dynamic BALANCED SEARCH TREES (NON-RANDOM)

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↳ always maintain $\Theta(\log n)$ height & update in $O(\log n)$ time

RED-BLACK trees

Structure: 1) nodes are colored red or black.

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the important rules

4) every red node has a black parent.
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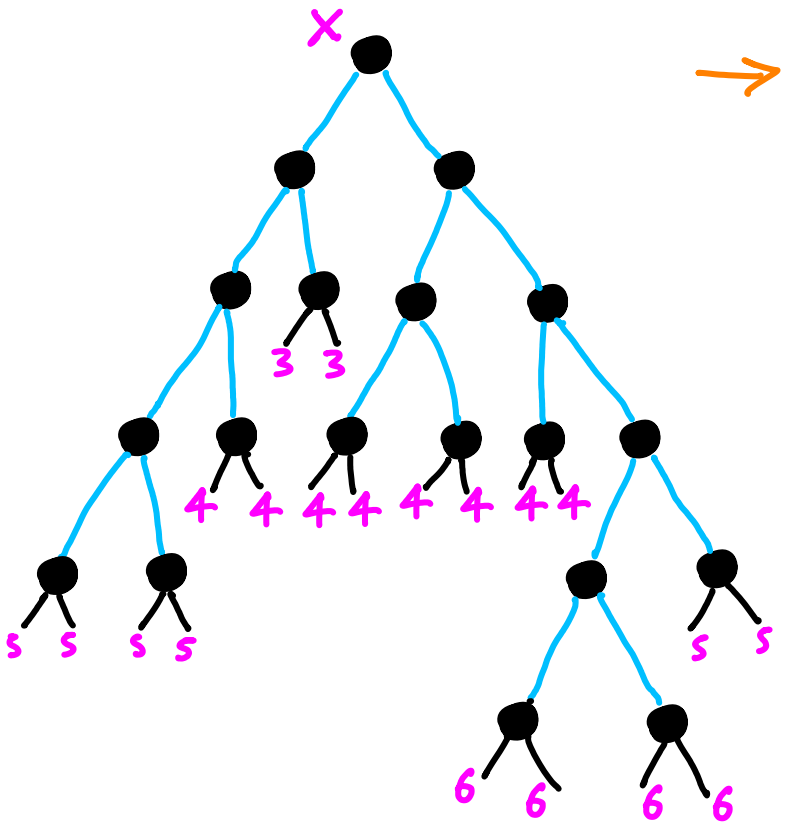

→ fails rule 5

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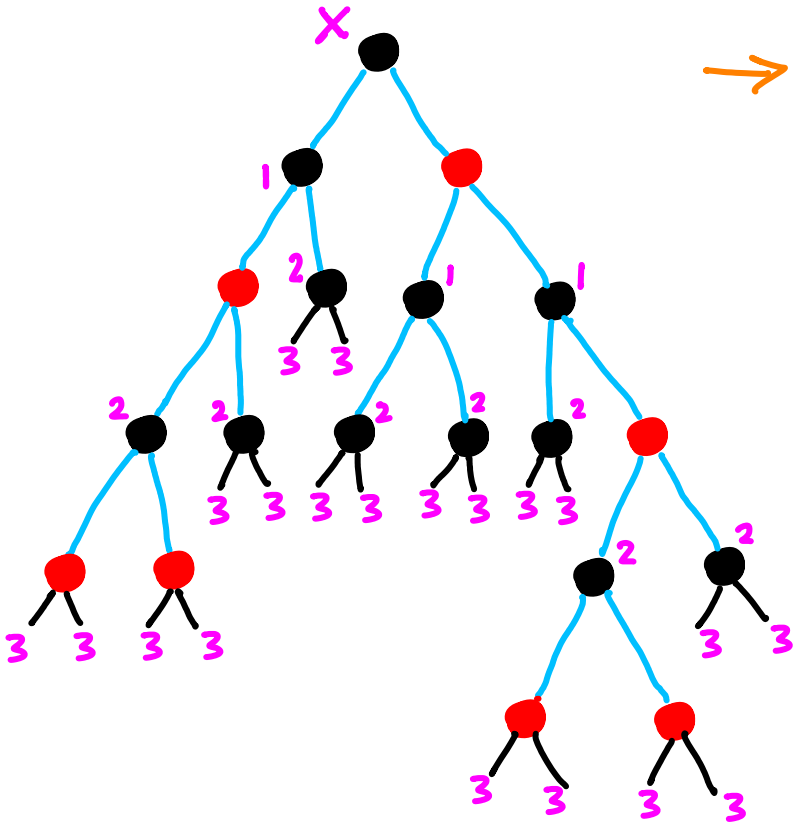
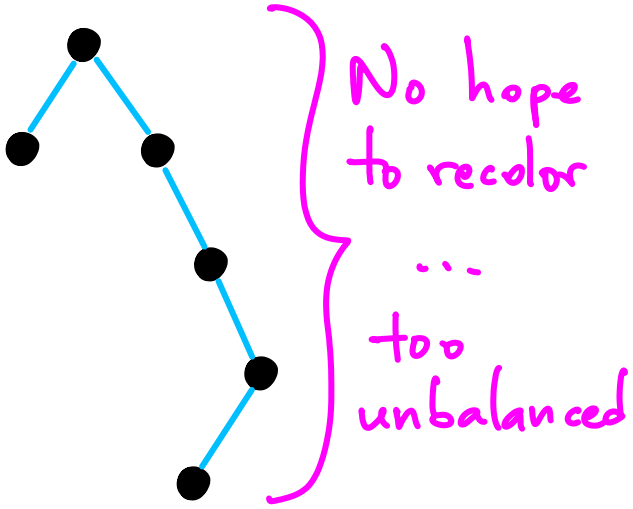
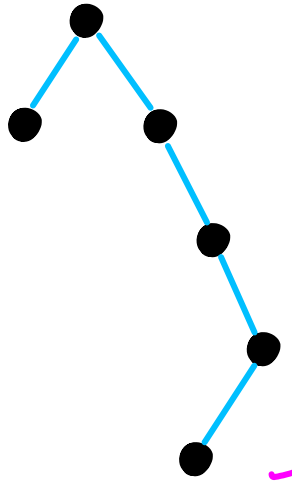


→ fails rule 5 ⇒ fix by making some nodes red.


No hope
to recolor
...
too
unbalanced



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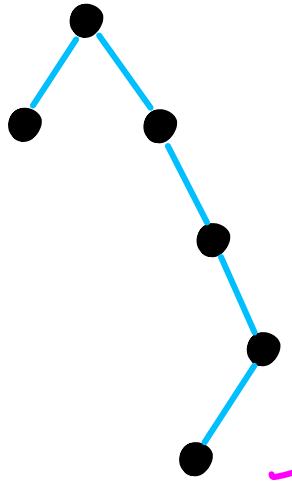
Any root→leaf path of size k must have $\geq \frac{k}{2}$ black nodes. } Rule 4



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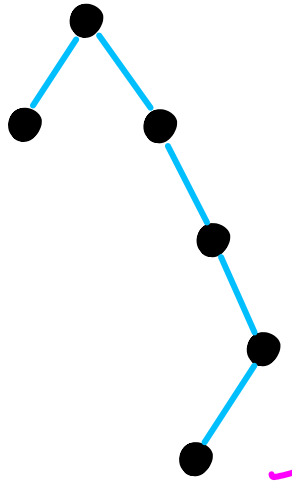
So if any path is >2 times longer than another, we can't make it RB.



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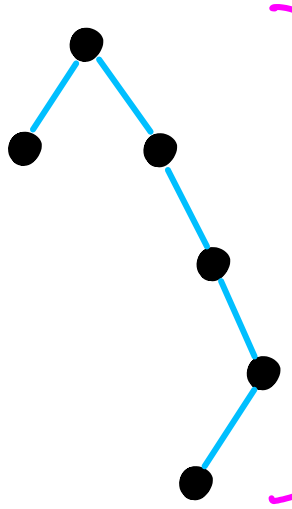
Proof that RB trees have height < 2logn



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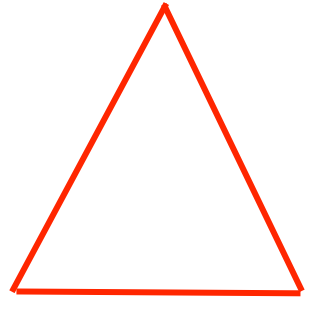

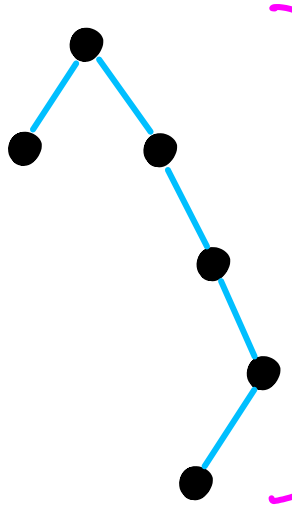

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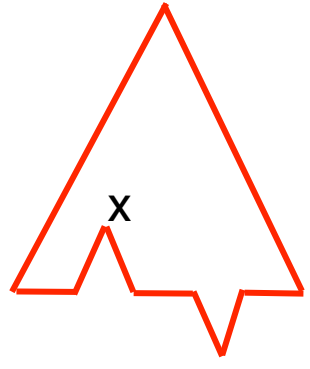

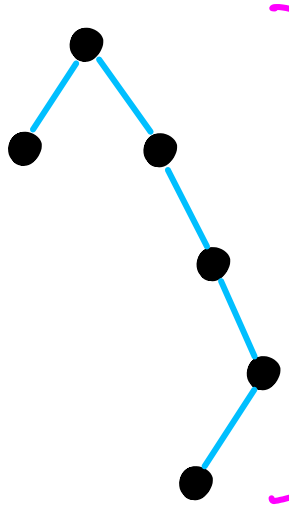

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Proof that RB trees have height < 2logn

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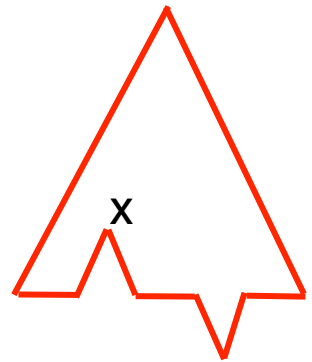
3) By our claim above there can't be any other node at depth > 2d



□

We have seen that RB trees are reasonably balanced : $\sim 2\log n$

↳ search, min, max, next, prev : $O(\log n)$ time.

Next: how to update RB trees (insert, delete)

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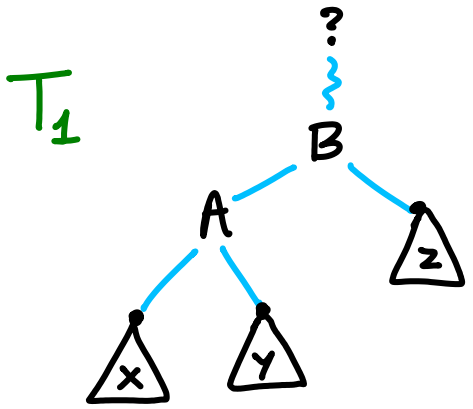

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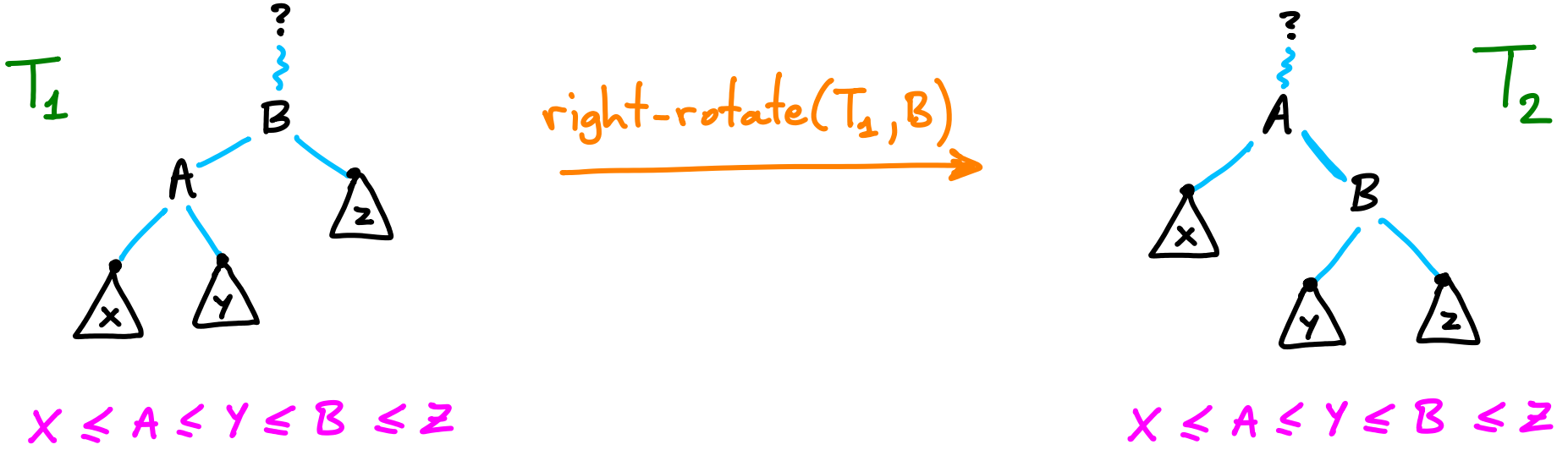
ROTATIONS in arbitrary BSTs



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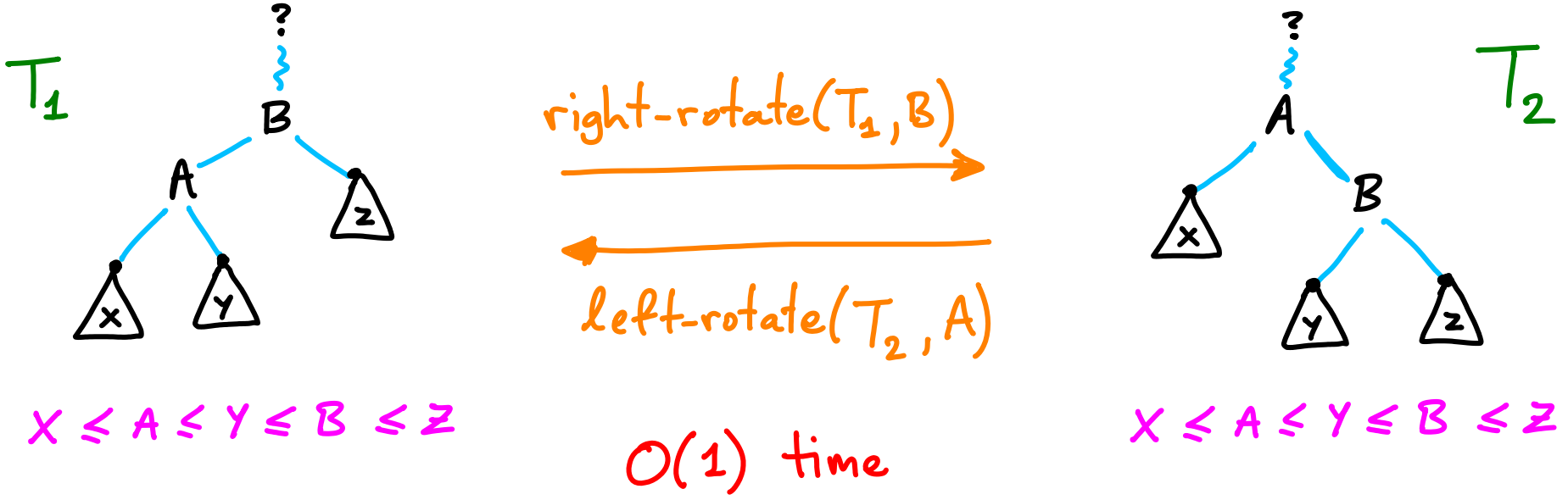

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$O(1)$ time

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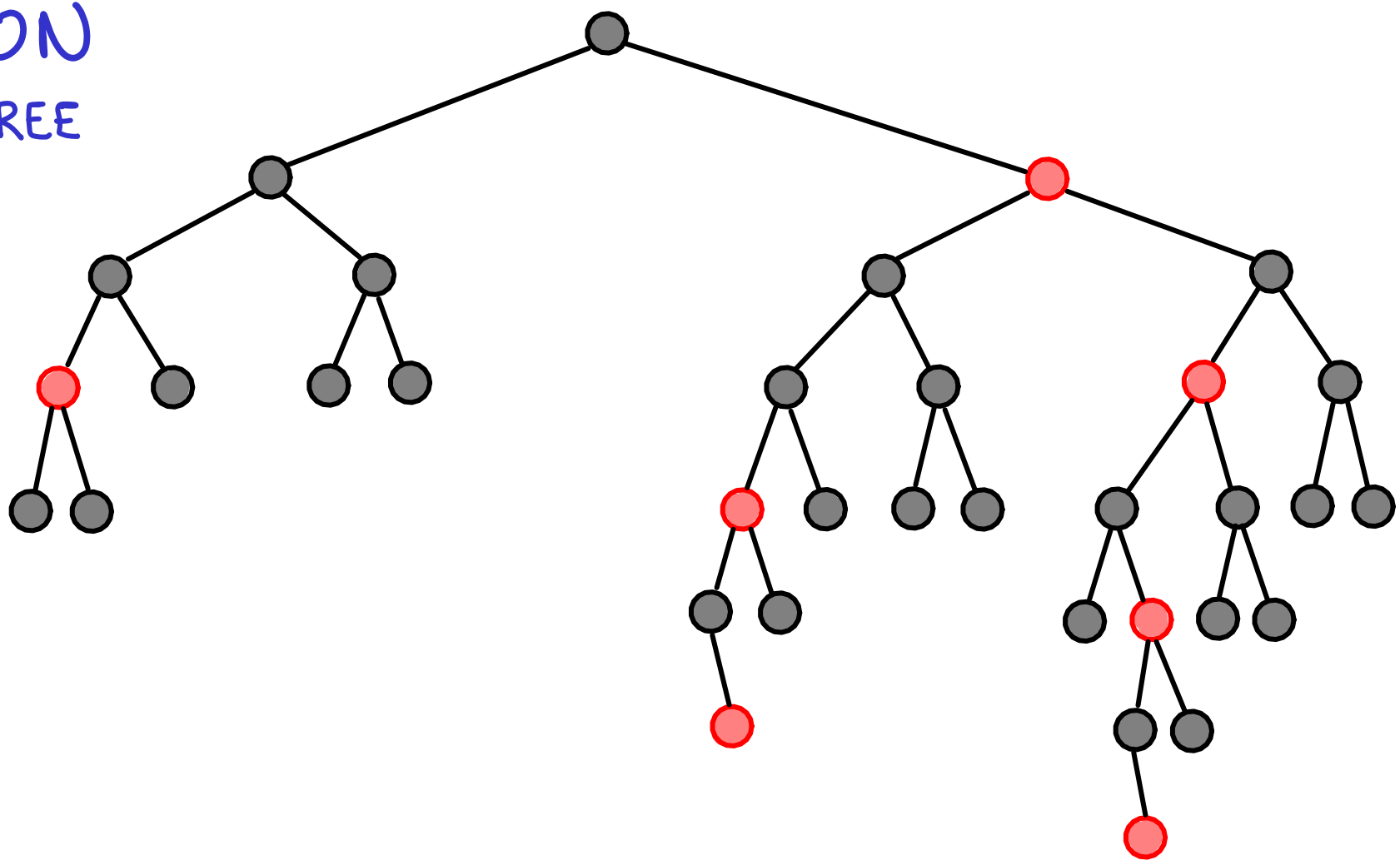
INSERTION

IN A R-B TREE



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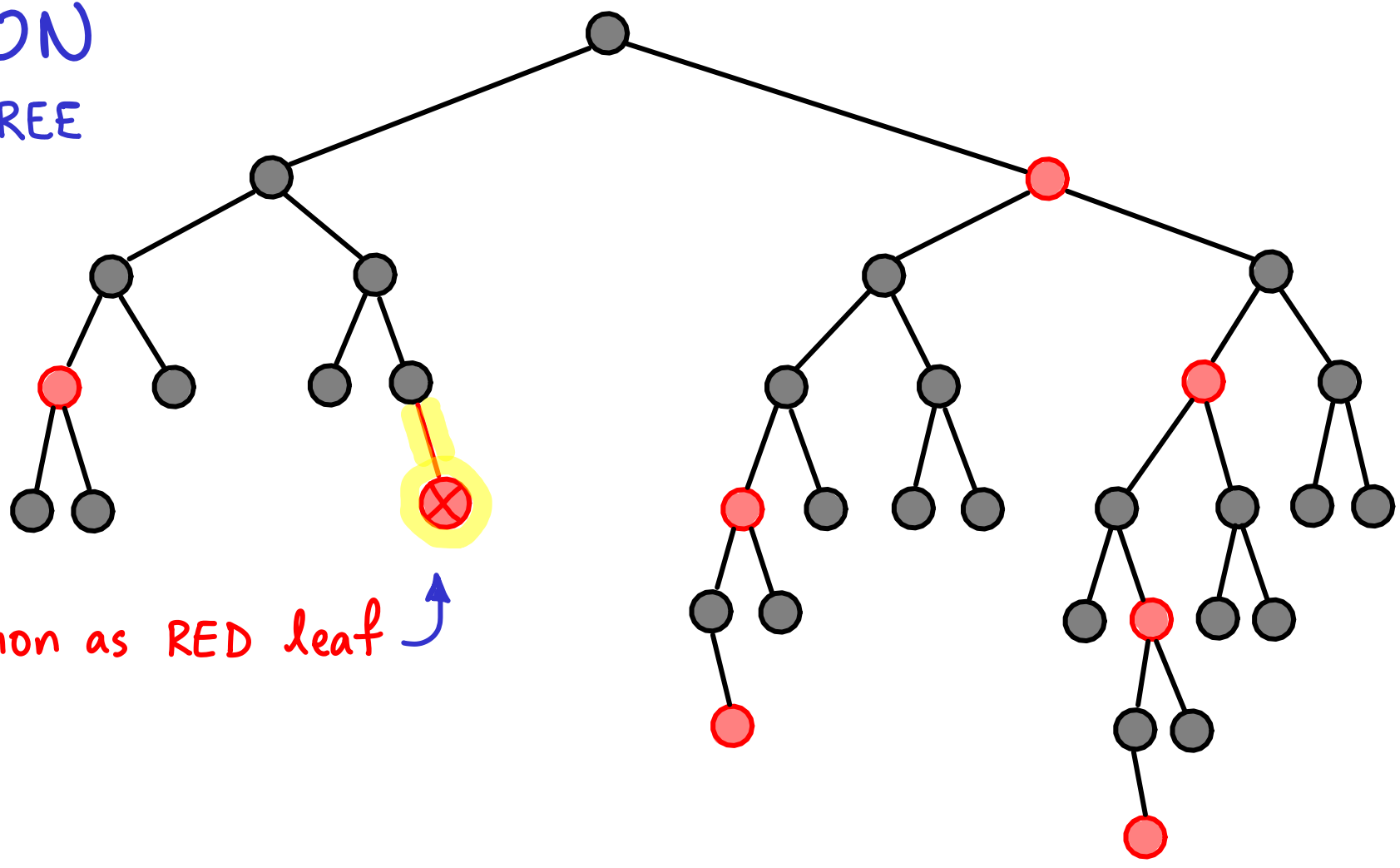

Regular insertion as RED leaf

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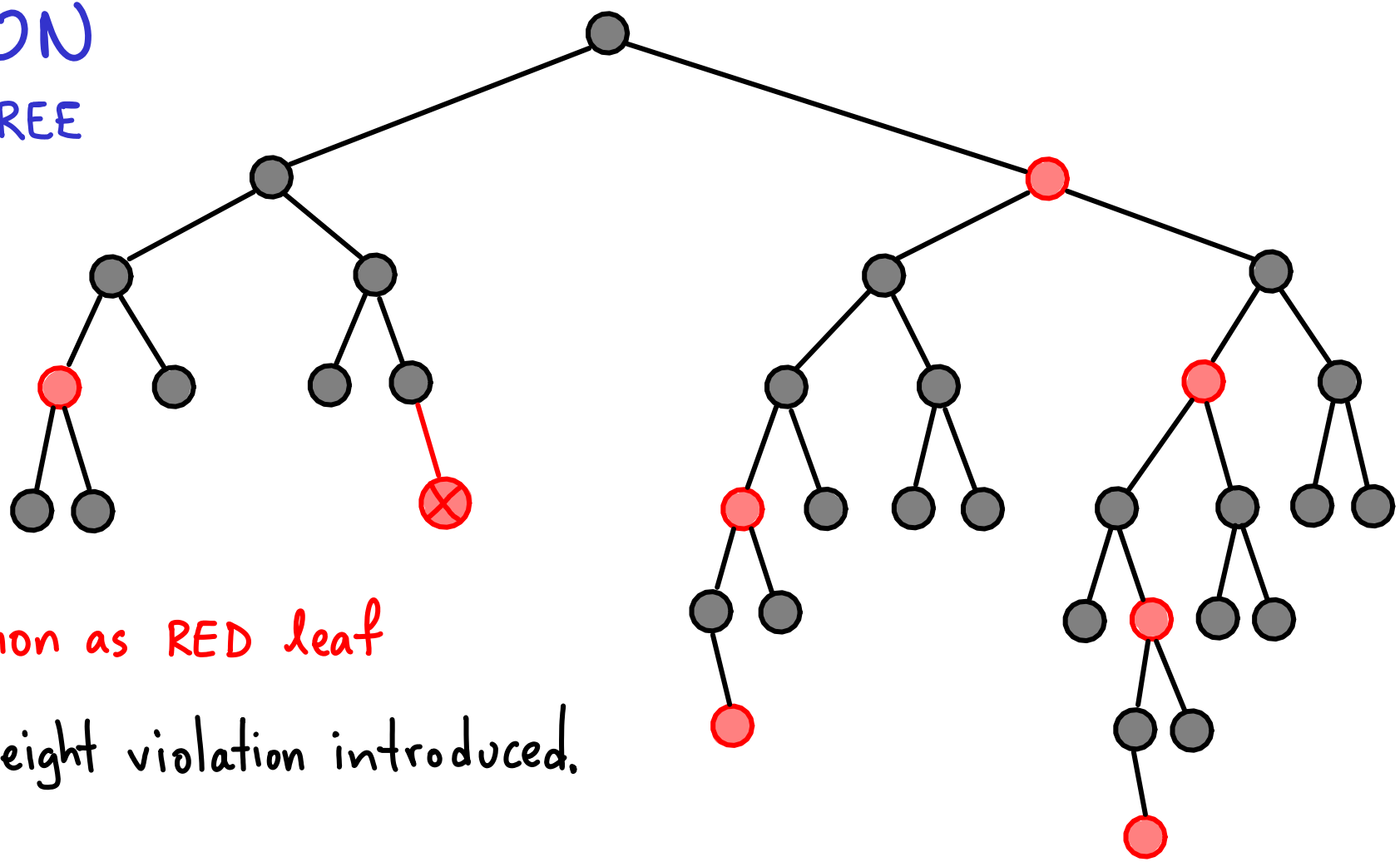

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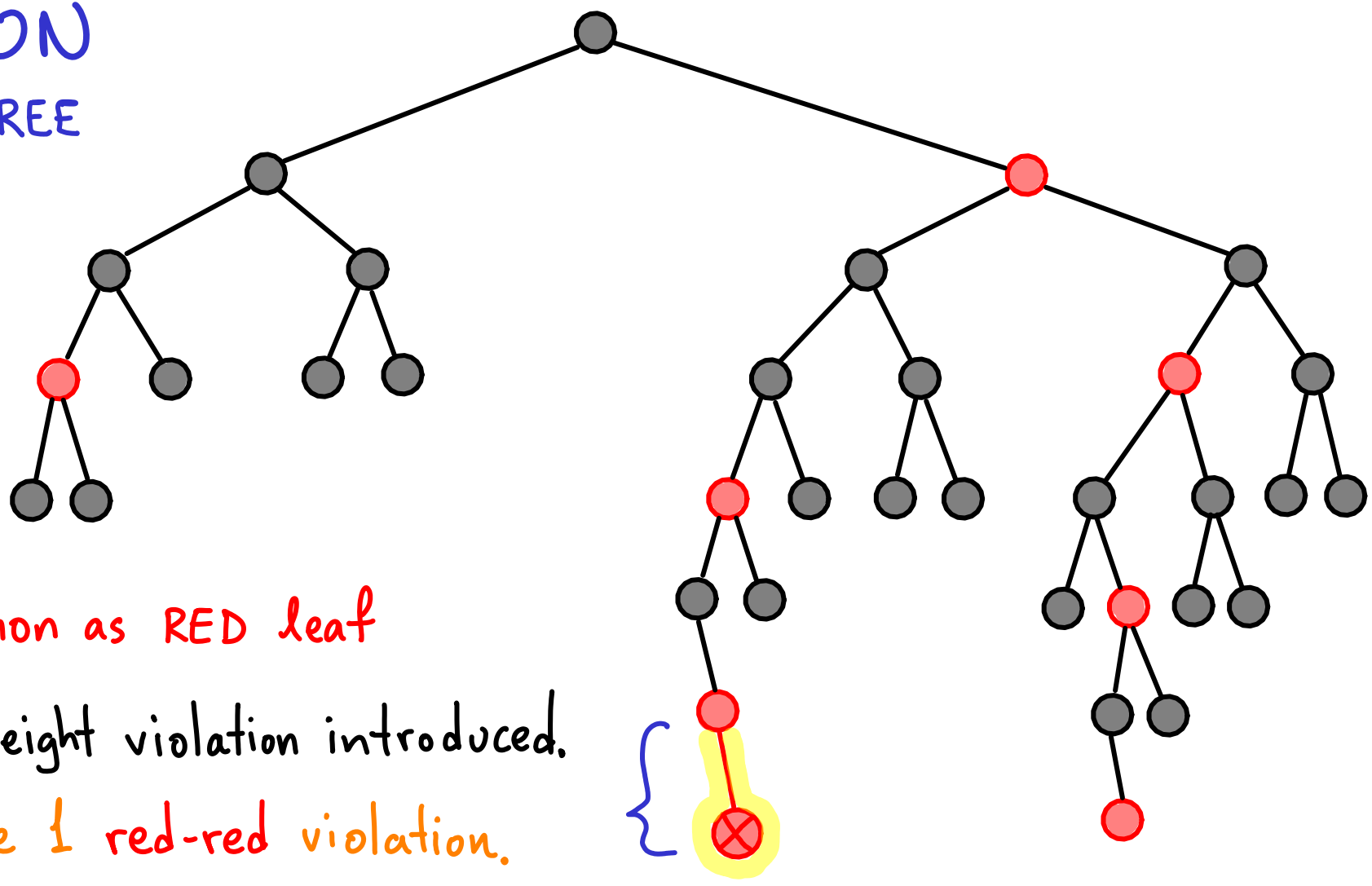

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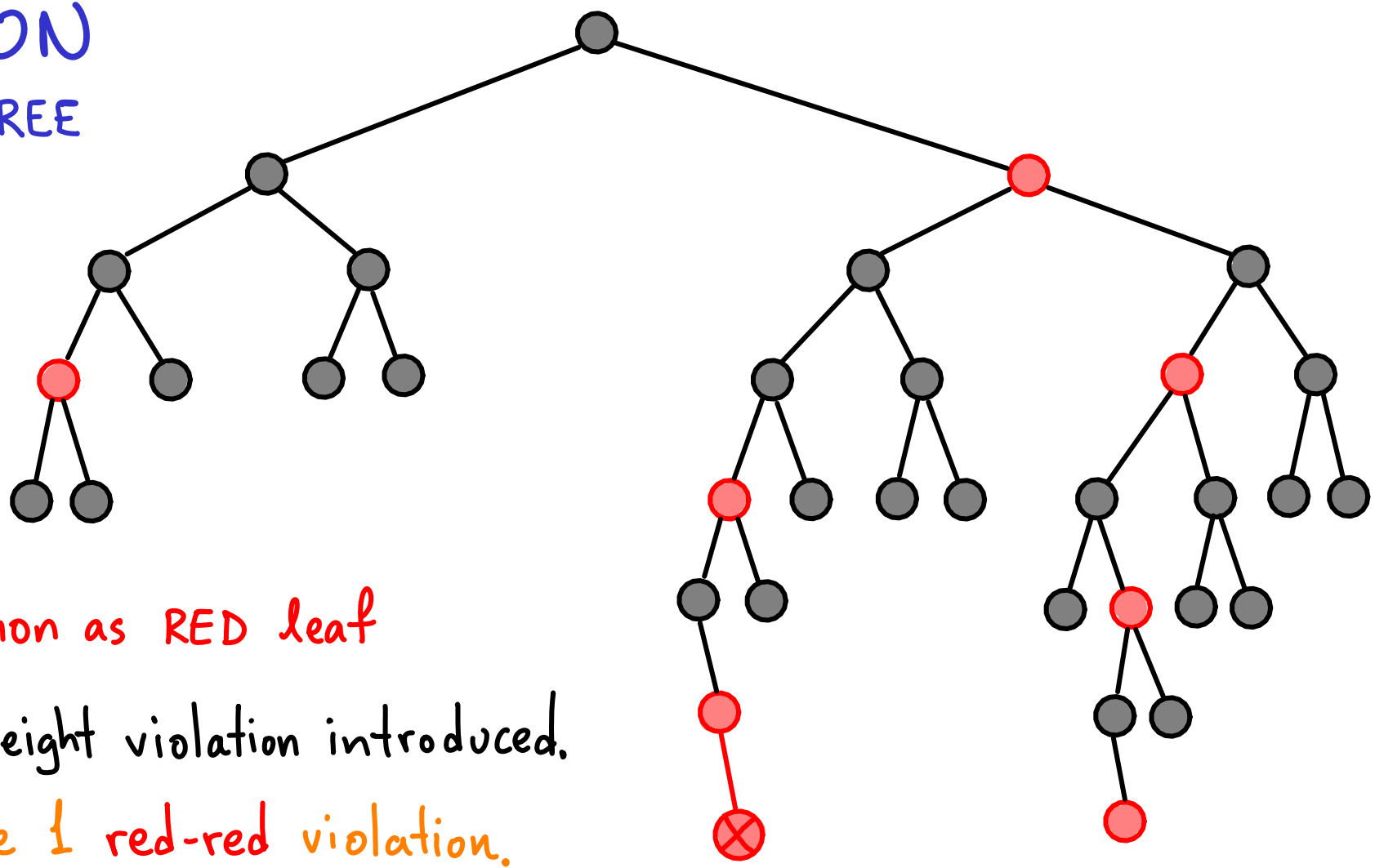

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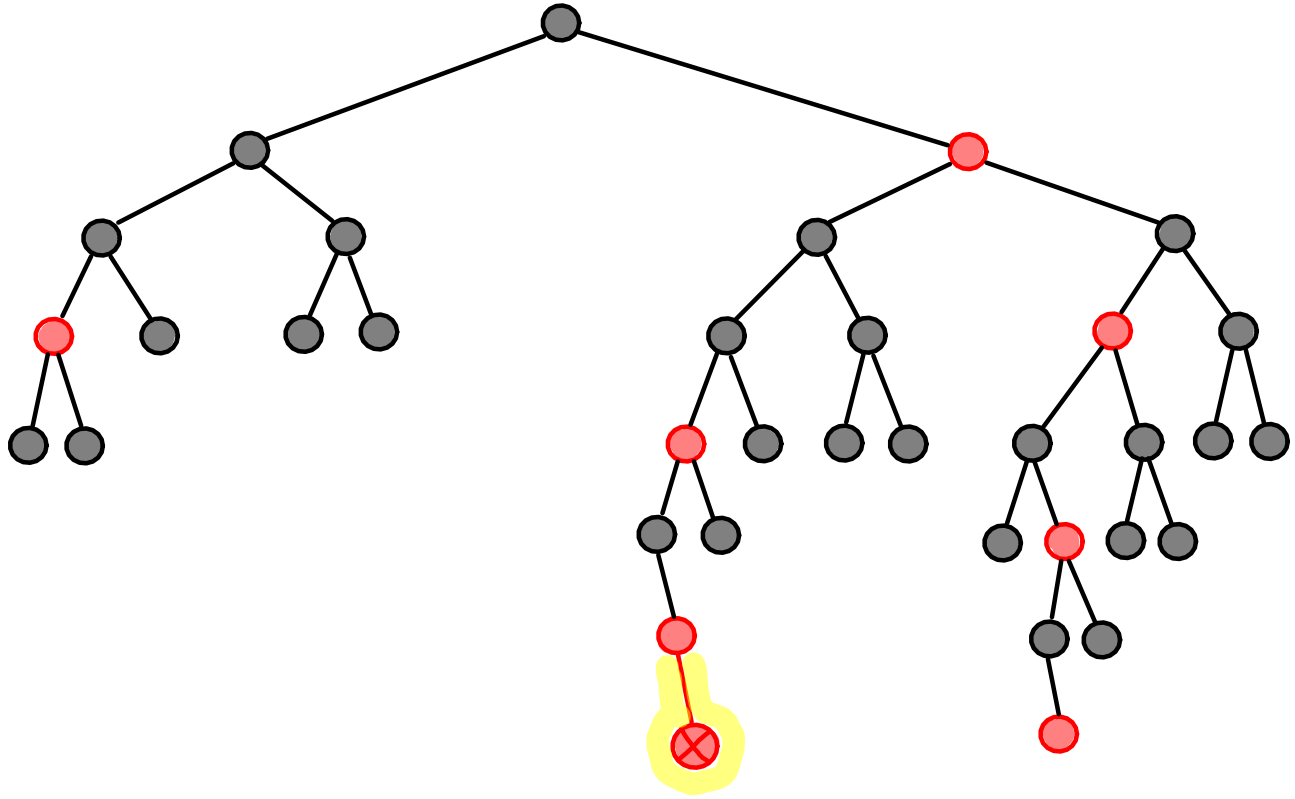
↳ If so, start fixing: locally & upward

fixing: locally & upward



fixing: locally & upward

Loop invariants:

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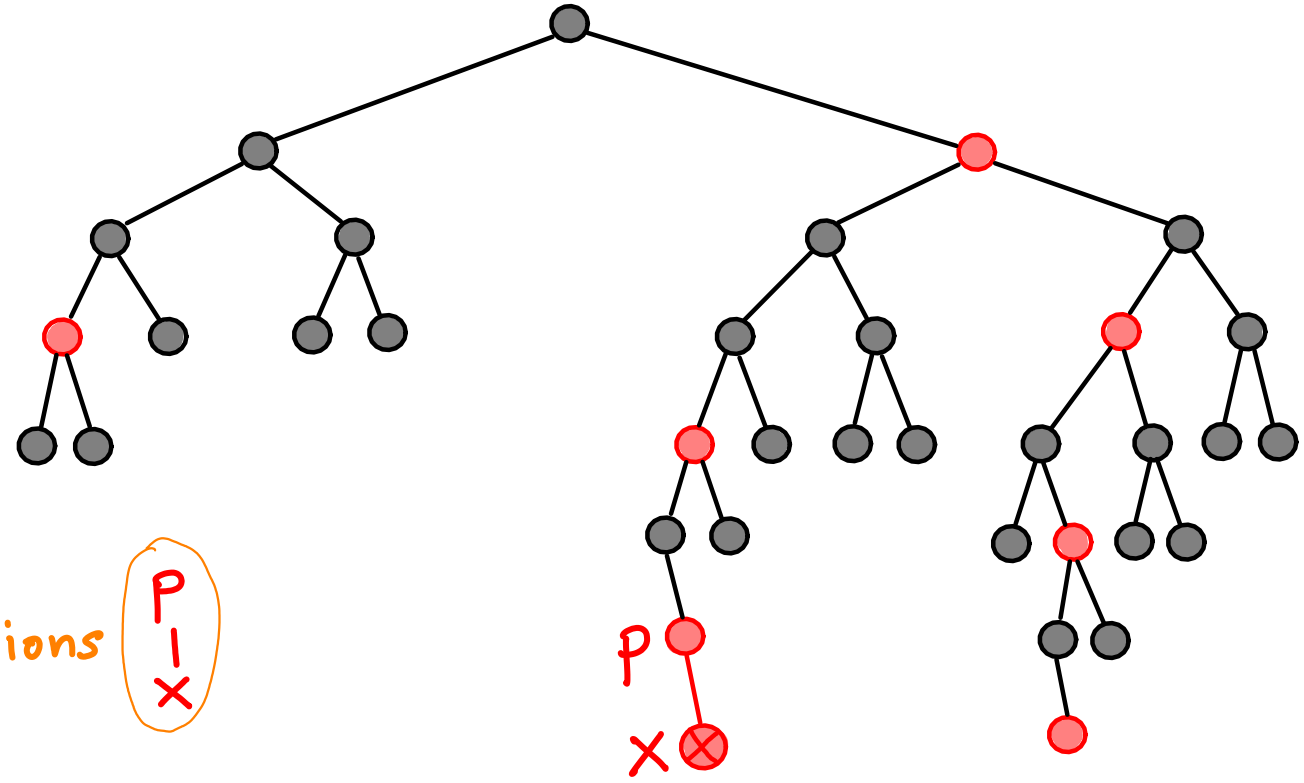

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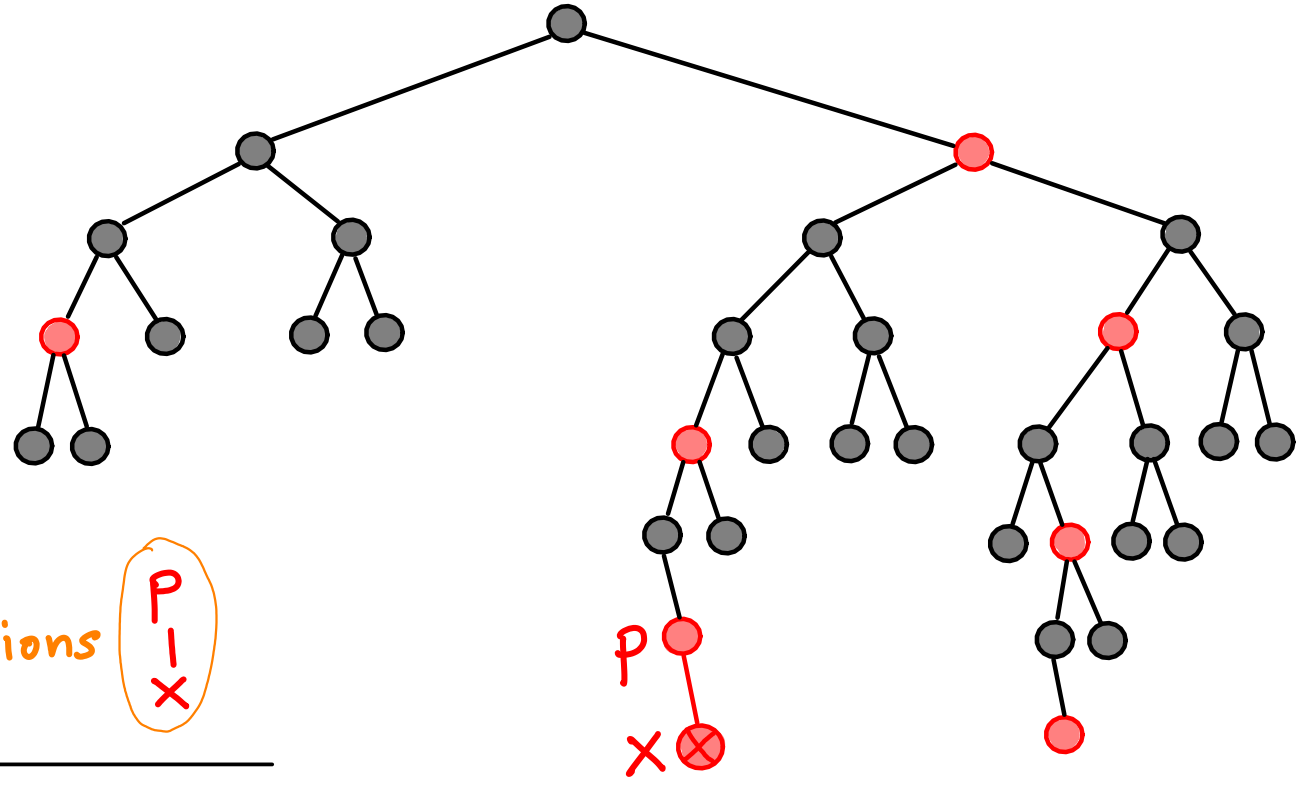

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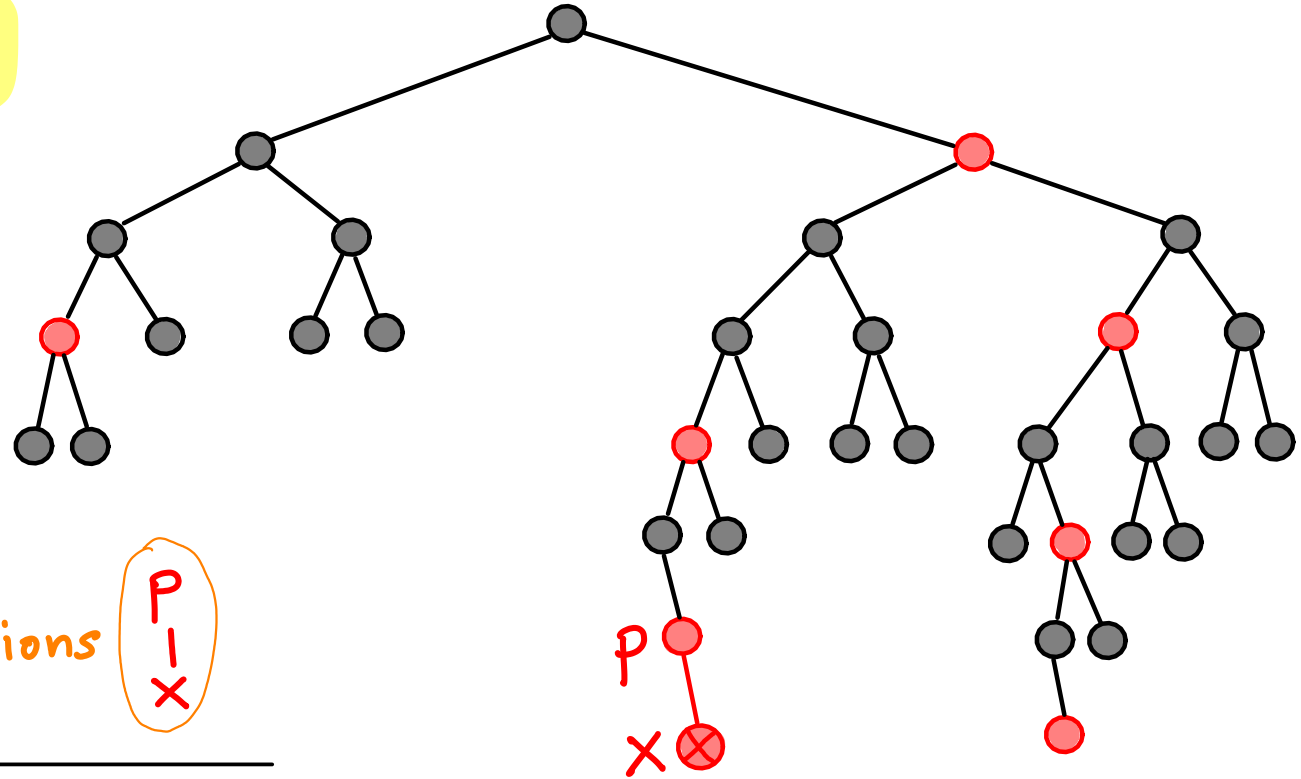
Loop outcomes:

Obtain R-B tree

OR

Redefine ancestor as x

fixing: locally & upward



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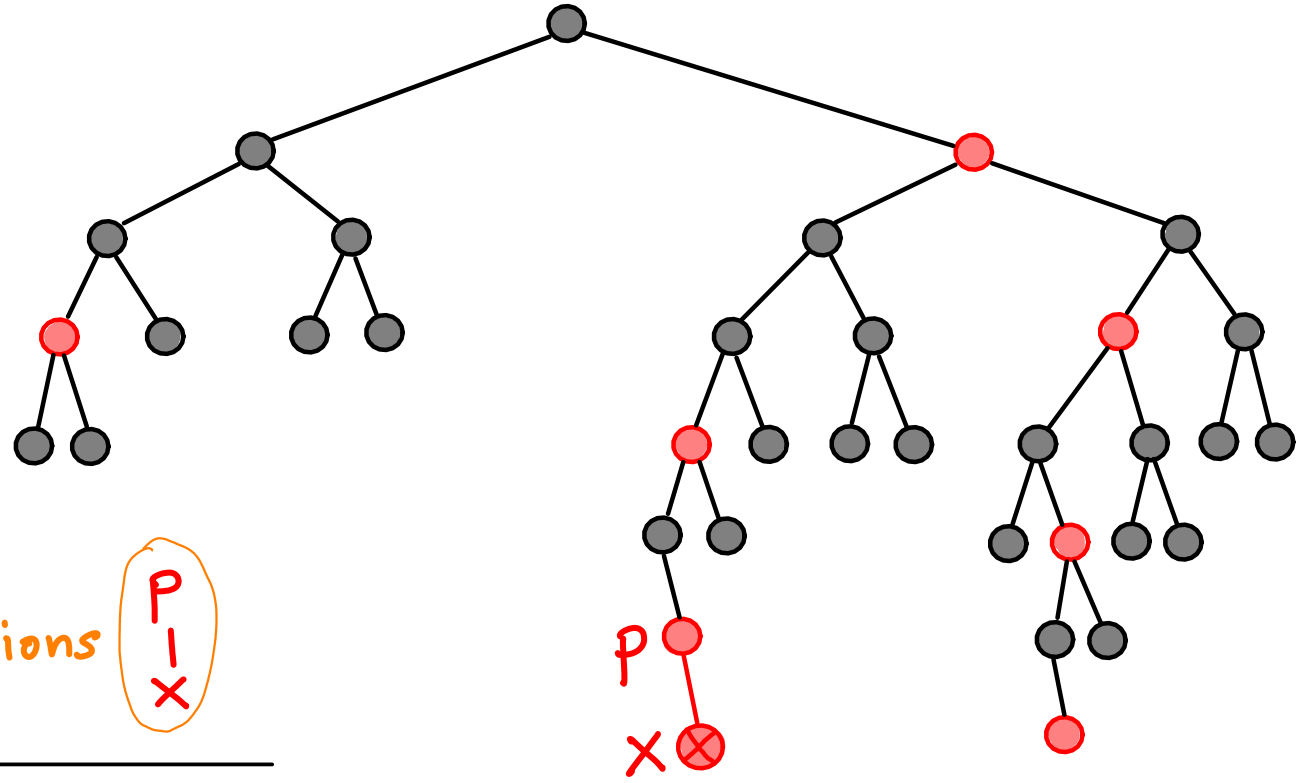

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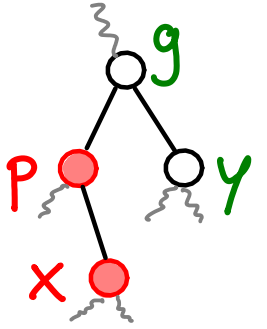

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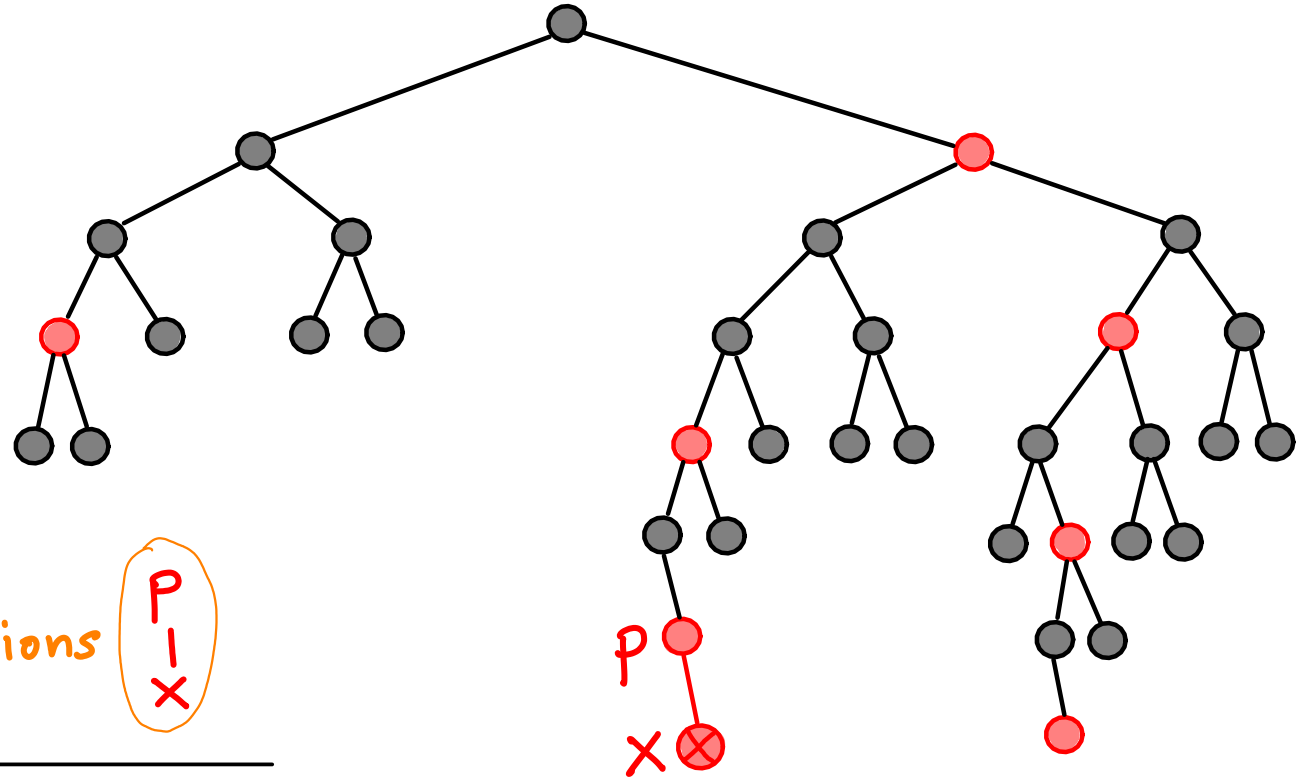

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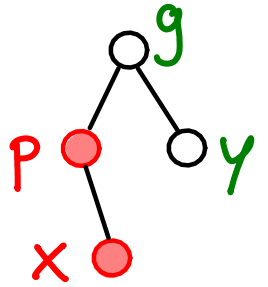
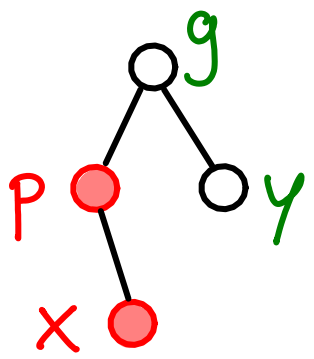


x & p : both red if we needed to continue loop

g: grandparent of x

y: "uncle" of x



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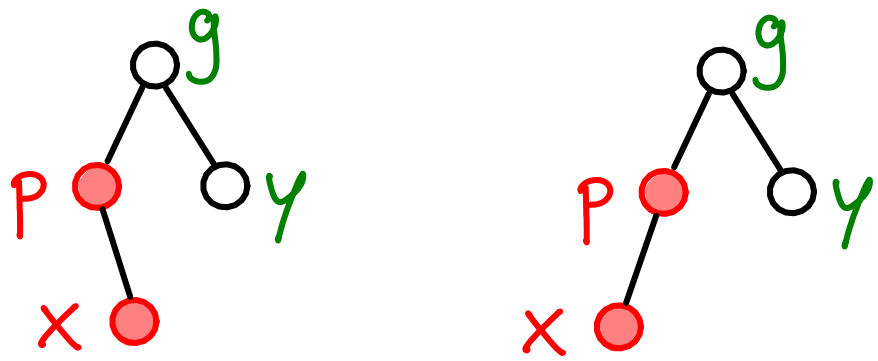
2 possible shapes...



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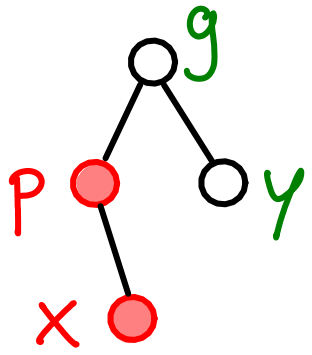
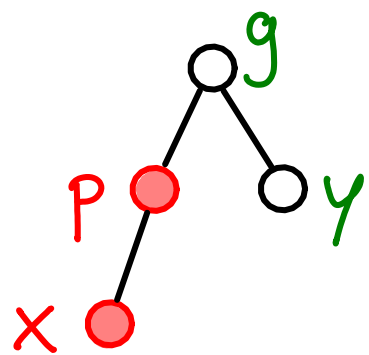
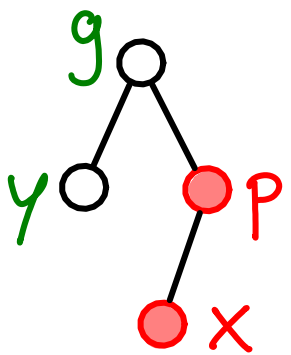
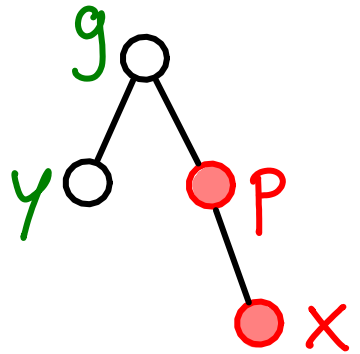




...and their mirror images





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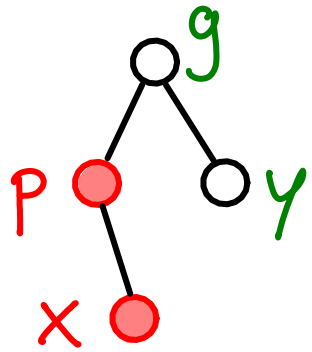

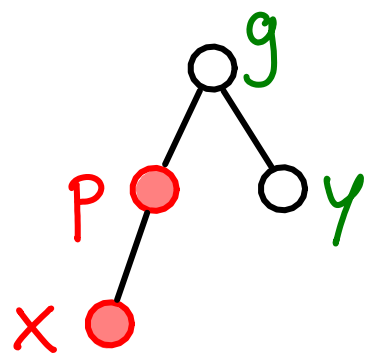

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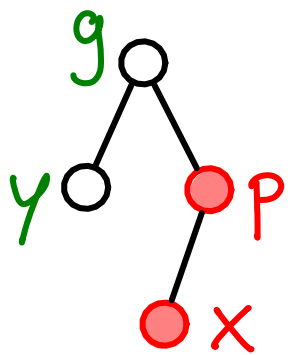

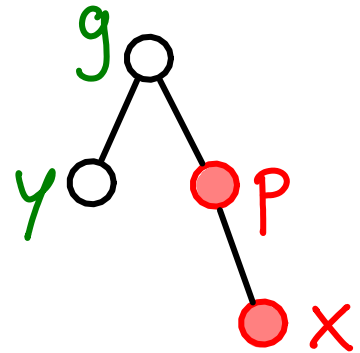

g must exist

because p can't be root

y must exist

(could be a fake leaf)

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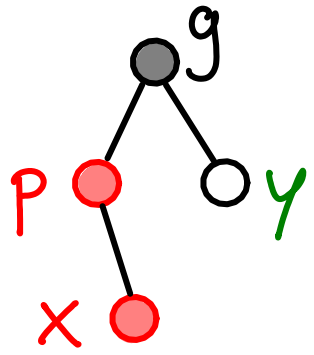

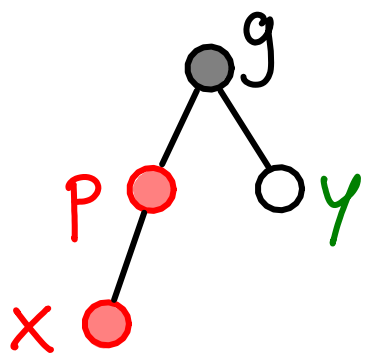

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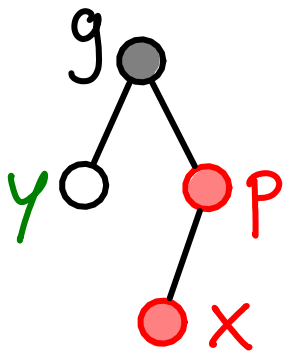

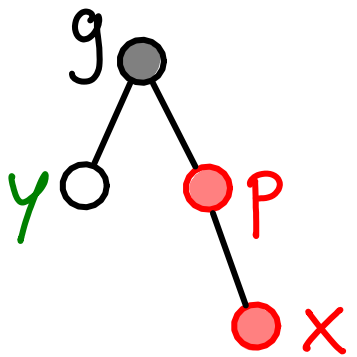

g must exist: must be black
because p can't be root

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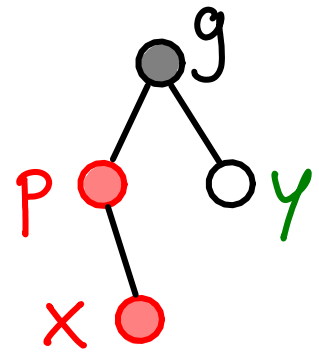
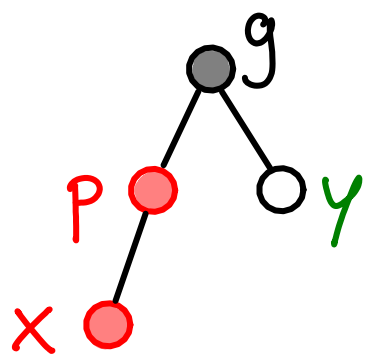
We will handle the 2 main shapes:

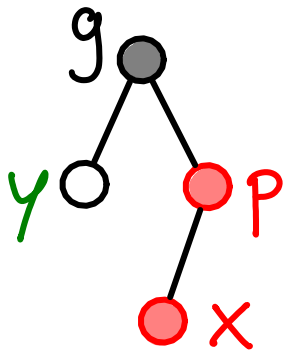
2 possible shapes...

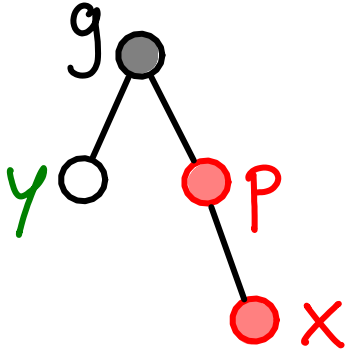




...and their mirror images

The others can be done by symmetry:





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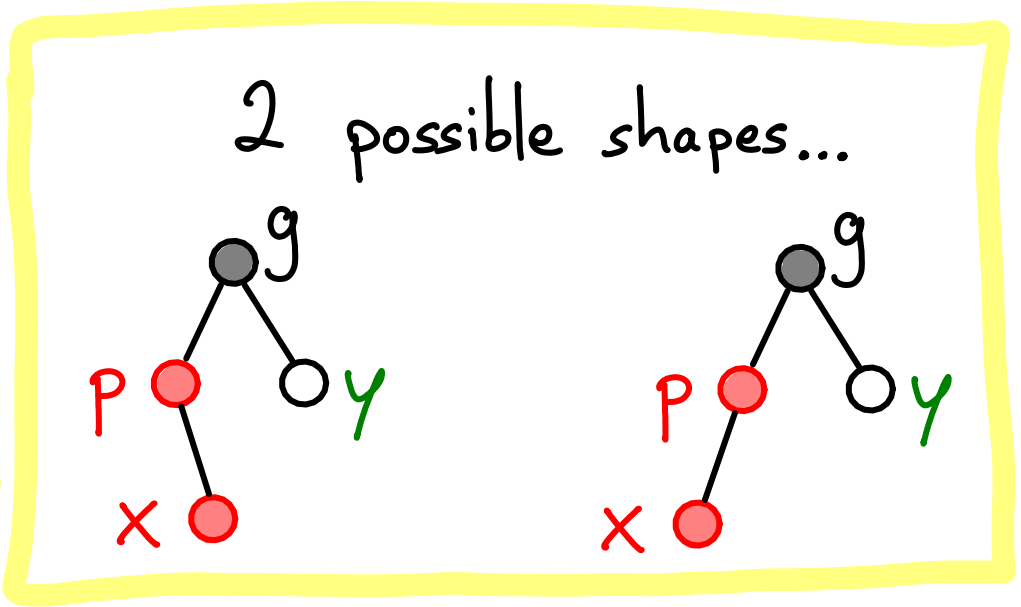
y: "uncle" of x

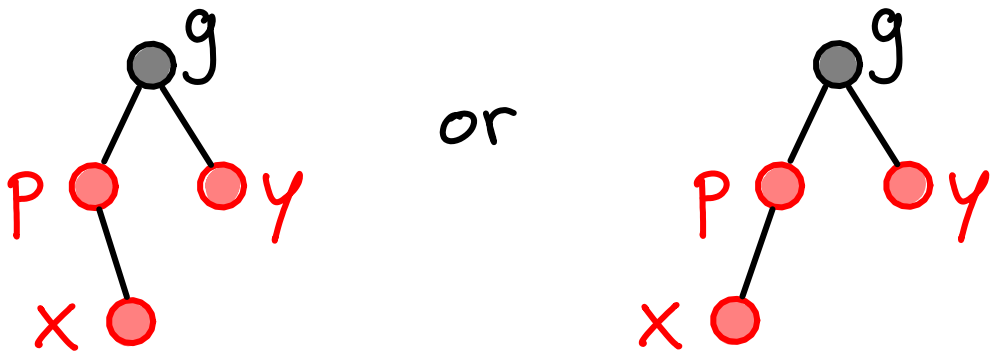
We will handle the 2 main shapes:



Case 1 : y is red

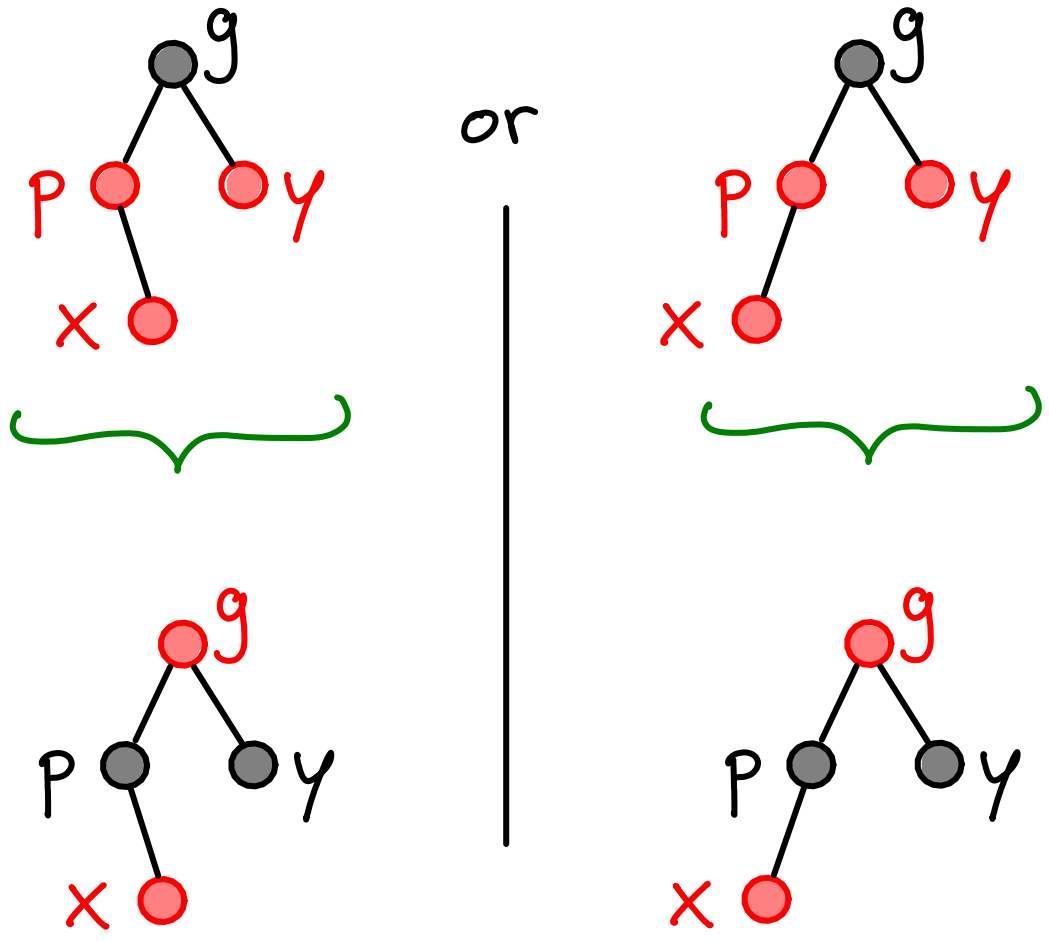
Cases 2 & 3 : y is black

Case 1 : y is red



Case 1 : y is red

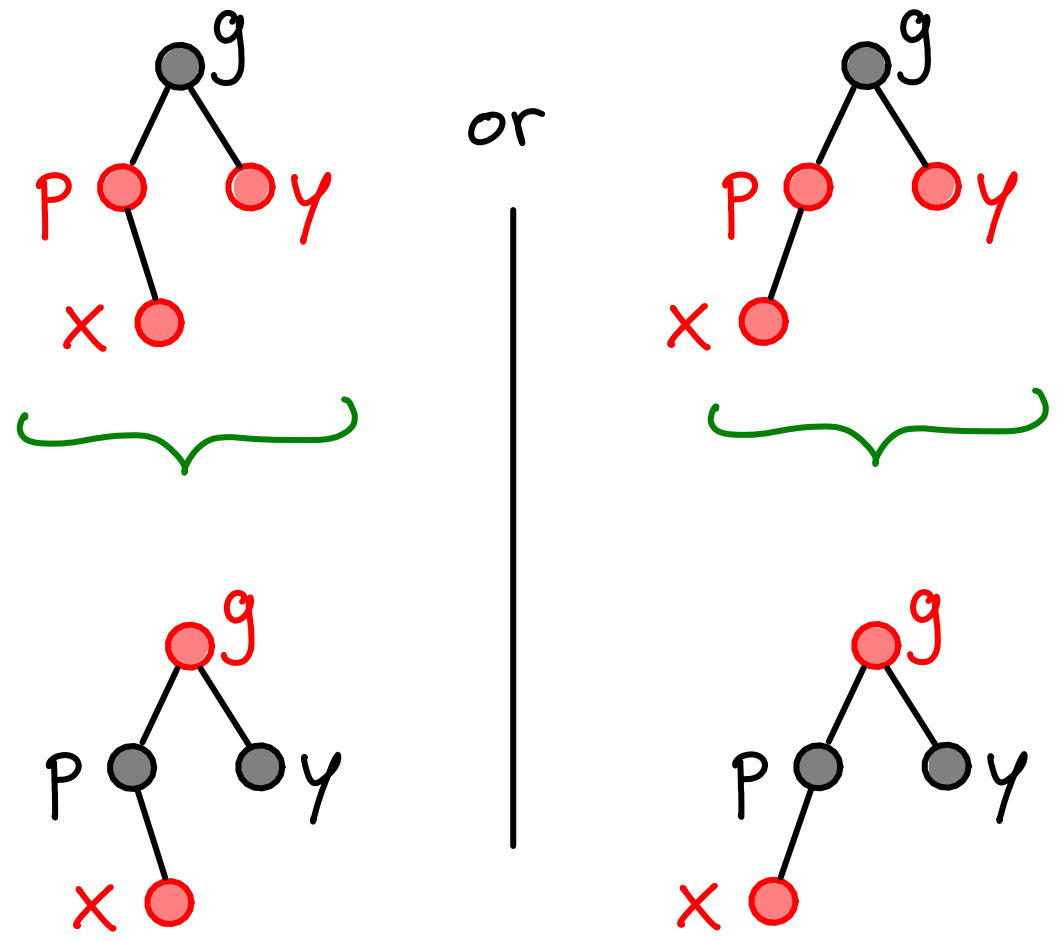
↳ Recolor p, g, y



Case 1 : y is red

↳ Recolor p, g, y

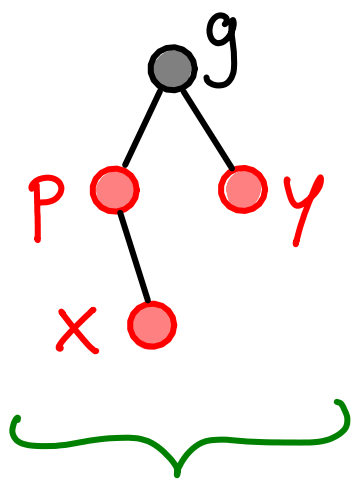
- Preserve black-height invariant.



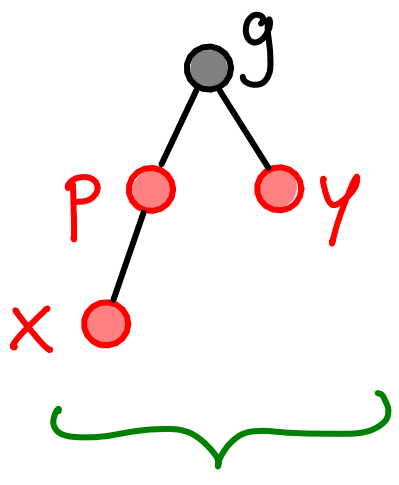
Case 1 : y is red

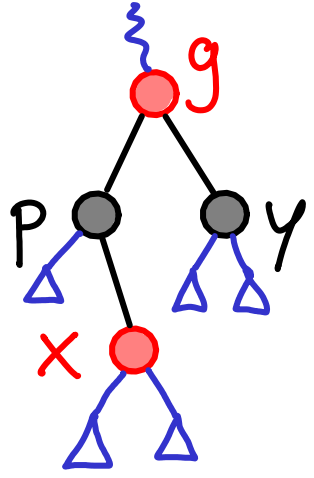
↳ Recolor p, g, y

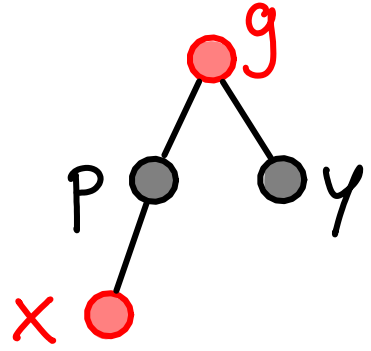
- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)



or



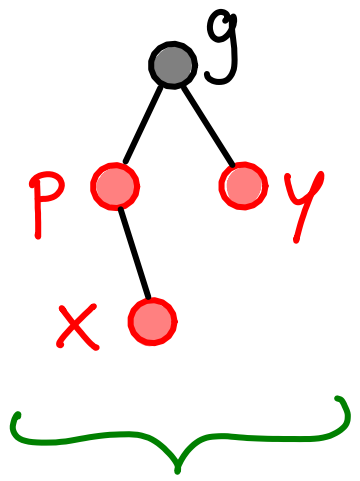




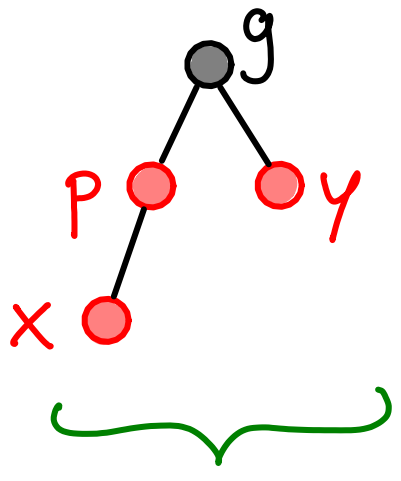
Case 1 : y is red

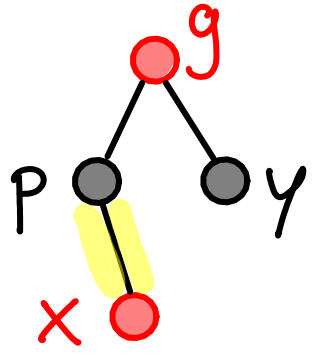
↳ Recolor p, g, y

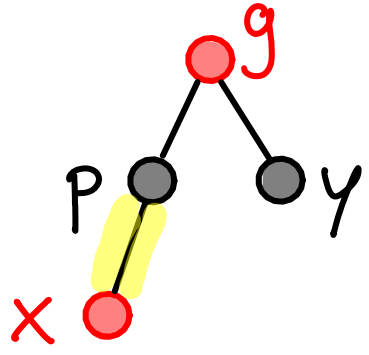
- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)
- Eliminate $\overset{P}{\underset{X}{|}}$ violation.



or



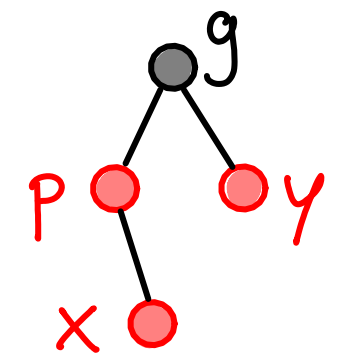


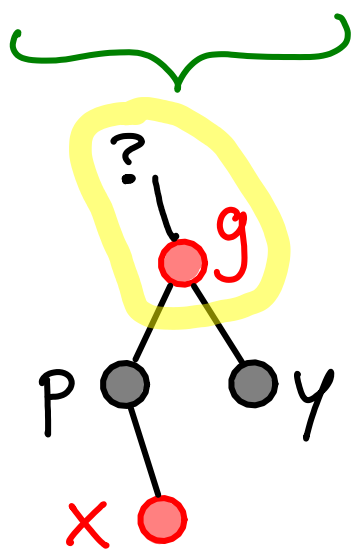


Case 1 : y is red

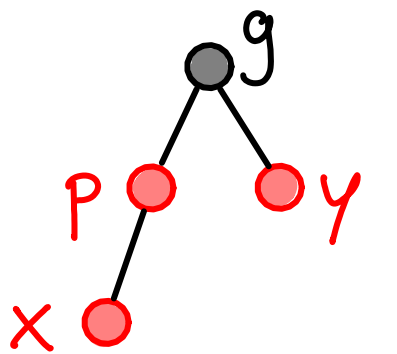
↳ Recolor p, g, y

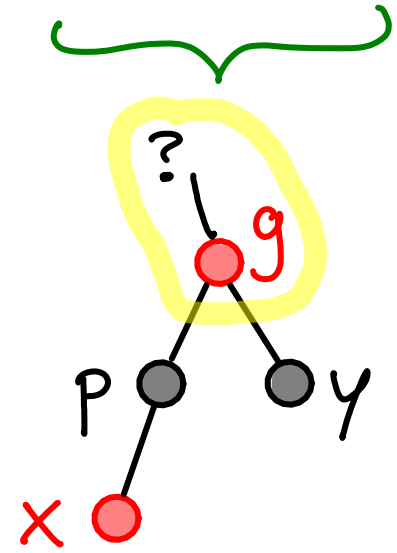
- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)
- Eliminate P / x violation.
- If no new violation (? / g) then DONE
  ↳ (note: if g = root color g black)





or



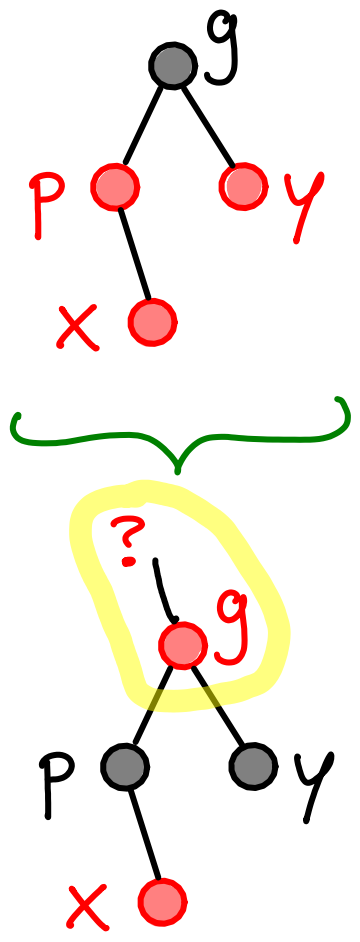


Case 1 : y is red

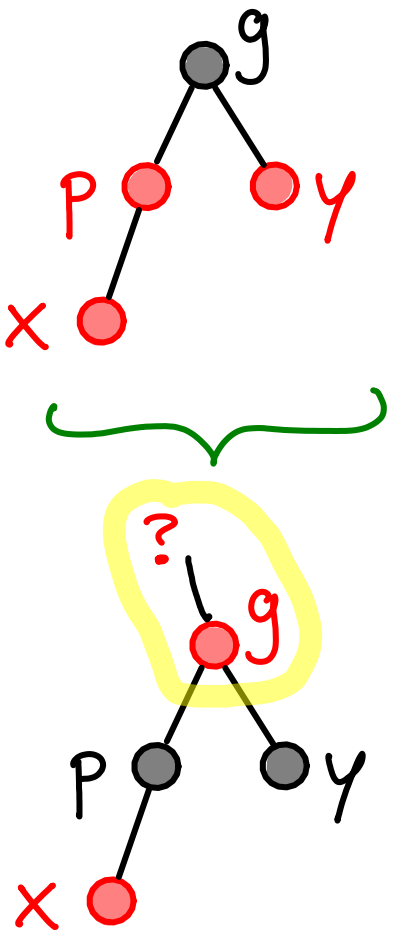
↳ Recolor p, g, y

- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)
- Eliminate P/X violation.
- Might cause violation ? ↕ g

otherwise DONE



or

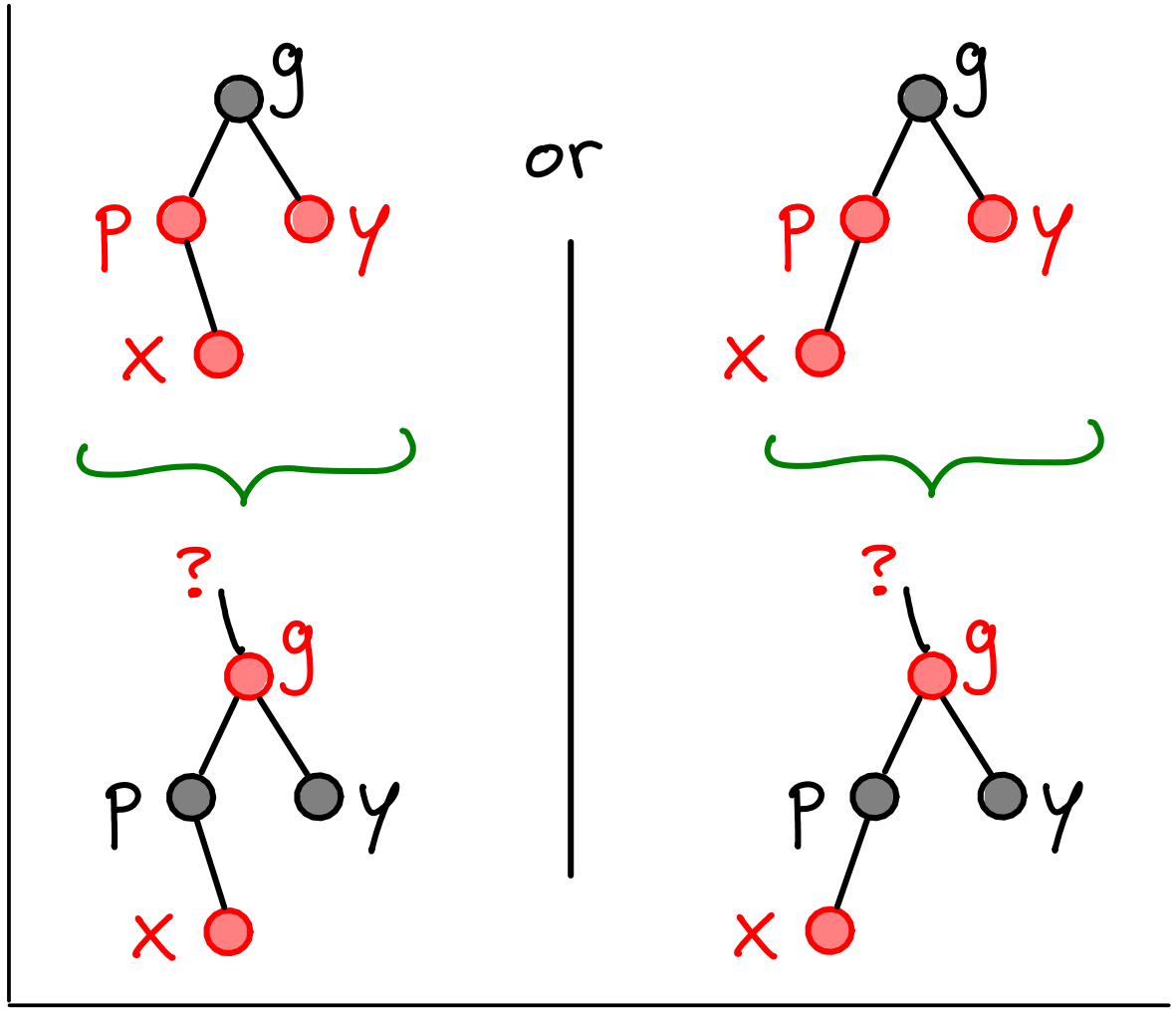


Case 1 : y is red

↳ Recolor p, g, y

- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)
- Eliminate P / x violation.
- Might cause violation ? / g

otherwise DONE



fix upward ↔ ? becomes p, g becomes x

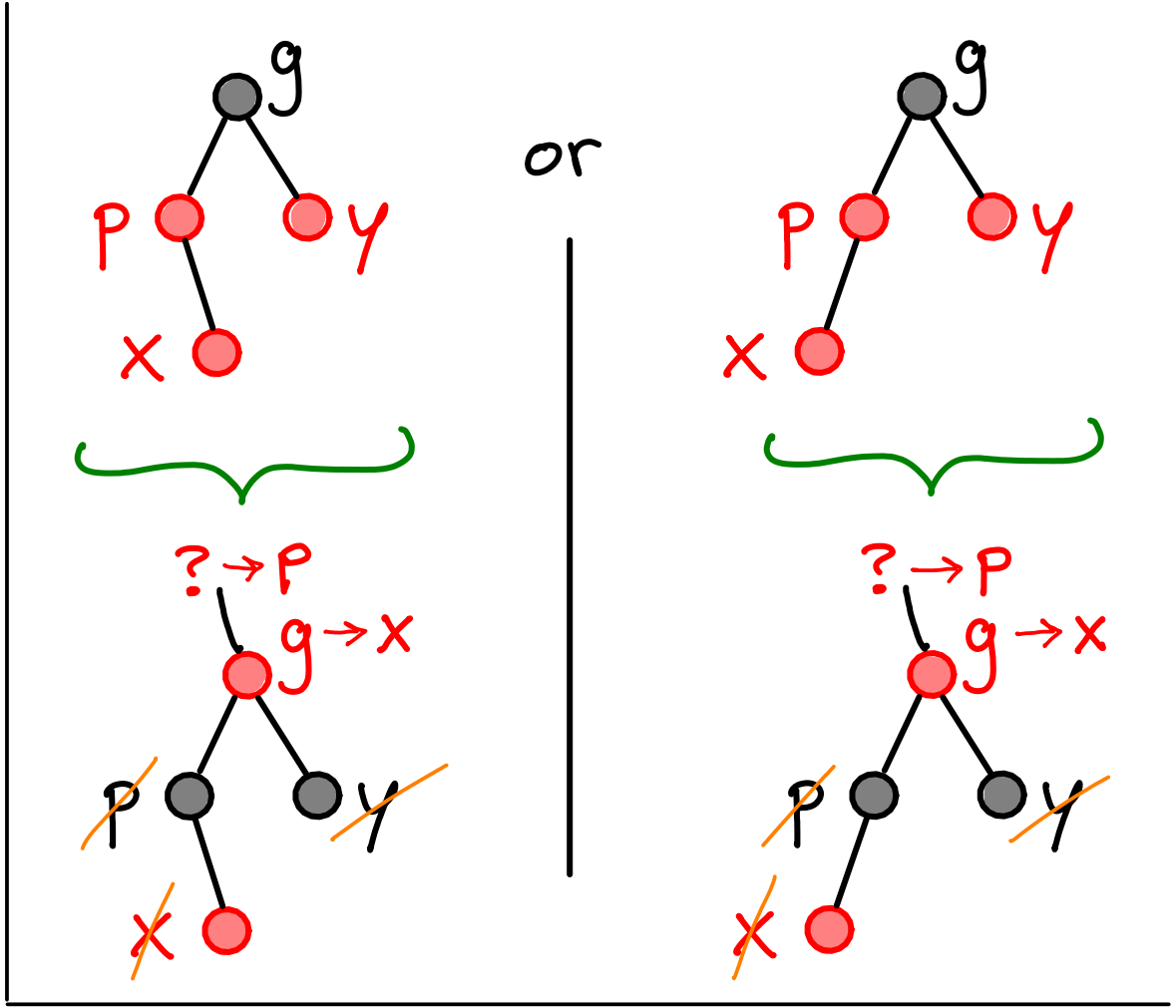
(Just changing labels)

Case 1 : y is red

↳ Recolor p, g, y O(1)

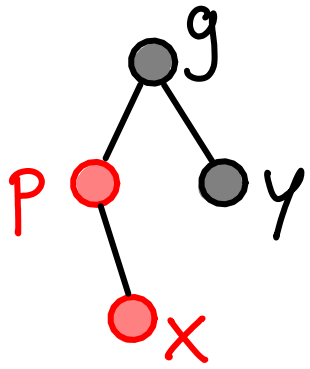
- Preserve black-height invariant.
  (total B nodes on any path from global root down to leaves)
- Eliminate p / x violation.
- Might cause violation ? / g

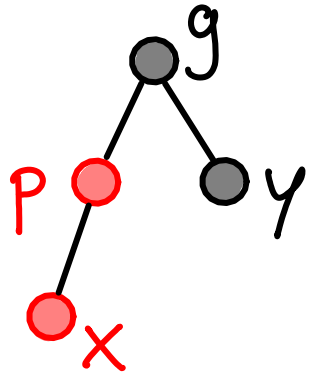
otherwise DONE



fix upward ↔ ? becomes p, g becomes x → Invariants restored

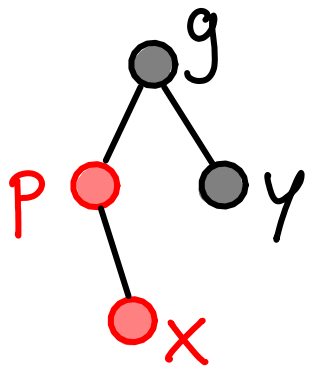
Cases 2 & 3: y is black



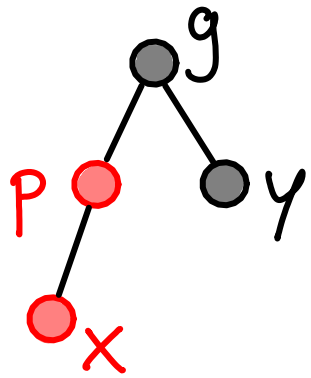


Cases 2 & 3: y is black

Case 2

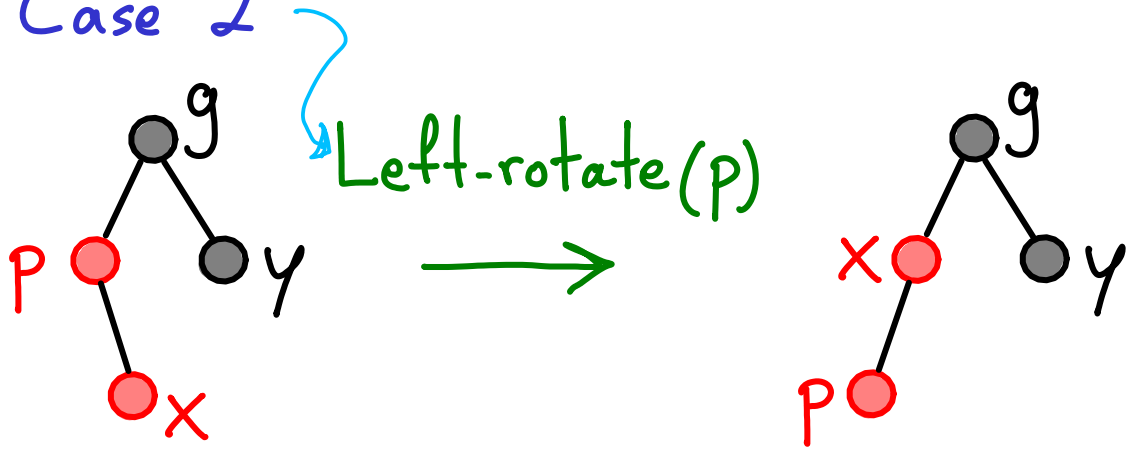


Case 3



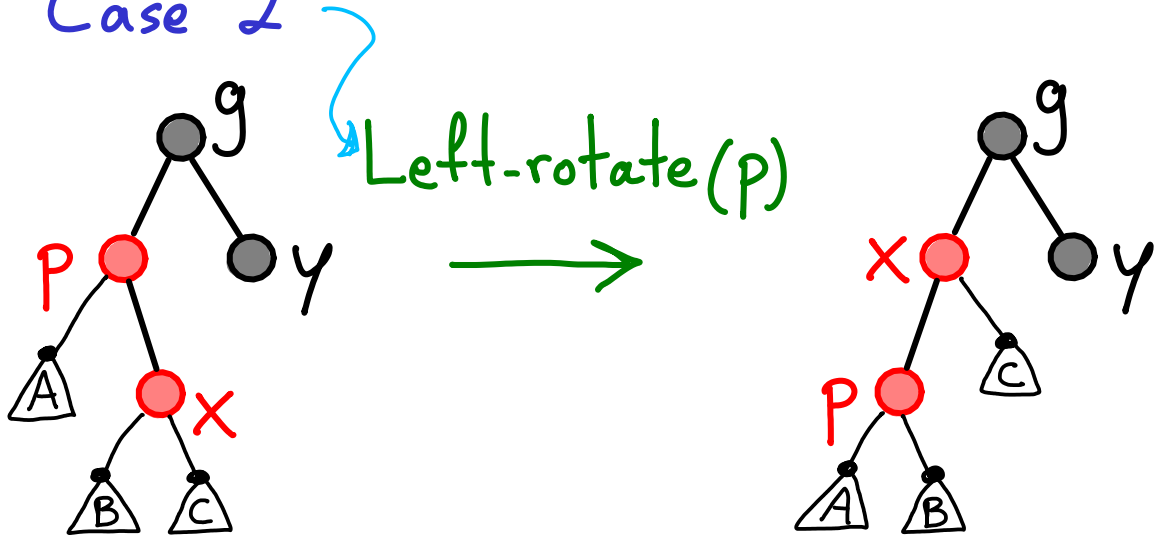
Cases 2 & 3: y is black

Case 2



Cases 2 & 3: y is black

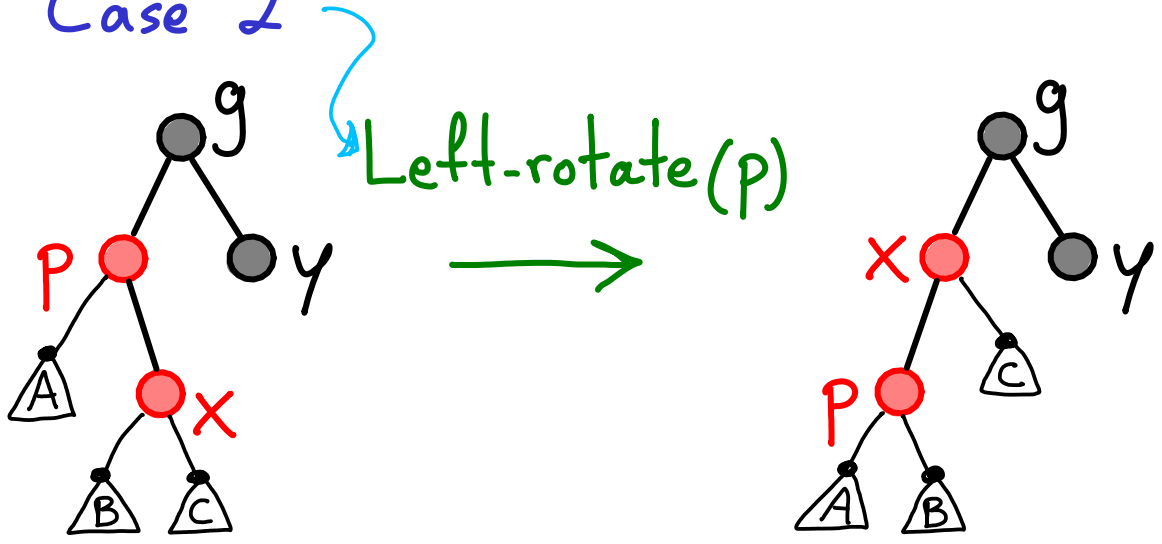
Case 2



Cases 2 & 3: y is black

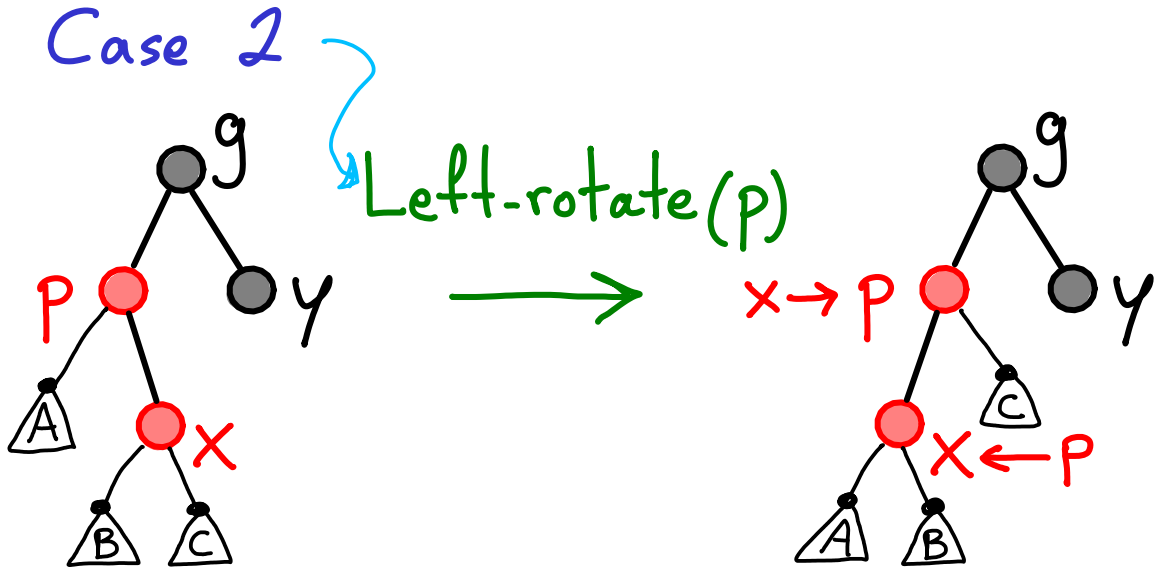
- Preserve black-height invariant.

Case 2



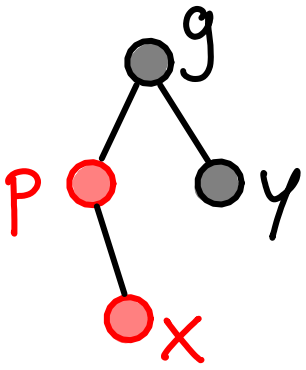
Cases 2 & 3: y is black

- Preserve black-height invariant.
- Swap labels $x \leftrightarrow p$
- No new violation

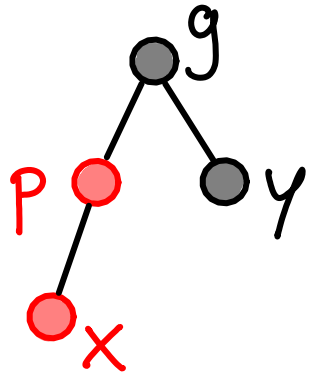


Cases 2 & 3 : y is black

Case 2

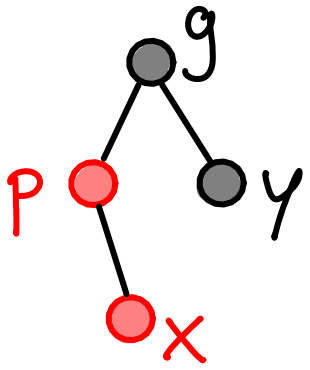


Case 3



Cases 2 & 3: y is black

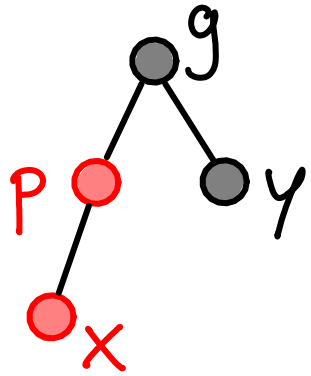
Case 2



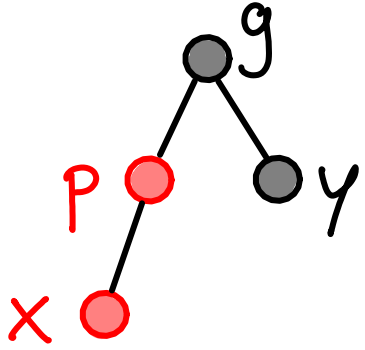
Left-rotate(p)

& swap(p,x)

Case 3

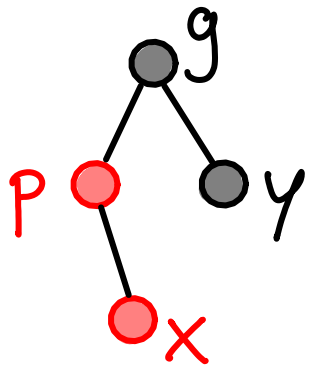


Handling Case 3:



Cases 2 & 3: y is black

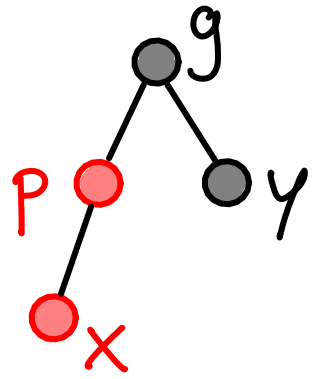
Case 2



Left-rotate(p)

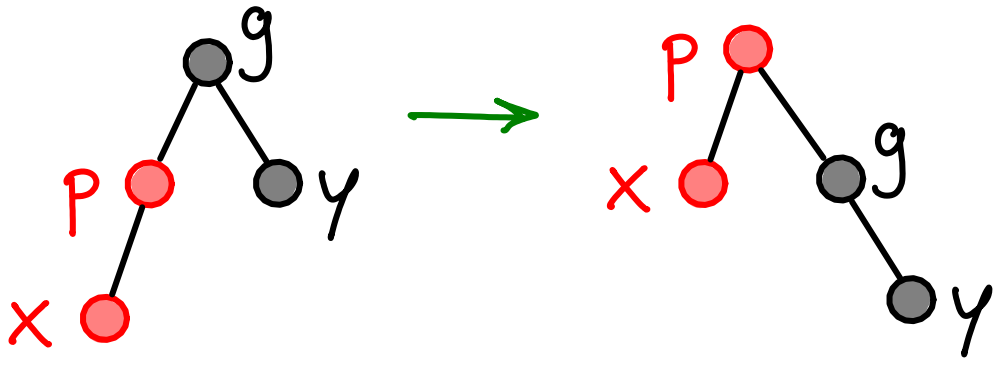
& swap(p,x)

Case 3



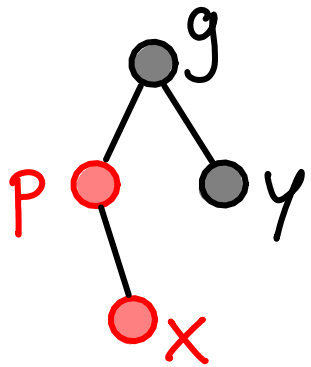
Handling Case 3:

Right-rotate(g)



Cases 2 & 3: y is black

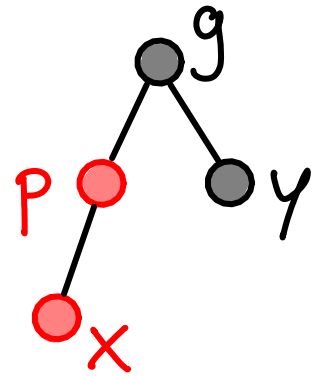
Case 2



Left-rotate(p)

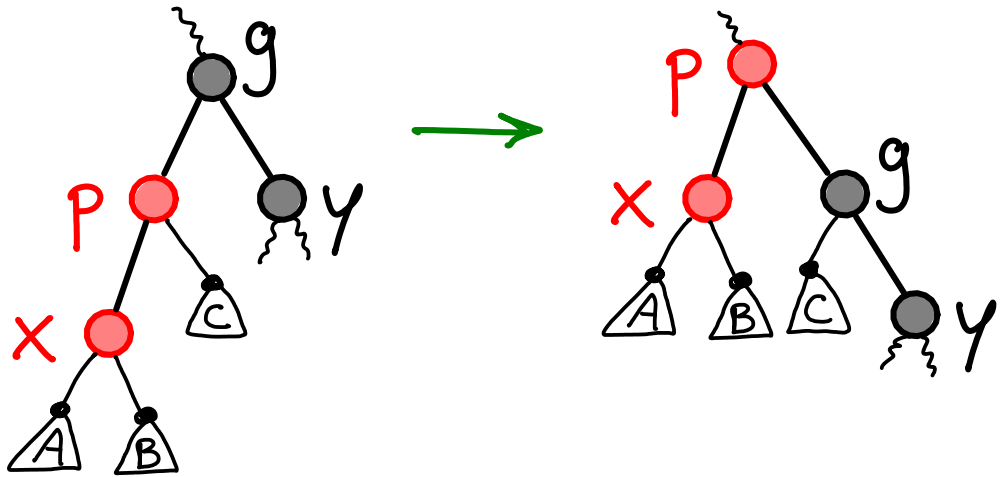
& swap(p,x)

Case 3



Handling Case 3:

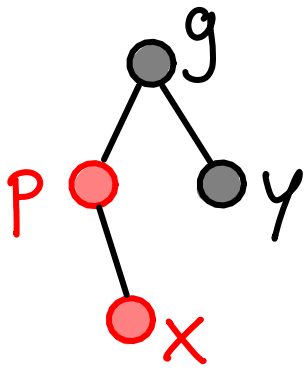
Right-rotate(g)



Cases 2 & 3: y is black

- Preserve black-height invariant.

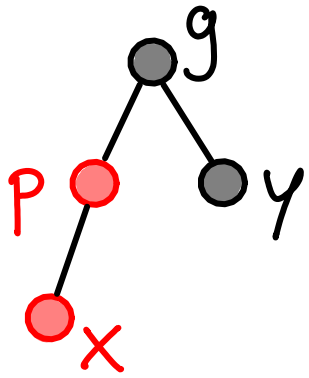
Case 2



Left-rotate(p)

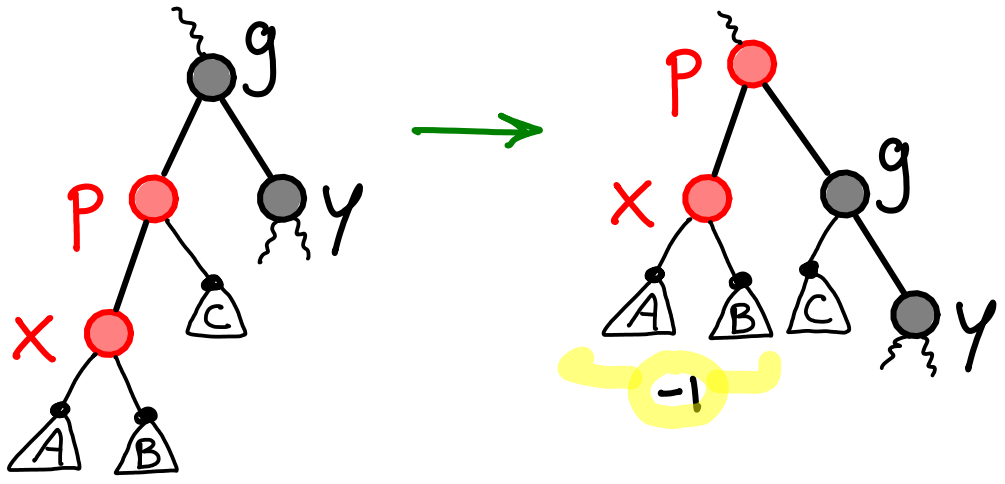
& swap(p,x)

Case 3



Handling Case 3:

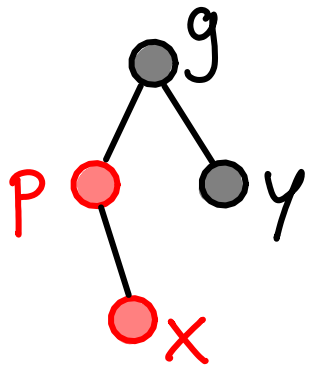
Right-rotate(g)



Cases 2 & 3: y is black

- Preserve black-height invariant. ✓

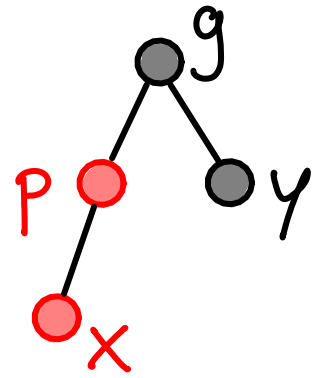
Case 2



Left-rotate(p)

& swap(p,x)

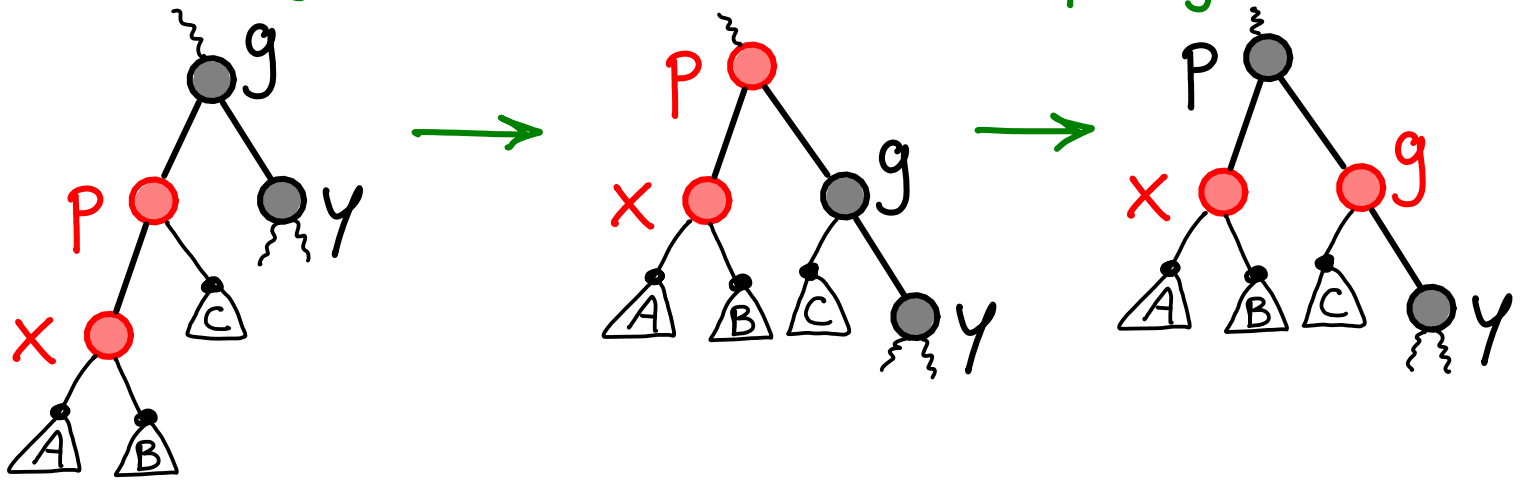
Case 3



Handling Case 3:

Right-rotate(g)

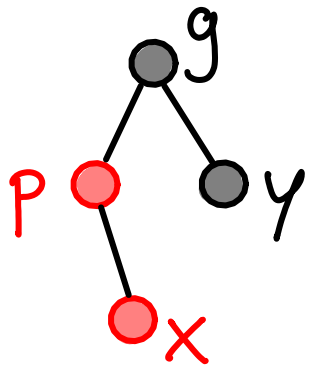
Recolor p & g



Cases 2 & 3: y is black

- Preserve black-height invariant.
- Eliminate $\begin{matrix} p \\ | \\ x \end{matrix}$ violation in case 3.

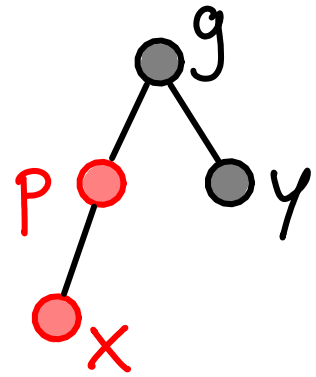
Case 2



Left-rotate(p)

& swap(p,x)

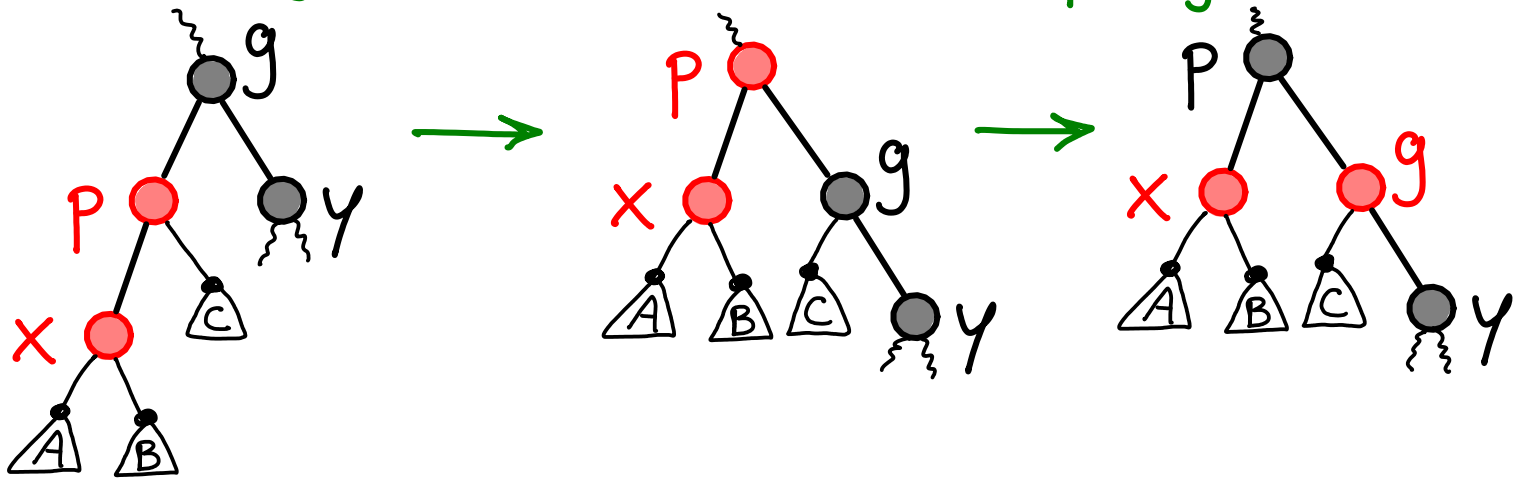
Case 3



Handling Case 3:

Right-rotate(g)

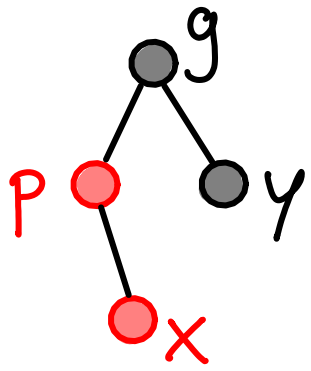
Recolor p & g



Cases 2 & 3: y is black

- Preserve black-height invariant.
- Eliminate $\begin{matrix} P \\ | \\ x \end{matrix}$ violation in case 3.
- No new violation

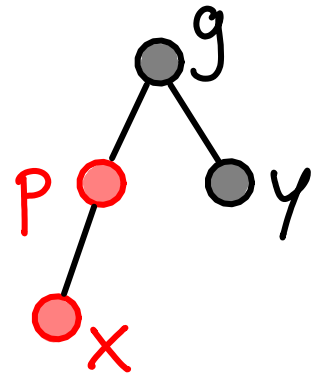
Case 2



Left-rotate(p)

& swap(p,x)

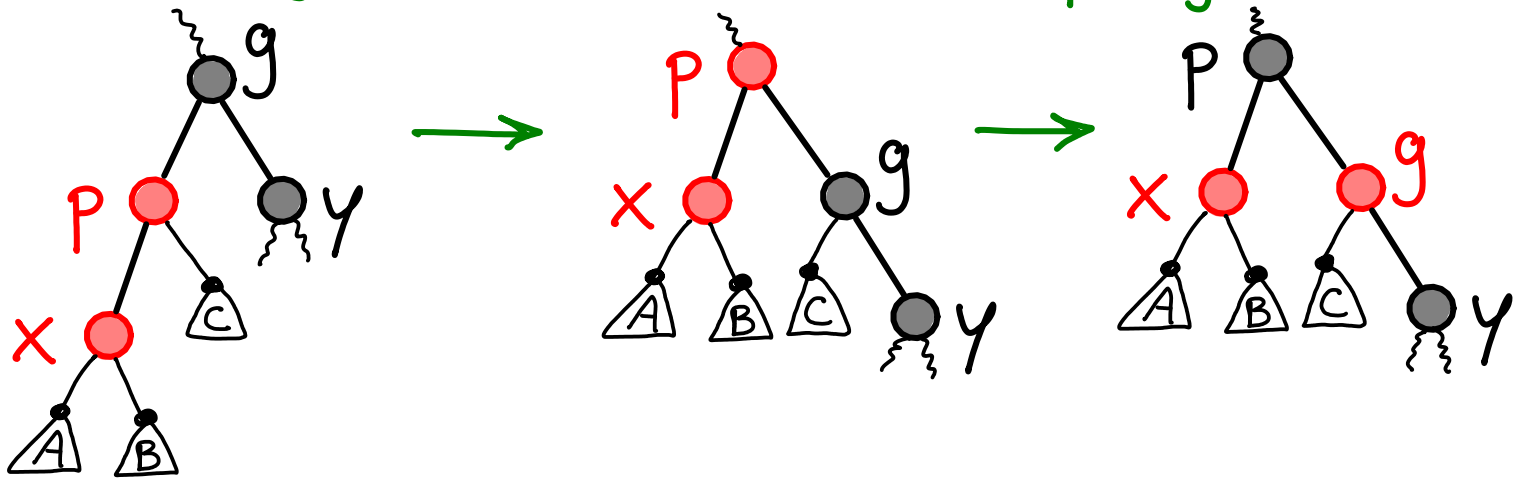
Case 3



Handling Case 3:

Right-rotate(g)

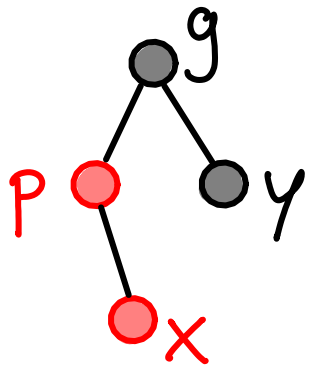
Recolor p & g



Cases 2 & 3: y is black

- Preserve black-height invariant.
- Eliminate $\begin{matrix} P \\ | \\ x \end{matrix}$ violation in case 3.
- No new violation

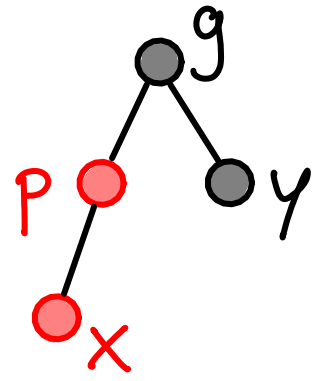
Case 2



Left-rotate(p)

& swap(p,x)

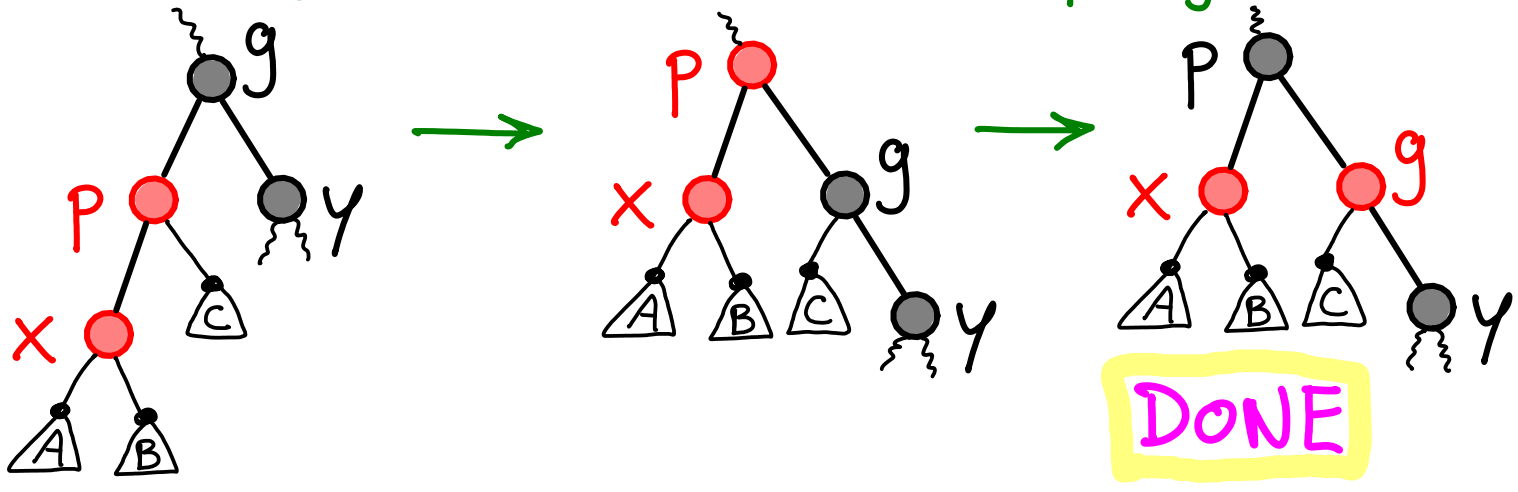
Case 3



Handling Case 3:

Right-rotate(g)

Recolor p & g



Each case takes O(1) time

All together O(logn) time

Each case takes O(1) time

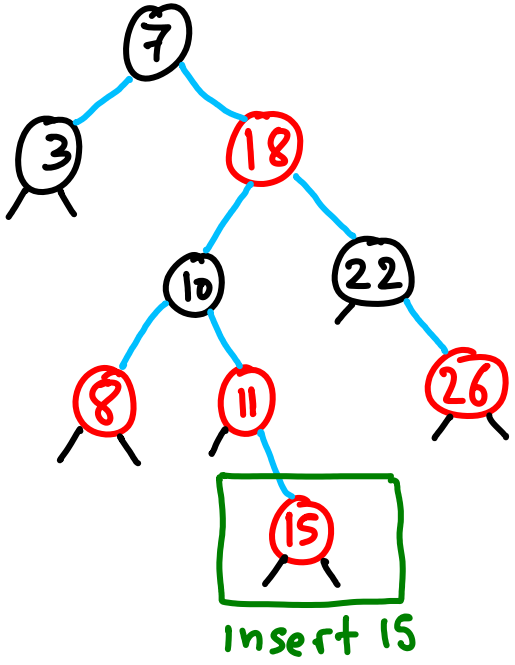
All together O(logn) time

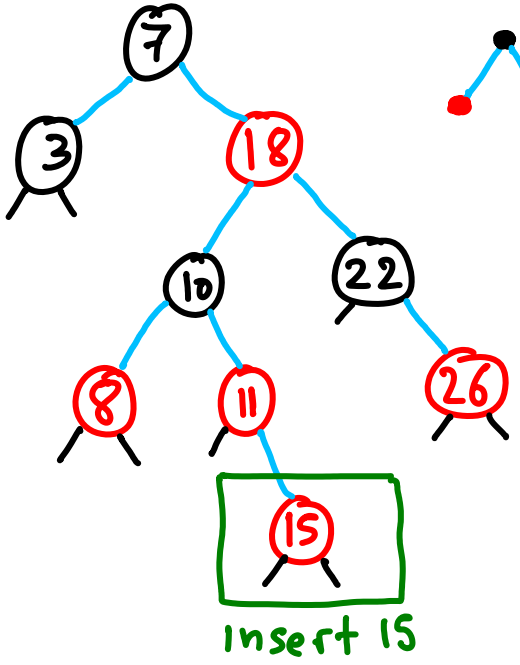
Possibly lots of recoloring (Case 1)

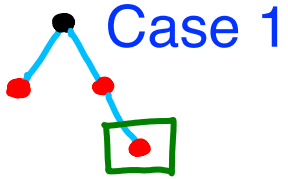
followed by Case 3 OR Case 2 —> 3

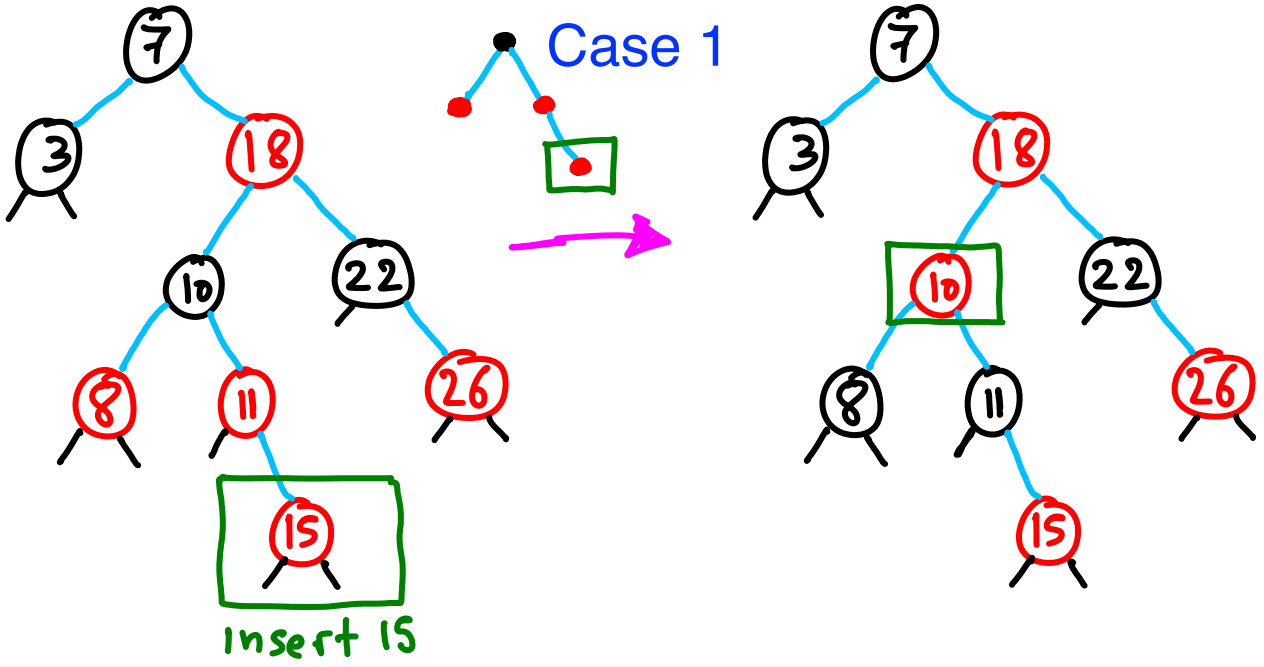
(1 or 2 rotations, total)

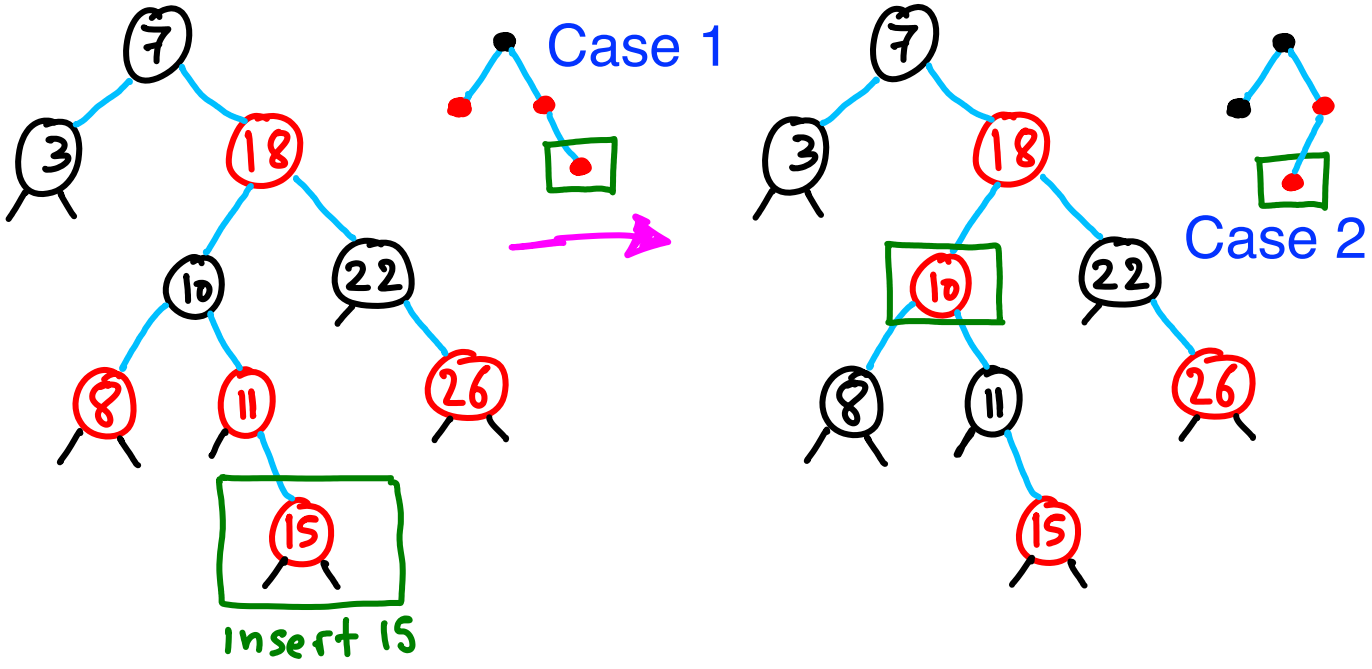
* useful property

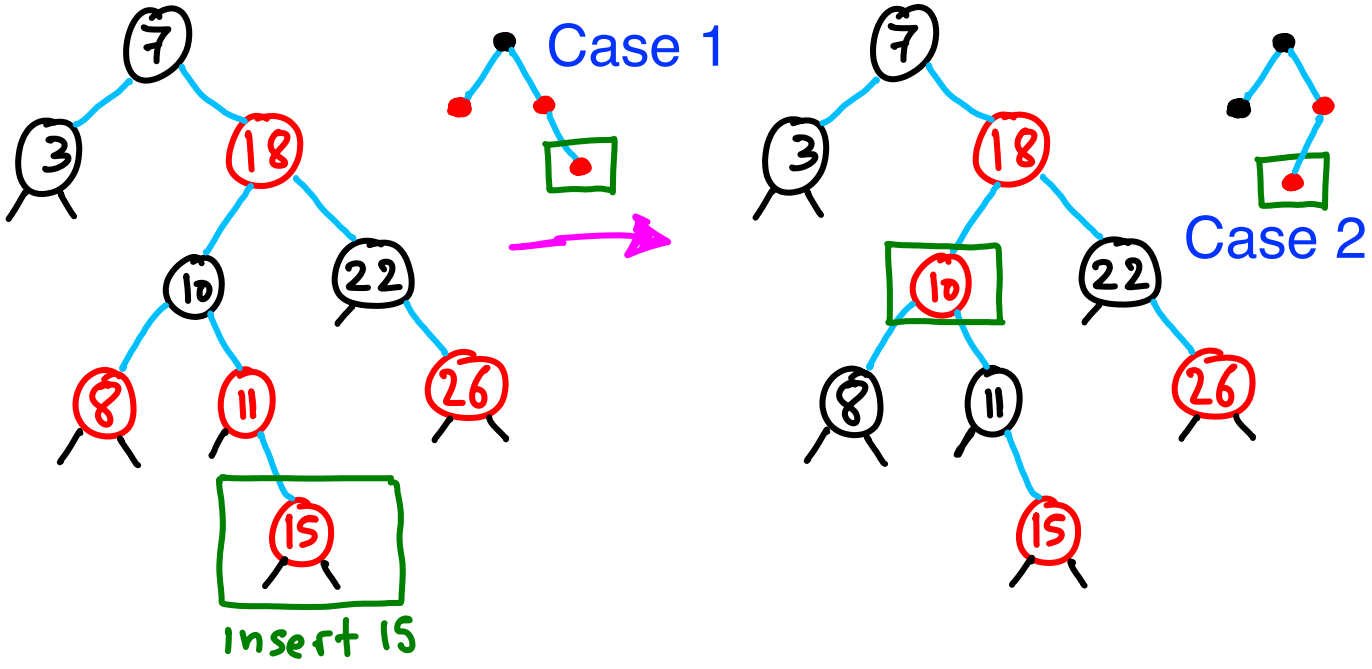

7
3
18
10
22
8
11
26
15
insert 15

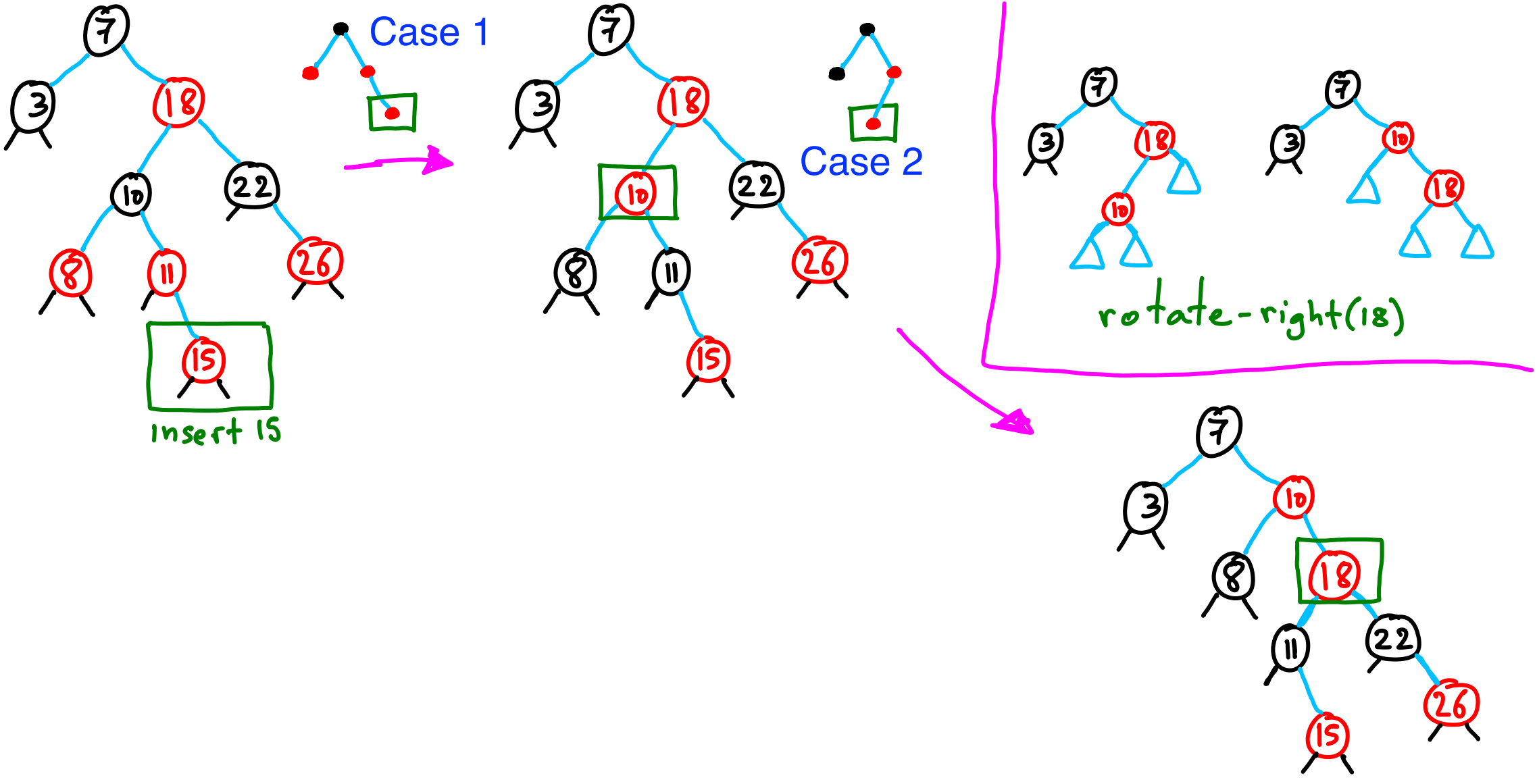

7
3
18
10
22
8
11
26
15
insert 15

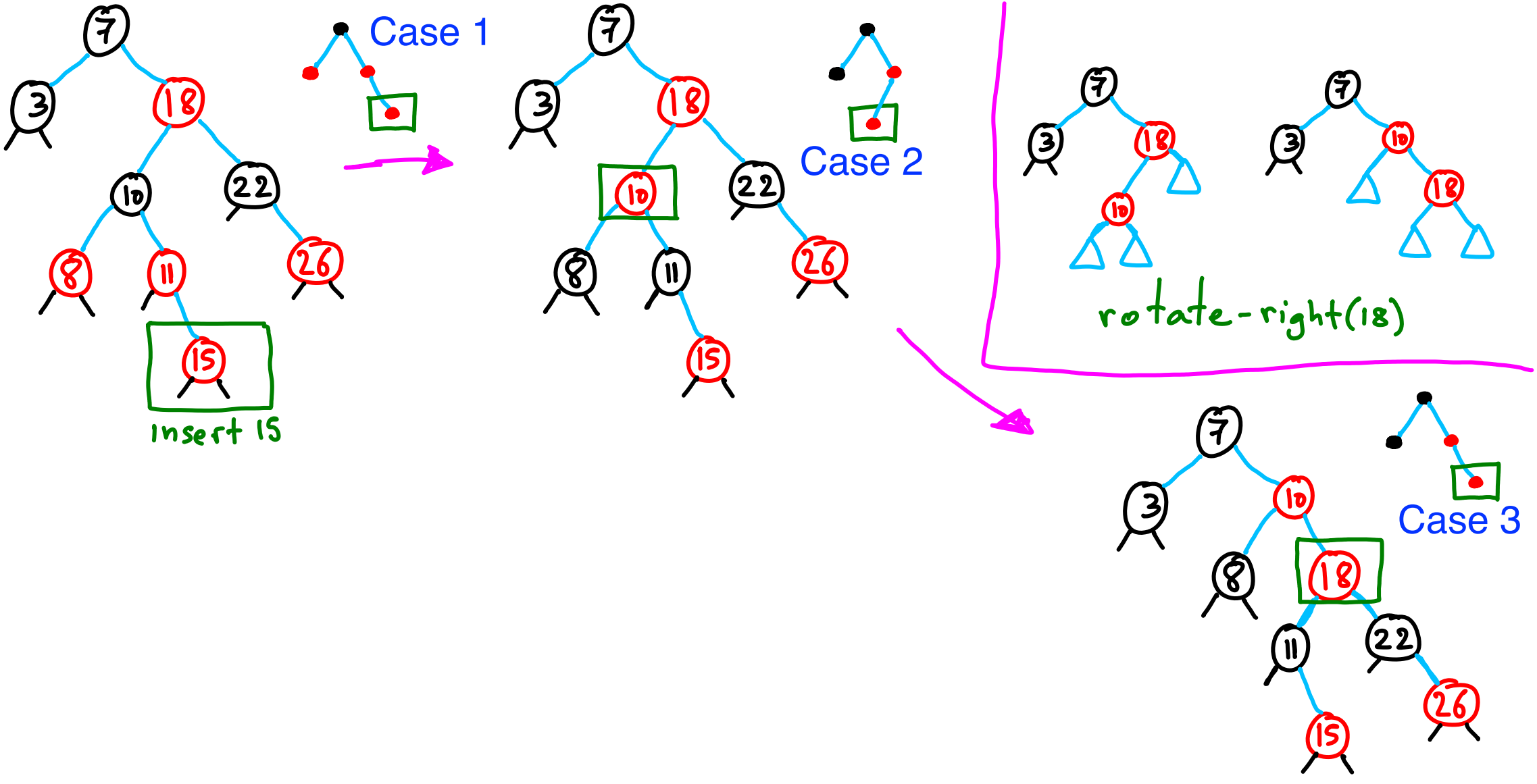

Case 1

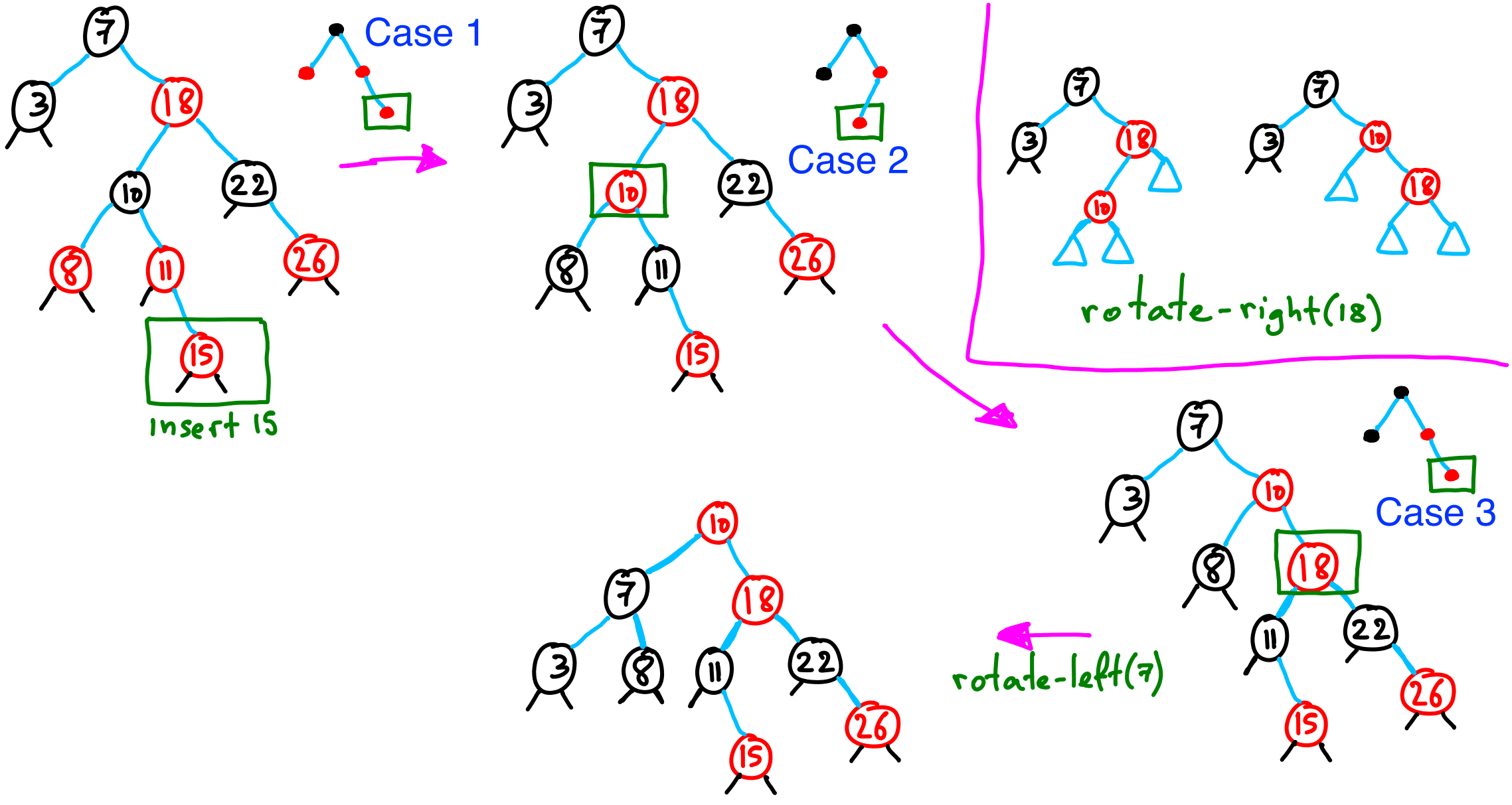

7
3
18
10
22
8
11
26
15
insert 15
Case 1
7
3
18
10
22
8
11
26
15

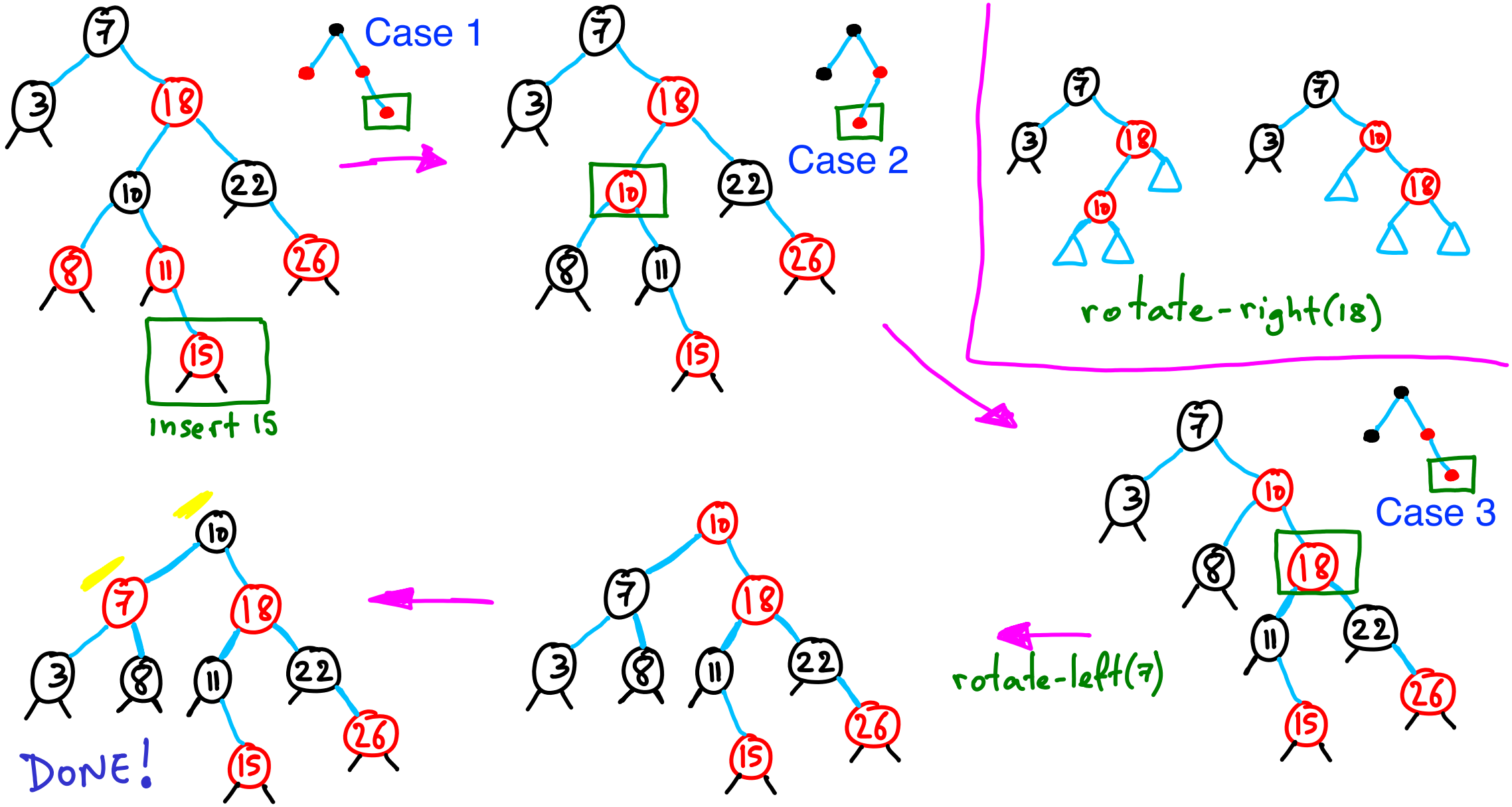

7
3
18
10
22
8
11
26
15
insert 15
Case 1
7
3
18
10
22
8
11
26
15
Case 2


7
3
18
10
22
8
11
26
15
insert 15
Case 1
7
3
18
10
22
8
11
26
15
Case 2
7
3
18
10
7
3
10
18
rotate-right(18)


Case 1
Case 2
insert 15
rotate-right(18)


Case 1
Case 2
Case 3
insert 15
rotate-right(18)


Case 1
Case 2
Case 3
insert 15
rotate-right(18)
rotate-left(7)


Case 1
Case 2
Case 3
insert 15
rotate-right(18)
rotate-left(7)
DONE!