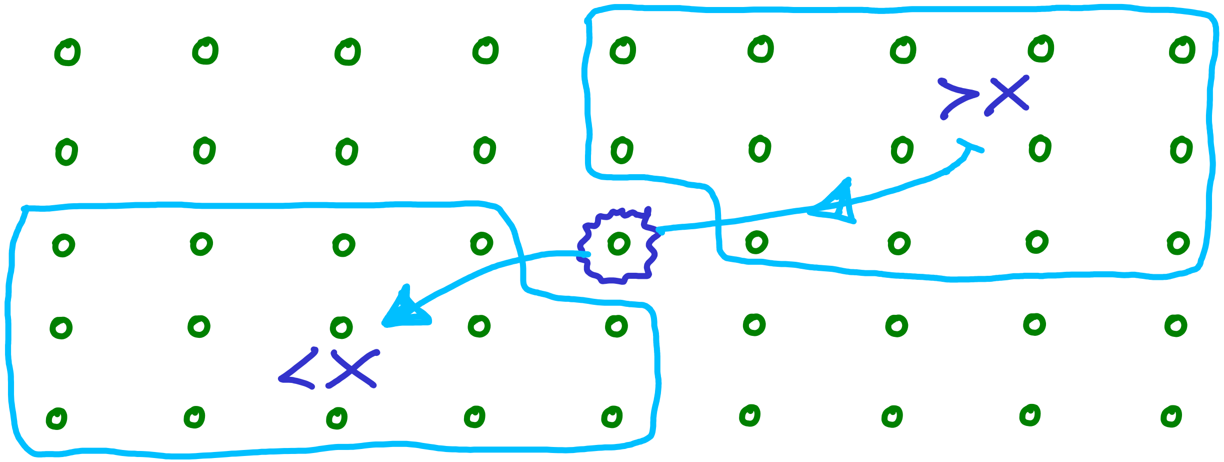
mean $A > B$



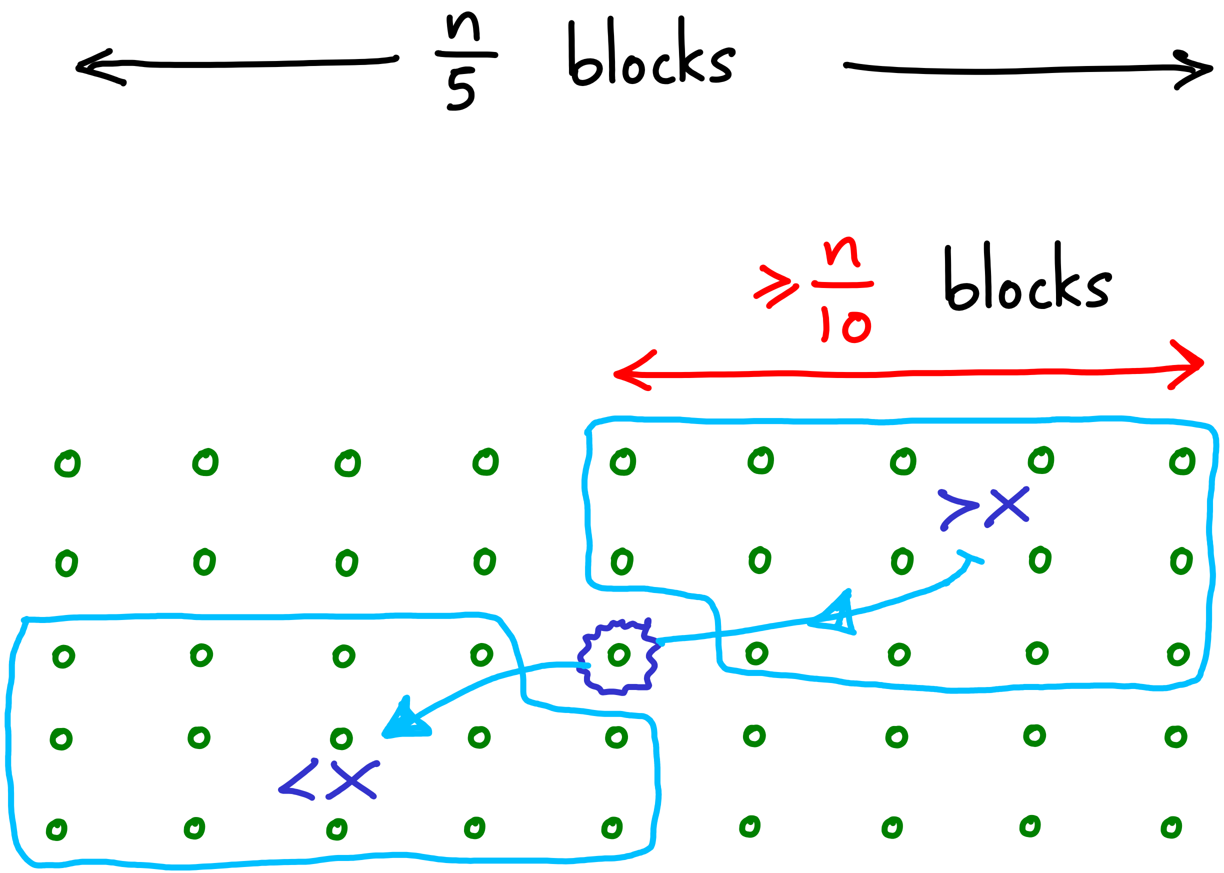
Let 

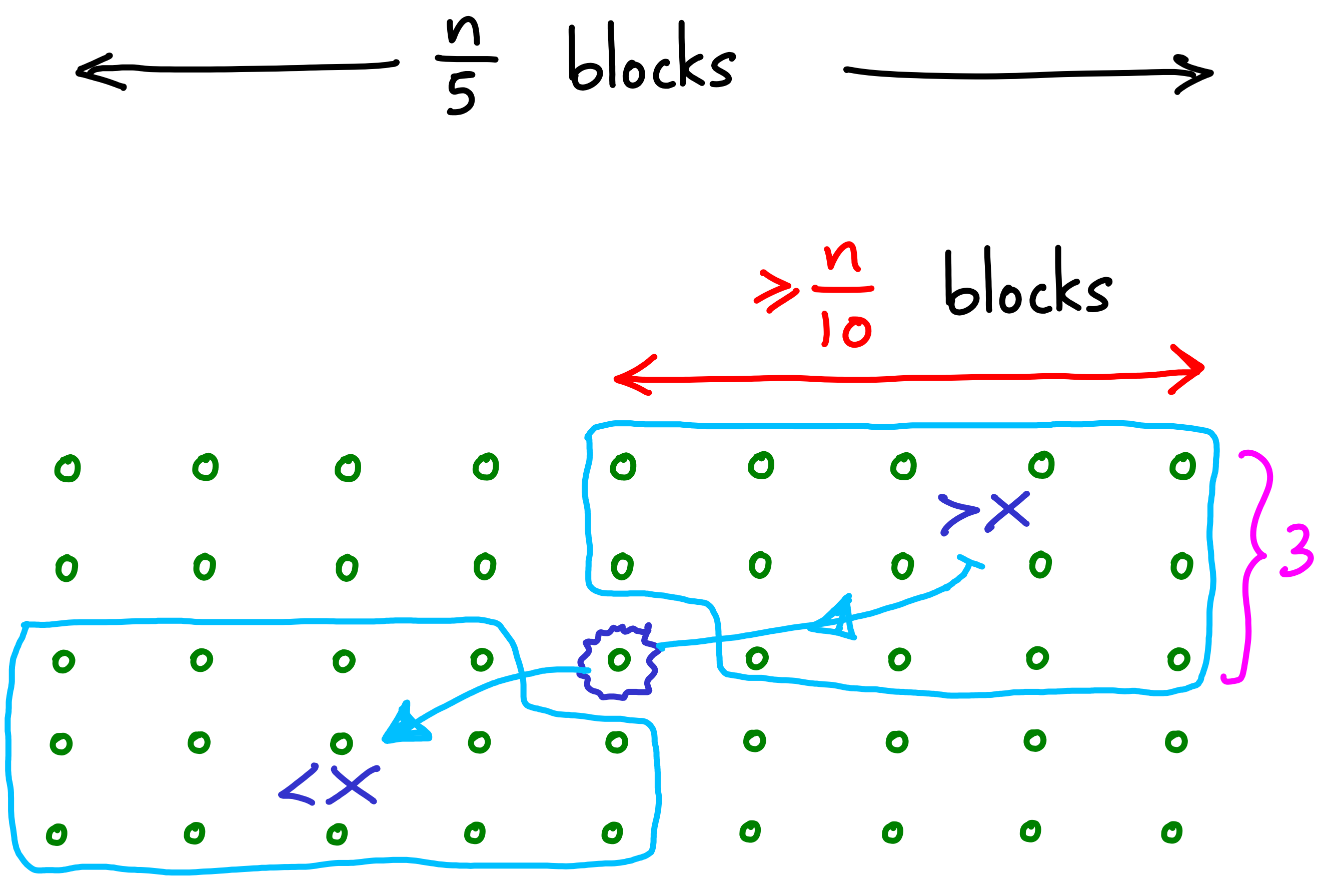
mean $A > B$





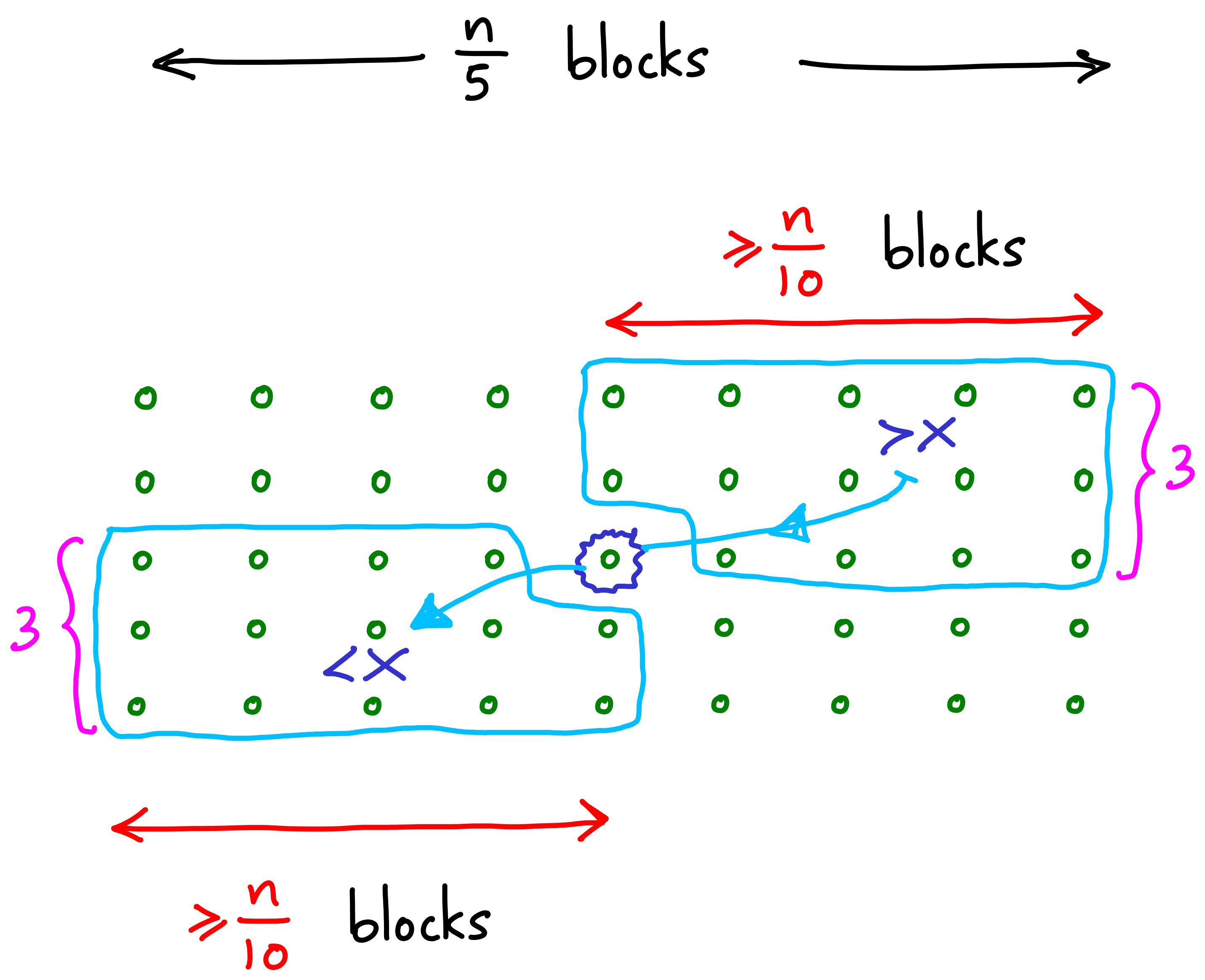
>x
<x


$\frac{n}{5}$ blocks
$\geq \frac{n}{10}$ blocks
$>x$
$<x$




#elements > x

$$\geq 3 \cdot \frac{n}{10}$$

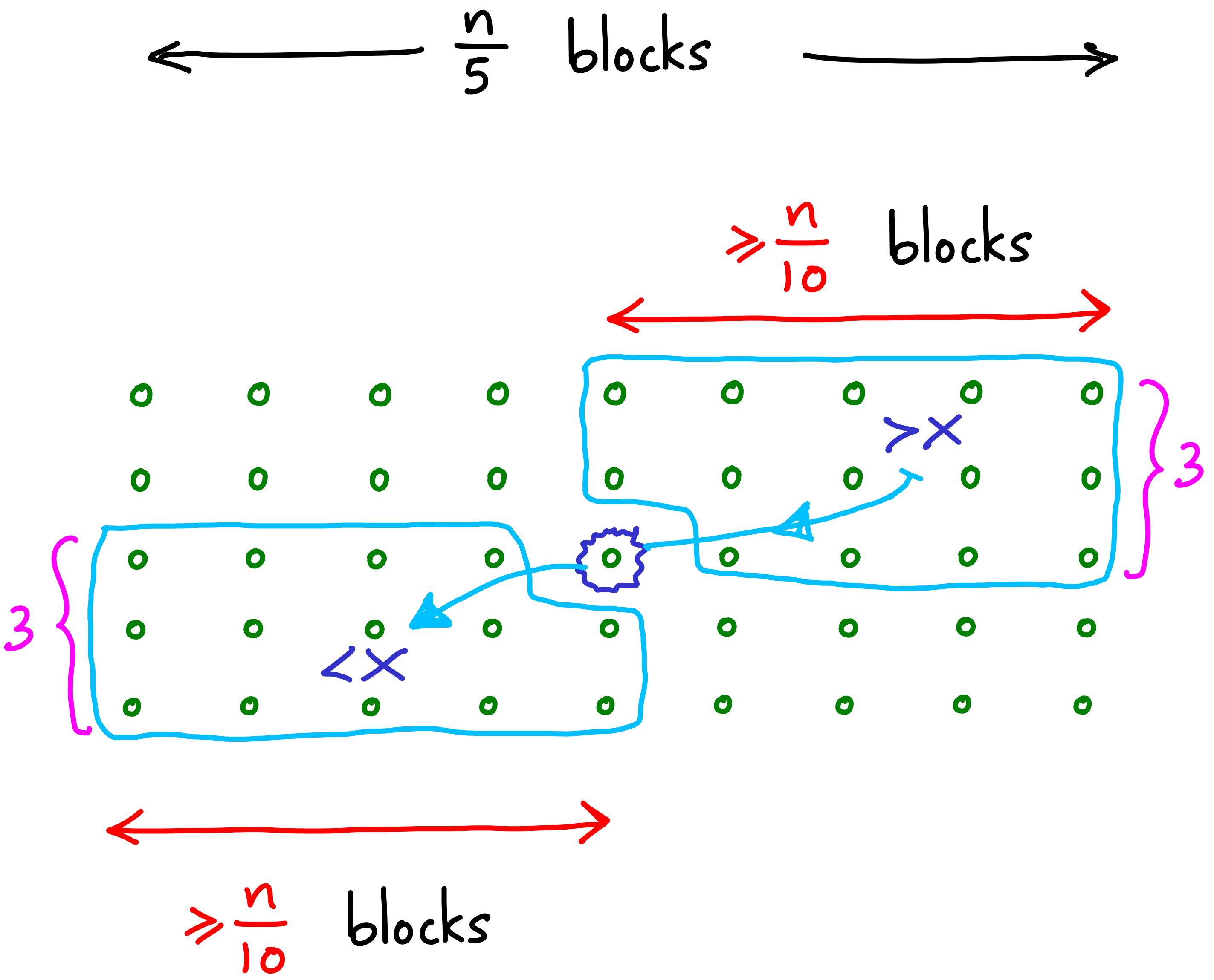



$\#\text{elements} > x$

$\geq 3 \cdot \frac{n}{10}$

$\#\text{elements} < x$

$\geq 3 \cdot \frac{n}{10}$




$\#$elements $> x$

$\geq 3 \cdot \frac{n}{10}$

$\#$elements $< x$

$\geq 3 \cdot \frac{n}{10}$

After partitioning, we recurse on at most $\frac{7n}{10}$ elements

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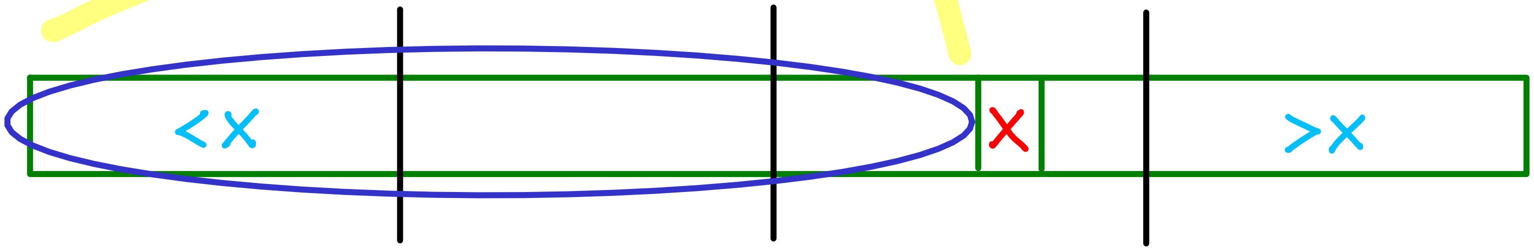

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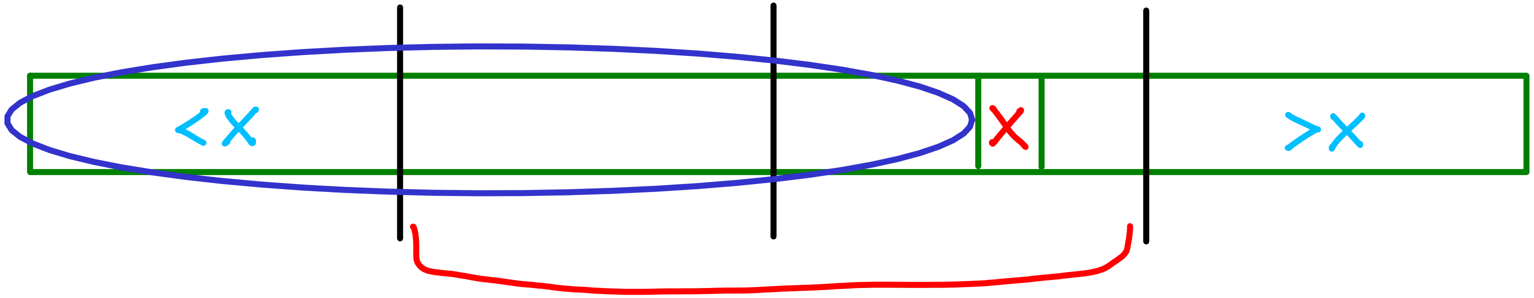
  ↳ $x$ has rank between $\frac{n}{4}$ and $\frac{3n}{4}$



$$T(n) \leq \Theta(n) + T\left(\frac{n}{5}\right) + T\left(\frac{3n}{4}\right)$$

1) find representatives
2) copy to new list

3) find $x$

4) partition using $x$

5) recurse left or right

$$T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{3n}{4}\right) + dn$$

assume $T(k) \leq ck$ for all $k < n$

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$$= \frac{19}{20} cn + dn$$

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$$= cn - \left(\frac{cn}{20} - dn\right)$$

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$$T(n) \leq c \cdot \frac{n}{5} + c \cdot \frac{3n}{4} + dn$$

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$$\leq cn$$ ...if $c = 20d$