Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

if $f(n) = n \Rightarrow$

$$n^{\log_b a} = n$$

Divide & Conquer

$T(n) = 2T(\frac{n}{2}) + f(n)$

if $f(n) = n \Rightarrow$ case2 $\Rightarrow$

$n^{\log_b a} = n$

Divide & Conquer

$T(n) = 2T(\frac{n}{2}) + f(n)$

if $f(n) = n \Rightarrow$ case2 $\Rightarrow \Theta(n \cdot \log n)$

$n^{\log_b a} = n$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = n \Rightarrow$ case2 $\Rightarrow \Theta(n \cdot \log n)$

if $f(n) = n \log^k n \Rightarrow$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = n \Rightarrow$ case 2 $\Rightarrow \Theta(n \cdot \log n)$

$(k=0)$

if $f(n) = n \log^k n \Rightarrow$ case 2 $\Rightarrow \Theta(n \log^{k+1} n)$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = n \Rightarrow$ case2 $\Rightarrow \Theta(n \cdot \log n)$

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if $f(n) = \log^c n \Rightarrow$

Divide & Conquer

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if $f(n) = n \log^k n \Rightarrow$ case 2 $\Rightarrow \Theta(n \log^{k+1} n)$

if $f(n) = \log^c n \Rightarrow$ case 1

$f(n) = O(n^{1-\varepsilon})$

Divide & Conquer

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$f(n) = O(n^{1-\varepsilon})$

if $f(n) = n^c$

~~for~~ $c > 1$

Divide & Conquer

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$f(n) = O(n^{1-\varepsilon})$

$$\left.\begin{array}{l} \text{if } f(n) = n^c \\ \text{for } c > 1 \end{array}\right\} f(n) = \Omega(n^{1+\varepsilon})$$

Divide & Conquer

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$f(n) = O(n^{1-\varepsilon})$

if $f(n) = n^c$ for $c > 1$ $\Big\}$ $f(n) = \Omega(n^{1+\varepsilon})$

AND $2f\left(\frac{n}{2}\right) = \frac{2}{2^c} n^c = \frac{1}{2^{c-1}} \cdot f(n)$

Divide & Conquer

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if $f(n) = n^c$ for $c > 1$ $\left\{ f(n) = \Omega(n^{1+\varepsilon}) \right.$

AND $2f\left(\frac{n}{2}\right) = \frac{2}{2^c} n^c = \frac{1}{2^{c-1}} \cdot f(n)$

... case 3 $\Rightarrow$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = n \Rightarrow$ case 2 $\Rightarrow \Theta(n \cdot \log n)$

$(k=0)$

if $f(n) = n \log^k n \Rightarrow$ case 2 $\Rightarrow \Theta(n \log^{k+1} n)$

if $f(n) = \log^c n \Rightarrow$ case 1 $\Rightarrow \Theta(n)$

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if $f(n) = n^c$ for $c > 1$ $\Big\}$ $f(n) = \Omega(n^{1+\varepsilon})$

AND $2f\left(\frac{n}{2}\right) = \frac{2}{2^c} n^c = \frac{1}{2^{c-1}} \cdot f(n)$

... case 3 $\Rightarrow \Theta(f(n))$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = n \Rightarrow$ case 2 $\Rightarrow \Theta(n \cdot \log n)$

$(k=0)$

if $f(n) = n \log^k n \Rightarrow$ case 2 $\Rightarrow \Theta(n \log^{k+1} n)$

if $f(n) = \log^c n \Rightarrow$ case 1 $\Rightarrow \Theta(n)$

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if $f(n) = n^c$ for $c > 1$ $\Big\}$ $f(n) = \Omega(n^{1+\varepsilon})$

AND $2f\left(\frac{n}{2}\right) = \frac{2}{2^c} n^c = \frac{1}{2^{c-1}} \cdot f(n)$

... case 3 $\Rightarrow \Theta(f(n))$

$$T(n) = 4T\left(\frac{n}{4}\right) + f(n)$$

?

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

if $f(n) = n \Rightarrow$ case 2 $\Rightarrow \Theta(n \cdot \log n)$

$(k=0)$

$$n^{\log_b a} = n$$

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if $f(n) = \log^c n \Rightarrow$ case 1 $\Rightarrow \Theta(n)$

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if $f(n) = n^c$ for $c > 1$ $\Big\}$ $f(n) = \Omega(n^{1+\varepsilon})$

AND $2f\left(\frac{n}{2}\right) = \frac{2}{2^c} n^c = \frac{1}{2^{c-1}} \cdot f(n)$

$$T(n) = 4T\left(\frac{n}{4}\right) + f(n)$$

SAME

... case 3 $\Rightarrow \Theta(f(n))$

COMPUTING THE (integer n) POWER OF A NUMBER x : x^n

$$\underbrace{x \cdot x \cdot x \cdot \cdots x \cdot x}_{n \text{ times}} \Big\} \ \Theta(n) \text{ time, assuming any multiplication costs } \Theta(1)$$

COMPUTING THE (integer n) POWER OF A NUMBER x : x^n

$\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n \text{ times}}$ } $\Theta(n)$ time, assuming any multiplication costs $\Theta(1)$

-OR-

for even n : $x^n = \underbrace{x^{n/2}} \cdot \underbrace{x^{n/2}}$: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(1)$

$\Theta(n)$

COMPUTING THE (integer n) POWER OF A NUMBER x : x^n

$\underbrace{x \cdot x \cdot x \cdot \dots \cdot x \cdot x}_{n \text{ times}}$ } $\Theta(n)$ time, assuming any multiplication costs $\Theta(1)$

-OR-

for even n : $x^n = \underbrace{x^{n/2}} \cdot x^{n/2}$:

$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(1)$

$T(n) = T\left(\frac{n}{2}\right) + \Theta(1)$

$O(n) \longrightarrow O(\log n)$

COMPUTING THE (integer n) POWER OF A NUMBER x : x^n

$\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n \text{ times}}$ } $\Theta(n)$ time , assuming any multiplication costs $\Theta(1)$

-OR-

(1) for even n : $x^n = x^{n/2} \cdot x^{n/2}$:

$T(n) = T(\frac{n}{2}) + \Theta(1)$

(2) for odd n : $x^{\frac{n-1}{2}} \cdot x^{\frac{n-1}{2}} \cdot x$ $T(n) = T(\frac{n-1}{2}) + \Theta(1)$

$\boxed{\Theta(\log n)}$

Fibonacci Number Computation

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2)$$

Fibonacci Number Computation

$F(0) = 0$

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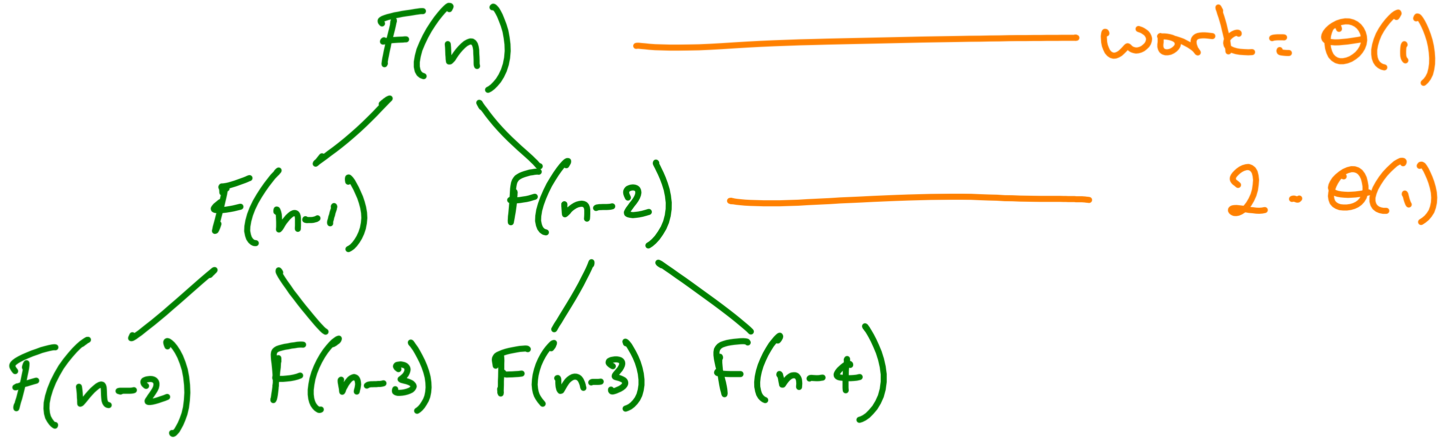
$$\begin{tikzcd}
 & F(n) \arrow[dl] \arrow[dr] \arrow[rr, no head] & & \text{work} = \Theta(1) \\
F(n-1) & & F(n-2) &
\end{tikzcd}$$

FIBONACCI NUMBER COMPUTATION

$F(0) = 0$

$F(1) = 1$

$F(n) = F(n-1) + F(n-2)$



Fibonacci Number Computation

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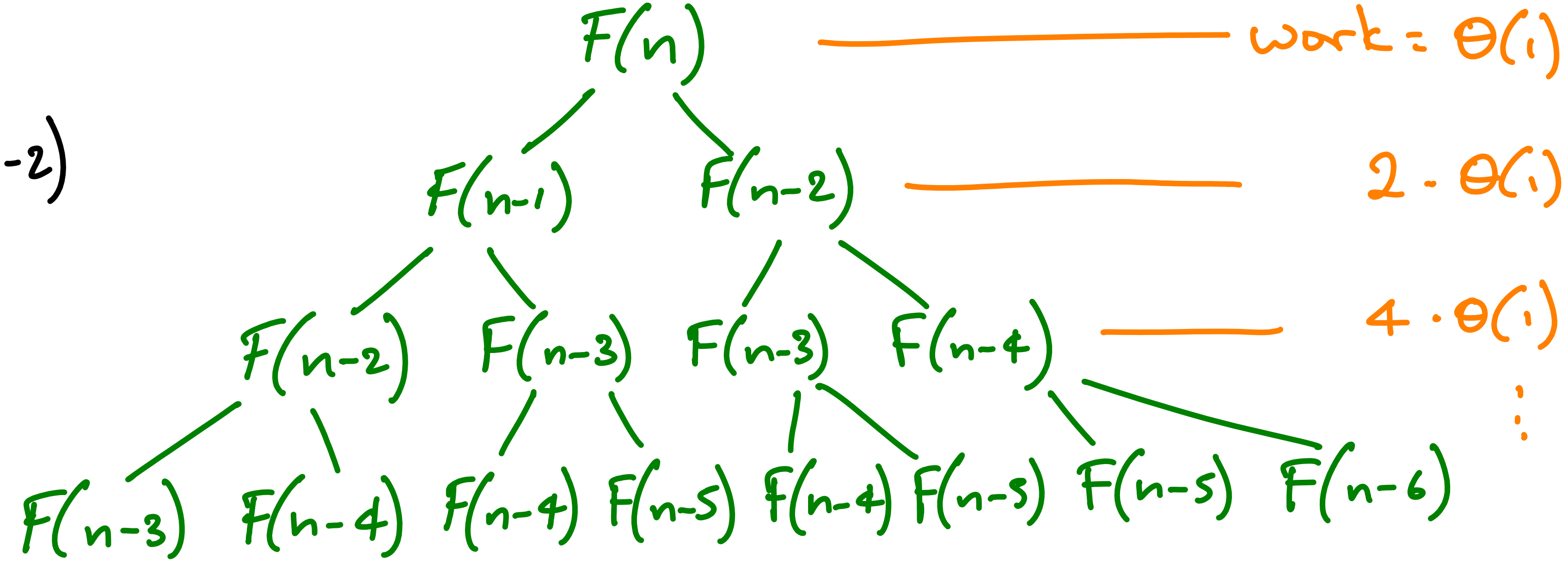

FIBONACCI NUMBER COMPUTATION

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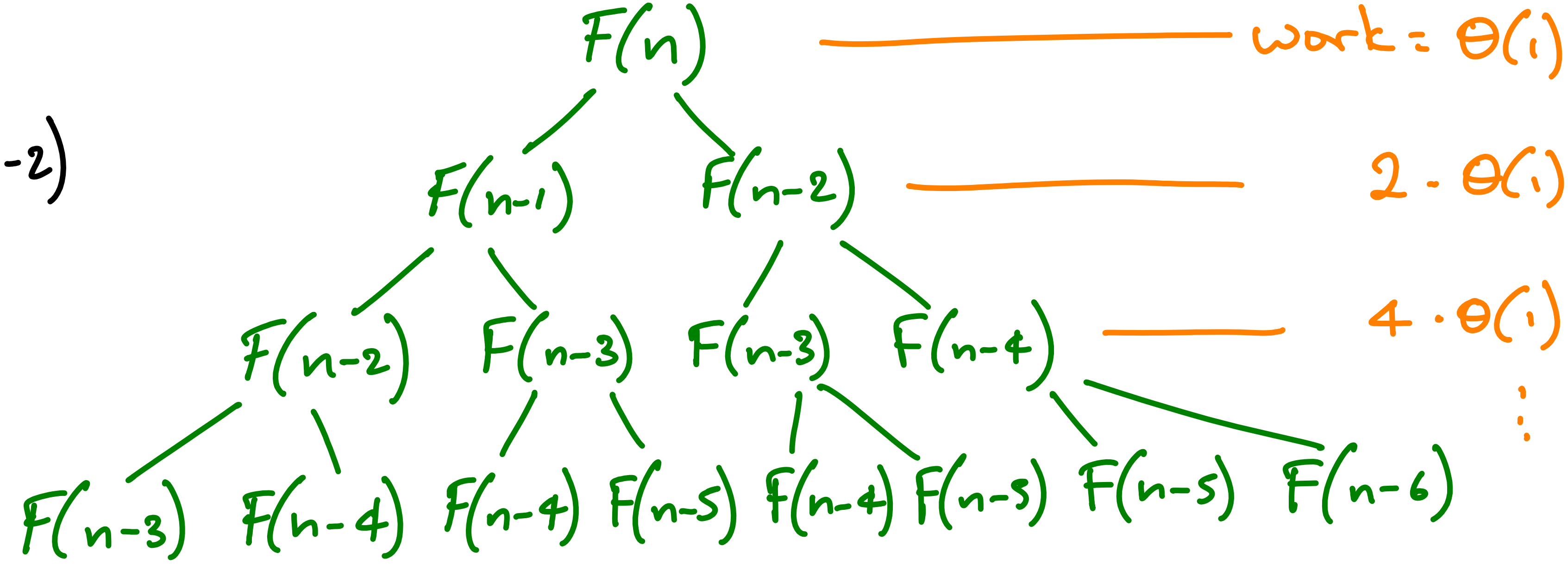

total work = # nodes

Fibonacci Number Computation

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$F(1) = 1$

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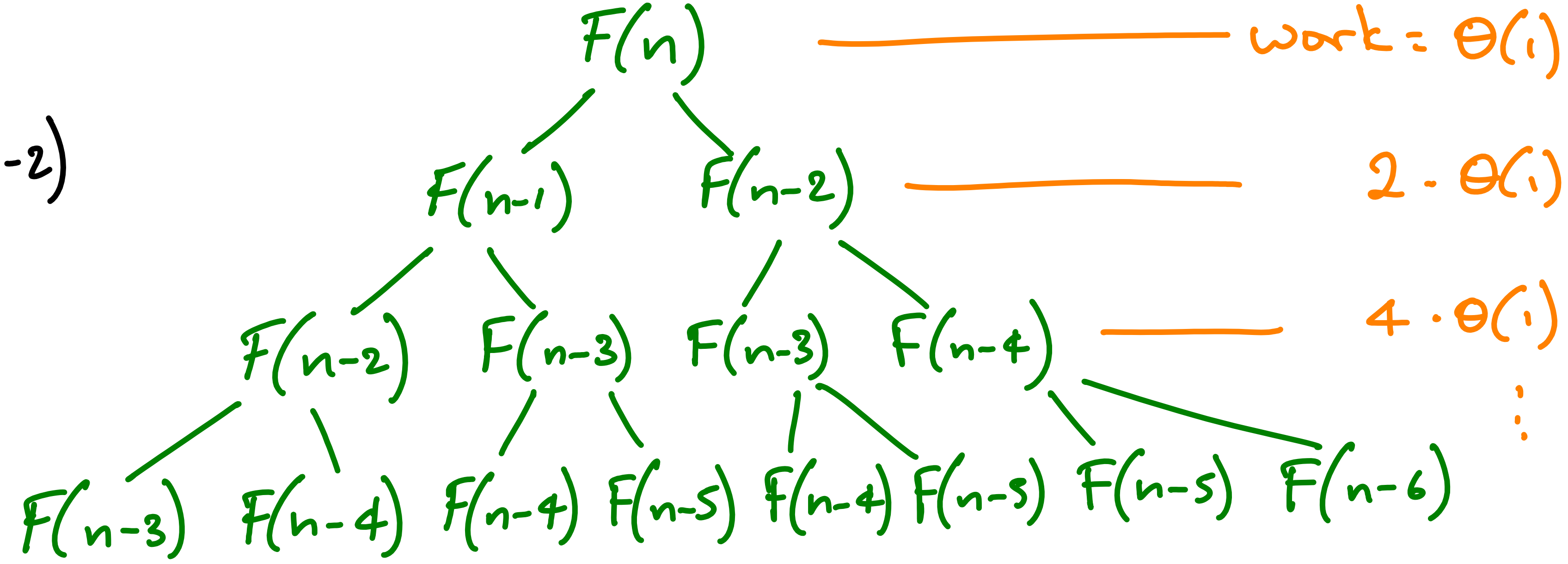

total work = # nodes

Which root→leaf path is shortest?

FIBONACCI NUMBER COMPUTATION

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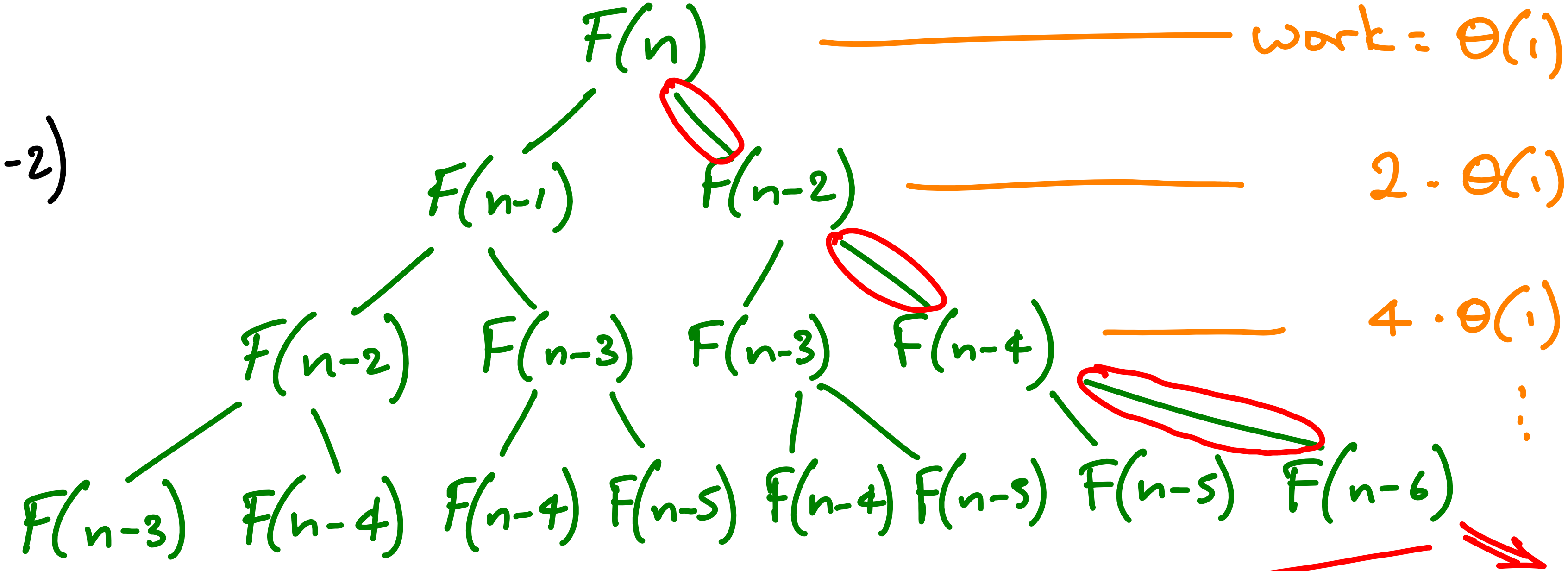
$F(1) = 1$

$F(n) = F(n-1) + F(n-2)$



total work = # nodes

Which root→leaf path is shortest?

RIGHTMOST

FIBONACCI NUMBER COMPUTATION

$F(0) = 0$

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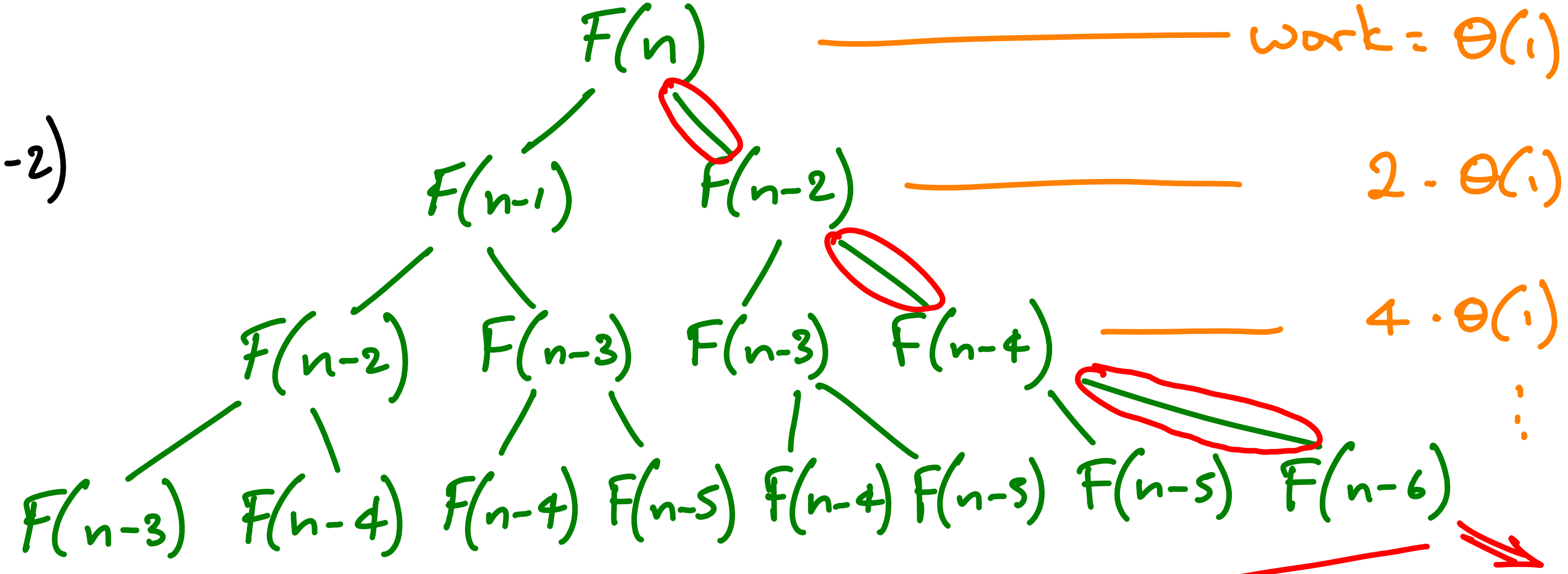
$F(n) = F(n-1) + F(n-2)$



total work = # nodes

Which root→leaf path is shortest?

RIGHTMOST

depth $\sim \frac{n}{2}$

so we have $\frac{n}{2}$ "full" levels: WORK $> 2^{n/2} = \Omega(\sqrt{2}^{\,n})$

Instead, just compute each $F(k)$ iteratively. (once)

$0, 1, F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5, F(6) = 8$ etc

$\Theta(n)$ time

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Fact: $F(n) = \frac{\Phi^n}{\sqrt{5}}$, rounded to integer

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Fact: $F(n) = \frac{\Phi^n}{\sqrt{5}}$, rounded to integer

Assume: $\left\{ \begin{array}{l} \text{1) you can compute \& store } \sqrt{5} \\ \text{2) you can round numbers quickly} \\ \text{3) you can multiply } \Phi \text{ powers quickly} \end{array} \right\}$ i.e quick arithmetic on "difficult" numbers

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↳ then by our x^n method we get $\Theta(\log n)$ time for $F(n)$

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$0, 1, F(2)=1, F(3)=2, F(4)=3, F(5)=5, F(6)=8$ etc

$\Theta(n)$ time

Fact: $F(n) = \dfrac{\Phi^n}{\sqrt{5}}$, rounded to integer

Assume:
1) you can compute & store $\sqrt{5}$
2) you can round numbers quickly
3) you can multiply Φ powers quickly

i.e quick arithmetic on "difficult" numbers

so this depends on the MODEL OF COMPUTATION

↳ then by our x^n method we get $\Theta(\log n)$ time for $F(n)$

Fibonacci Number Computation

Finale

FIBONACCI NUMBER COMPUTATION

FINALE

$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

FIBONACCI NUMBER COMPUTATION

FINALE

matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ $\{$ true for $n=1\}$ $\begin{matrix} F_2 & F_1 \\ F_1 & F_0 \end{matrix} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

FIBONACCI NUMBER COMPUTATION

FINALE

matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ $\left\}$ true for $n=1 \right\}$ $\begin{matrix} F_2 & F_1 \\ F_1 & F_0 \end{matrix} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

General proof

Induction

$$\underbrace{\begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}}$$

assume holds

i.e $= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}$

FIBONACCI NUMBER COMPUTATION

FINALE

matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ $\}$ true for $n=1$ $\}$ $\begin{matrix} F_2 & F_1 \\ F_1 & F_0 \end{matrix} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

General proof Induction

$$\underbrace{\begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}}_{\text{assume holds}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_n \\ F_{n-1} + F_{n-2} & F_{n-1} \end{bmatrix}$$

i.e $= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}$

FIBONACCI NUMBER COMPUTATION

FINALE

matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ } true for $n=1$ } $\begin{matrix} F_2 & F_1 \\ F_1 & F_0 \end{matrix} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

General proof Induction

$$\underbrace{\begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}}_{\text{assume holds i.e } = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_n \\ F_{n-1} + F_{n-2} & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

Q.E.D.

FIBONACCI NUMBER COMPUTATION

FINALE

matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ } true for $n=1$ } $\begin{matrix} F_2 & F_1 \\ F_1 & F_0 \end{matrix} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

General proof — Induction

$$\underbrace{\begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}}_{\text{assume holds}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_n \\ F_{n-1} + F_{n-2} & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \quad \text{Q.E.D.}$$

i.e $= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n$ is like x^n : $T(n) \simeq T\left(\frac{n}{2}\right) + \Theta(1)$

$\hookrightarrow F_n$ in $\Theta(\log n)$ time

$n \times n$ SQUARE MATRIX MULTIPLICATION

$C = A \times B$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

$n \times n$ SQUARE MATRIX MULTIPLICATION

$C = A \times B \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

$n \times n$ SQUARE MATRIX MULTIPLICATION

$C = A \times B$ $\quad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ $\Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

Divide & Conquer:
$$\underset{C}{\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right]} = \underset{A}{\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]} \times \underset{B}{\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]}$$

$n \times n$ SQUARE MATRIX MULTIPLICATION

$C = A \times B$ $\quad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ $\Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

Divide & Conquer:

$$\underset{C}{\begin{bmatrix} r & s \\ t & u \end{bmatrix}} = \underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} \quad \begin{cases} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{cases}$$

$n \times n$ SQUARE MATRIX MULTIPLICATION

$C = A \times B$ $\quad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

Divide & Conquer:

$$\underset{C}{\begin{bmatrix} r & s \\ t & u \end{bmatrix}} = \underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} \quad \left\{ \begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned} \right\} \quad T(n) = ?$$

nxn # SQUARE MATRIX MULTIPLICATION

$C = A \times B$ $\qquad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

Divide & Conquer:

$$\underset{C}{\begin{bmatrix} r & s \\ t & u \end{bmatrix}} = \underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} \quad \left\{ \begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned} \right\}$$

$$T(n) = 8T\left(\frac{n}{2}\right) + \underbrace{\Theta(n^2)}_{\text{4x matrix addition}}$$

$n \times n$ SQUARE MATRIX MULTIPLICATION

$C = A \times B$ $\qquad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

Divide & Conquer:

$$\underbrace{\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right]}_{C} = \underbrace{\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]}_{A} \times \underbrace{\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]}_{B} \quad \left\{\begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array}\right\} \quad T(n) = 8T\left(\tfrac{n}{2}\right) + \underbrace{\Theta(n^2)}_{\text{4x matrix addition}}$$

Master theorem: $\quad f(n) = \Theta(n^2) \qquad n^{\log_b a} = n^3$

so $f(n) = O(n^{3-\varepsilon})$ for $0 < \varepsilon \leqslant 1$

CASE 1 $\rightarrow T(n) = \Theta(n^3)$:(

nxn **SQUARE MATRIX MULTIPLICATION**

$C = A \times B \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \Rightarrow \Theta(n)$ per element $\Rightarrow \Theta(n^3)$ total

Divide & Conquer:

$$\underbrace{\left[\begin{array}{c:c} r & s \\ \hdashline t & u \end{array}\right]}_{C} = \underbrace{\left[\begin{array}{c:c} a & b \\ \hdashline c & d \end{array}\right]}_{A} \times \underbrace{\left[\begin{array}{c:c} e & f \\ \hdashline g & h \end{array}\right]}_{B} \quad \left\{\begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array}\right\} \quad T(n) = 8T\left(\frac{n}{2}\right) + \underbrace{\Theta(n^2)}_{\text{4x matrix addition}}$$

Master theorem: $f(n) = \Theta(n^2) \qquad n^{\log_b a} = n^3$

so $f(n) = O(n^{3-\varepsilon})$ for $0 < \varepsilon \leq 1$

CASE1 $\rightarrow T(n) = \Theta(n^3)$:(

Trying smaller blocks won't help

Strassen's Algorithm

$$\left[\begin{array}{c:c} r & s \\ \hdashline t & u \end{array}\right] = \left[\begin{array}{c:c} a & b \\ \hdashline c & d \end{array}\right] \times \left[\begin{array}{c:c} e & f \\ \hdashline g & h \end{array}\right] \left\{\begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array}\right.$$

Strassen's Algorithm

$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \times \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right] \left\{\begin{array}{l} r = ae + bg = p_4 + p_5 + p_6 - p_2 \\ s = af + bh = p_1 + p_2 \\ t = ce + dg = p_3 + p_4 \\ u = cf + dh = p_1 + p_5 - p_3 - p_7 \end{array}\right.$$

note: matrix multip. is not commutative

$p_1 = a(f-h)$

$p_2 = (a+b) \cdot h$

$p_3 = (c+d) \cdot e$

$p_4 = d \cdot (g-e)$

$p_5 = (a+d)(e+h)$

$p_6 = (b-d)(g+h)$

$p_7 = (a-c)(e+f)$

Strassen's Algorithm

$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \times \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]$$

$$\left\{\begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array}\right. \begin{array}{l} = p_4 + p_5 + p_6 - p_2 \\ = p_1 + p_2 \\ = p_3 + p_4 \\ = p_1 + p_5 - p_3 - p_7 \end{array}$$

8 additions

note: matrix multip. is not commutative

$$\underbrace{\begin{array}{l} p_1 = a(f-h) \\ p_2 = (a+b)\cdot h \\ p_3 = (c+d)\cdot e \\ p_4 = d\cdot(g-e) \\ p_5 = (a+d)(e+h) \\ p_6 = (b-d)(g+h) \\ p_7 = (a-c)(e+f) \end{array}}$$

7 multiplications

& 10 additions

$\Theta(n^2)$ time

additions to compute p_i

& to add p_i to get r, s, t, u

Strassen's Algorithm

$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \times \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right] \quad \begin{cases} r = ae + bg = p_4 + p_5 + p_6 - p_2 \\ s = af + bh = p_1 + p_2 \\ t = ce + dg = p_3 + p_4 \\ u = cf + dh = p_1 + p_5 - p_3 - p_7 \end{cases}$$

note: matrix multip. is not commutative

$$\underbrace{\begin{aligned} p_1 &= a(f-h) \\ p_2 &= (a+b)\cdot h \\ p_3 &= (c+d)\cdot e \\ p_4 &= d\cdot(g-e) \\ p_5 &= (a+d)(e+h) \\ p_6 &= (b-d)(g+h) \\ p_7 &= (a-c)(e+f) \end{aligned}}_{\text{7 multiplications \& O(1) additions}}$$

Divide & Conquer

$$T(n) = \underbrace{7T\left(\tfrac{n}{2}\right)}_{\text{multiply to get } p_i} + \underbrace{\Theta(n^2)}_{\text{additions to compute } p_i \text{ \& to add } p_i \text{ to get } r,s,t,u}$$

STRASSEN'S ALGORITHM

$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \times \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]$$

$$\begin{cases} r = ae + bg & = P_4 + P_5 + P_6 - P_2 \\ s = af + bh & = P_1 + P_2 \\ t = ce + dg & = P_3 + P_4 \\ u = cf + dh & = P_1 + P_5 - P_3 - P_7 \end{cases}$$

note: matrix multip. is not commutative

$$\underbrace{\begin{aligned} p_1 &= a(f-h) \\ p_2 &= (a+b)\cdot h \\ p_3 &= (c+d)\cdot e \\ p_4 &= d\cdot(g-e) \\ p_5 &= (a+d)(e+h) \\ p_6 &= (b-d)(g+h) \\ p_7 &= (a-c)(e+f) \end{aligned}}_{\text{7 multiplications \& } O(1) \text{ additions}}$$

DIVIDE & CONQUER

$$T(n) = \underbrace{7T\left(\frac{n}{2}\right)}_{\text{multiply to get } p_i} + \underbrace{\Theta(n^2)}_{\text{additions to compute } p_i \text{ \& to add } p_i \text{ to get } r,s,t,u}$$

still CASE 1

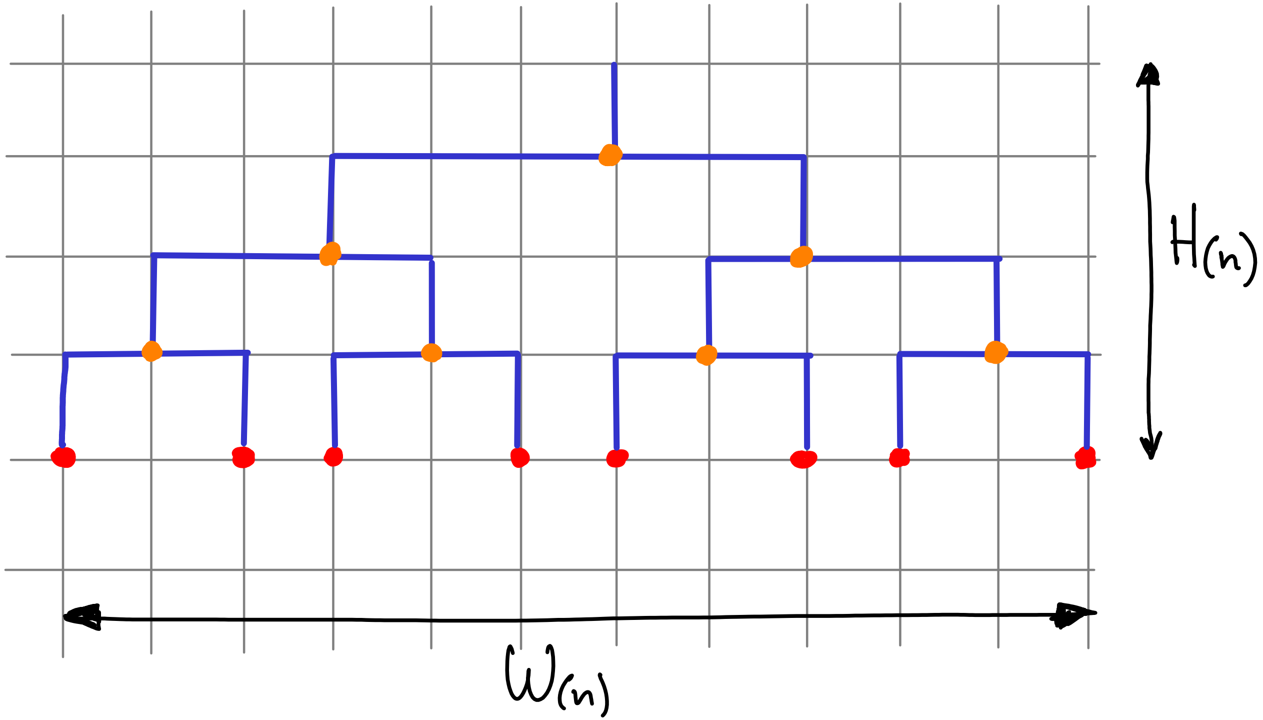
$$T(n) = \Theta(n^{\log 7}) = O(n^{2.81})$$

- worthwhile for $n > 30$
- there is a $n^{2.376}$ algo
  ↳ practically never used.

VLSI layout [very large scale integrated chips]

e.g.: embed a complete binary tree w/ n leaves on a grid, w/o crossings & MINIMIZE AREA

(edges can be any length, but on grid)

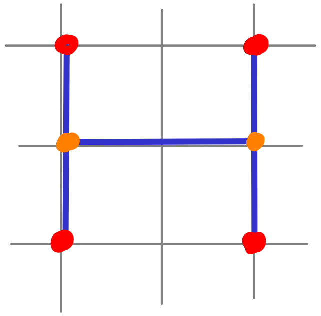


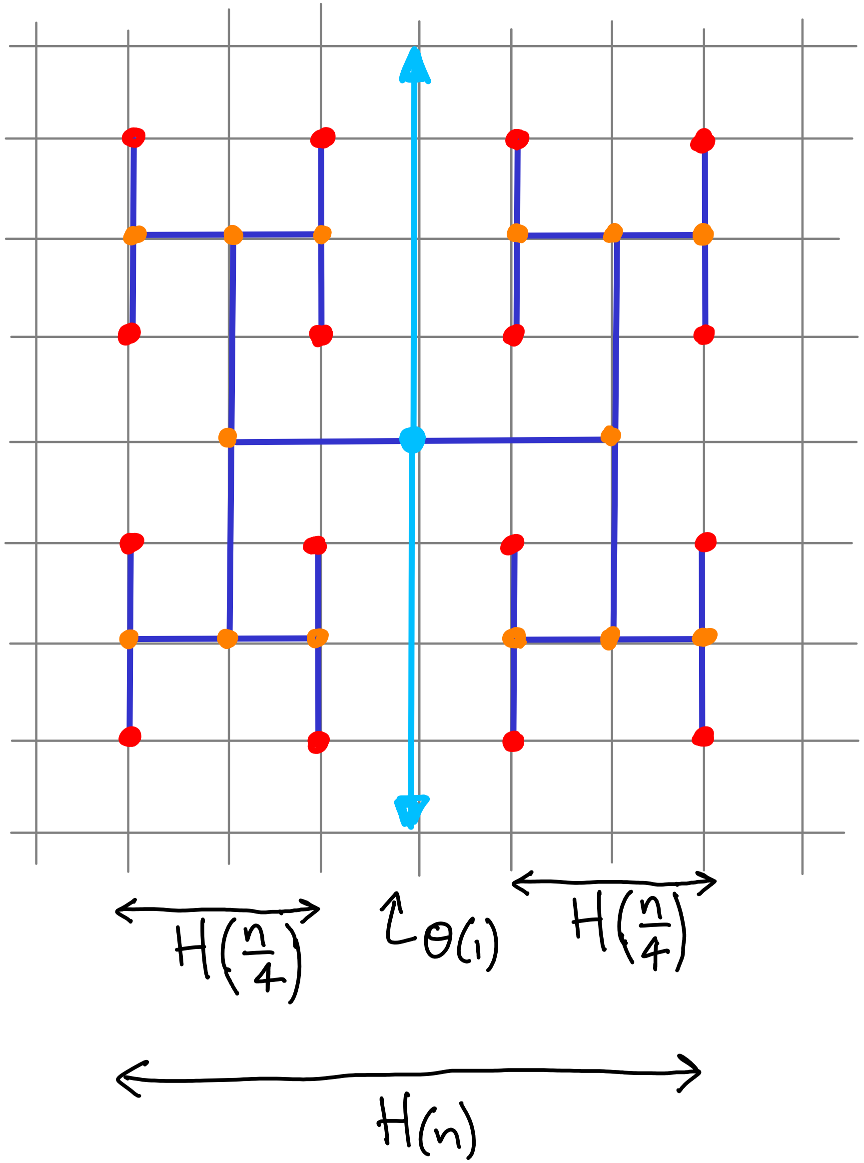
$$H(n) = H\left(\tfrac{n}{2}\right) + \Theta(1) = \Theta(\log n)$$

$$\times$$

$$W(n) = 2W\left(\tfrac{n}{2}\right) + O(1) = \Theta(n)$$

$\Theta(n \log n)$ area





$H(n) = W(n) = 2H(\frac{n}{4}) + \Theta(1)$

Master: $H(n) = a \cdot H(\frac{n}{b}) + f(n)$

$a = 2 \qquad b = 4$

$n^{\log_b a} = n^{1/2}$

$f(n) = O(n^{1/2 - \varepsilon})$ case 1

$H(n) = \Theta(\sqrt{n})$

$\text{Area} = \Theta(n)$