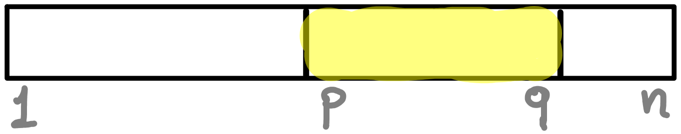
RANDSELECT (randomized Selection)

Given n unsorted elements in an array (or linked list)

find the one with rank r → r-th smallest.

- For simplicity, assume no duplicates → Easy to handle.
- If necessary shuffle data to make random order.

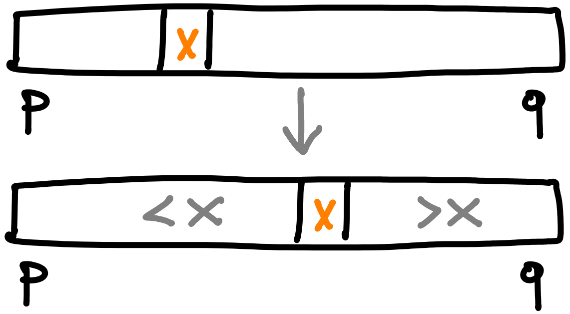
Recursive function: RandSelect(k, p, q)



returns k-th smallest in subarray from index p to index q.

We start with RandSelect$(r, 1, n)$

// Find k-th smallest within $[p,q]$

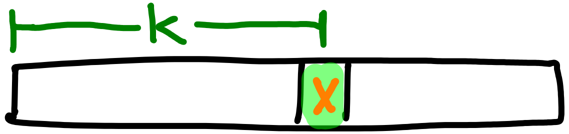


RandSelect(k, p, q)

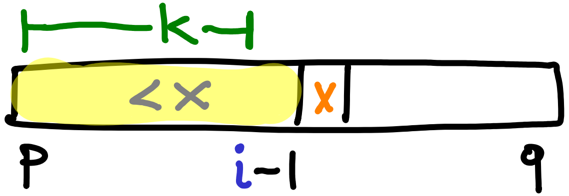
1) Use a random pivot x to partition $[p,q]$

2) Calculate rank of x within $[p,q]$ = 1 + #elements smaller than X

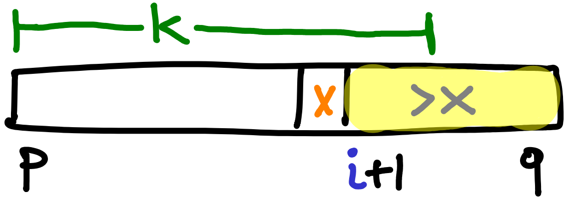
3) if $\text{rank}(x) = k$, return x



if $k < \text{rank}(x)$, RandSelect$(k, p, i-1)$



if $\text{rank}(x) < k$, RandSelect$(k - \text{rank}(x), i+1, q)$



Example: Find 7th smallest

$k = r = 7, \quad p = 1, \quad q = n = 12$

11	10	8	13	9	3	2	6	5	1	12	15
1	2	3	4	5	6	7	8	9	10	11	12

RandSelect(7, 1, 12) →

5	10	8	1	9	3	2	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: pivot = x = 11

$7 = k < \text{Rank}(11) = 9$

RandSelect(7, 1, 8) →

1	2	3	5	9	8	10	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: $x = 5$

$4 = \text{Rank}(5) < k = 7$

RandSelect(3, 5, 8) →

1	2	3	5	6	8	9	10	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: $x = 9$

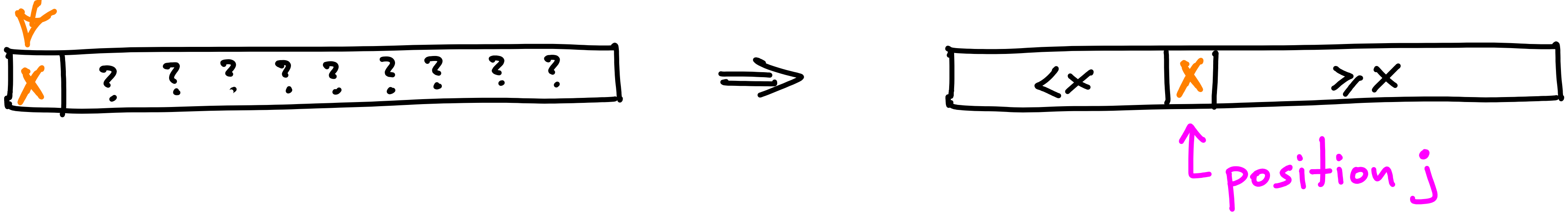
return 9

$\text{Rank}(9) = 3$

RandSelect recap

- If necessary shuffle data to make random order.

- choose a pivot & partition. → $\Theta(n)$



- in the worst case, RandSelect the larger side.

$$T(n) = \Theta(n) + \max\{T(j-1), T(n-j)\}$$

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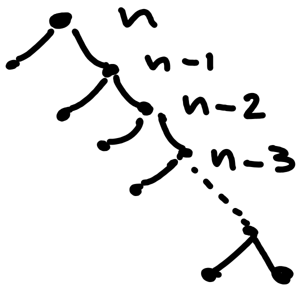
What is the worst-case time complexity, and why?

↳ already sorted input, reverse-sorted, nearly sorted...

$$T(n) = T(n-1) + \Theta(n) \qquad = \Theta(n^2)$$



What would be ideal? (assuming we must actually recurse)

↳ ~ balanced partition, every time

$$T(n) = T\left(\frac{n}{2}\right) + \Theta(n) \qquad = \Theta(n)$$

What if we always have a "sort-of-balanced" partition?

e.g., $T(n) = T\left(\frac{9n}{10}\right) + \Theta(n) \quad = \Theta(n)$

Expected time: call a split balanced if pivot ranks in $\left[\frac{n}{4} \ldots \frac{3n}{4}\right]$

unbalanced otherwise

Worst case if balanced split: $T(n) \leq T\left(\frac{3n}{4}\right) + dn$

Worst case if unbalanced split: $T(n) \leq T(n-1) + dn < T(n) + dn$

Each split has a 50% chance of being balanced

$$T(n) \leq 0.5\left(T(n)+dn\right) + 0.5 \cdot \left(T\left(\frac{3n}{4}\right) + dn\right)$$

$$0.5\,T(n) \leq dn + 0.5 \cdot T\left(\frac{3n}{4}\right)$$

$$T(n) \leq T\left(\frac{3n}{4}\right) + 2dn = \Theta(n)$$

$$2dn \cdot \frac{1}{1-3/4} = 8dn$$

Define $X_k \begin{cases} 1 & \text{if RandPartition gives } k \text{ vs } n-k-1 \text{ split.} \\ 0 & \text{otherwise} \end{cases}$

always assuming worst scenario

$$T(n) \leq \Theta(n) + \text{one of}: \begin{cases} T(\max\{0, n-1\}) & k=0 \\ T(\max\{1, n-2\}) & \\ T(\max\{2, n-3\}) & \\ \vdots & \\ T(\max\{n-1, 0\}) & k=n-1 \end{cases} \Bigg\} \; n \text{ possible outcomes}$$

Expected time complexity of RandSelect

More refined calculation

Define $X_k \begin{cases} 1 & \text{if RandPartition gives } k \text{ vs } n-k-1 \text{ split.} \\ 0 & \text{otherwise} \end{cases}$

always assuming worst scenario

$$T(n) \le \underbrace{\Theta(n) + \text{one of : } \begin{cases} T(\max\{0, n-1\}) \\ T(\max\{1, n-2\}) \\ T(\max\{2, n-3\}) \\ \vdots \\ T(\max\{n-1, 0\}) \end{cases}}_{\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})}$$

$$E[T(n)] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})\right]$$

$$= E[\Theta(n)] + \sum_{k=0}^{n-1} E[X_k \cdot T(\max\{k, n-k-1\})]$$ by linearity of expectation

$$= \Theta(n) + \sum E[X_k] \cdot E[T(\max\{k, n-k-1\})]$$ by independence

$$= \Theta(n) + \sum_{k=0}^{n-1} \cdot \frac{1}{n} \cdot E[T(\max\{k, n-k-1\})]$$ by random choice

$$\leq \Theta(n) + \frac{2}{n} \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

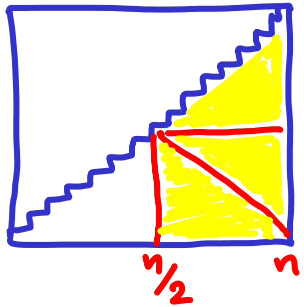
$$E[T(n)] \le \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \le c\cdot n$

↳ assume true for $k<n$

$$\le \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} ck = \Theta(n) + \frac{2c}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k$$

$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k \le \frac{3}{8}n^2$$



$$\le \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \underline{\Theta(n)} + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$= cn - \underbrace{\left(\frac{c}{4} - d\right)}_{\text{positive if } c>4d}n$$

$$E[T(n)] \le 4dn$$