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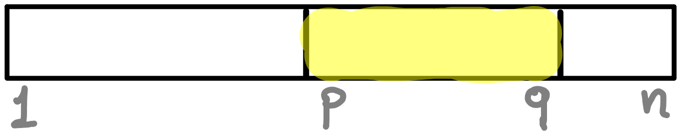
Recursive function: RandSelect(k, p, q)

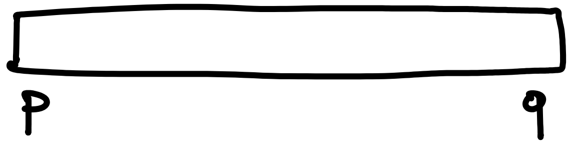


returns k-th smallest in subarray from index p to index q.

We start with RandSelect$(r, 1, n)$

// Find k-th smallest within [p,q]

RandSelect(k, p, q)



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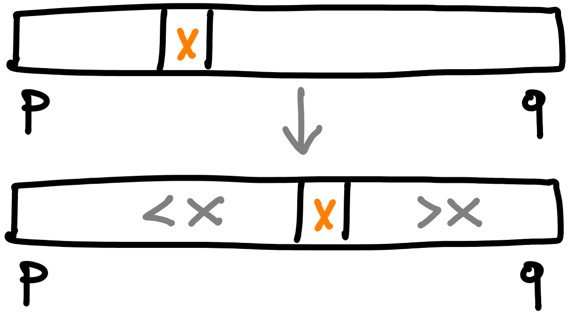

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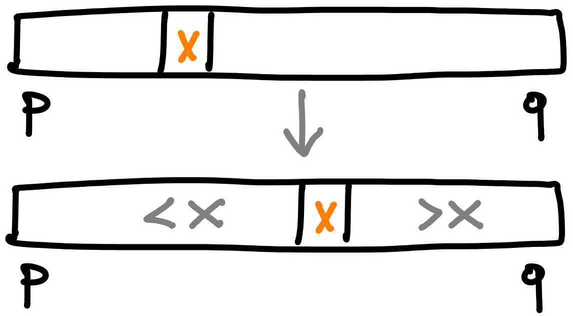

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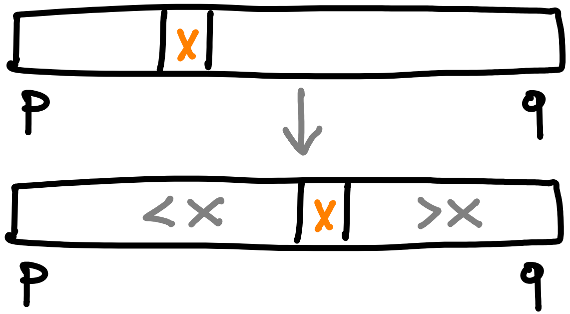
2) Calculate rank of x within [p,q]

// Find k-th smallest within [p,q]



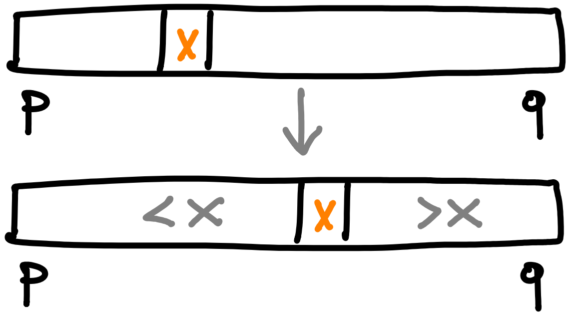
RandSelect(k, p, q)

1) Use a random pivot x to partition [p,q]

2) Calculate rank of x within [p,q] : 1 + #elements smaller than x

// Find k-th smallest within [p,q]

RandSelect(k, p, q)



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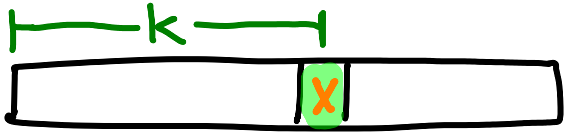
3) if rank(x) = k, return x



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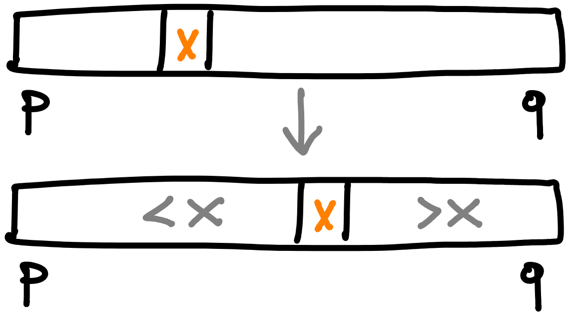

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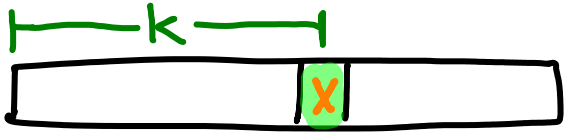
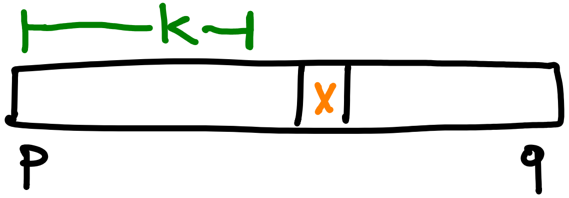


if k < rank(x) ... ?



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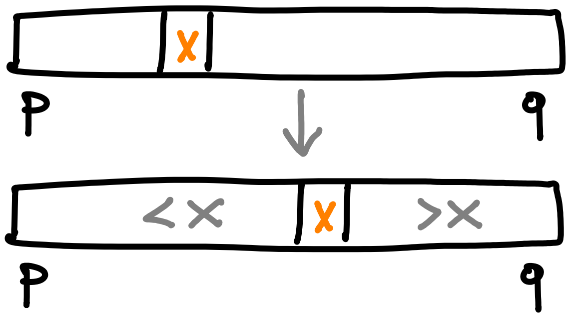

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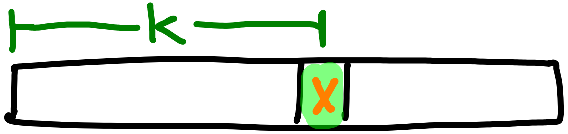
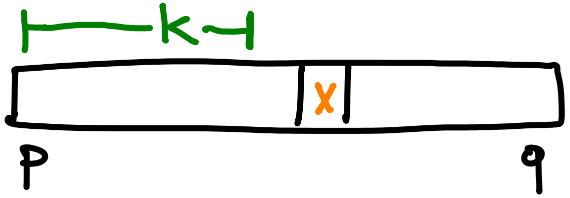


if k < rank(x), RandSelect(?, ?, ?)



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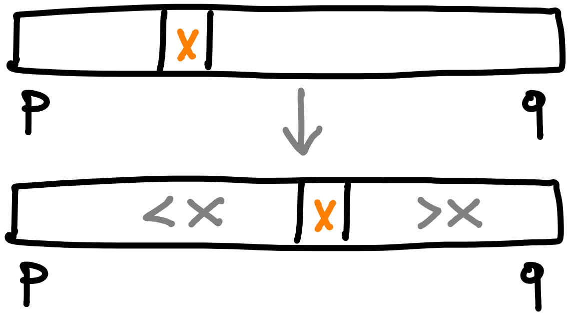

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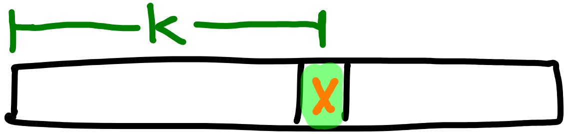

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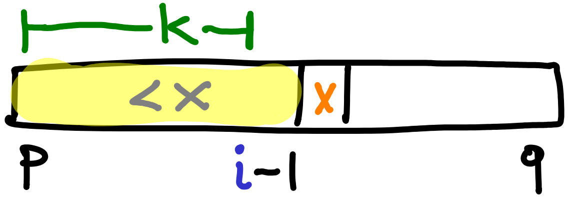

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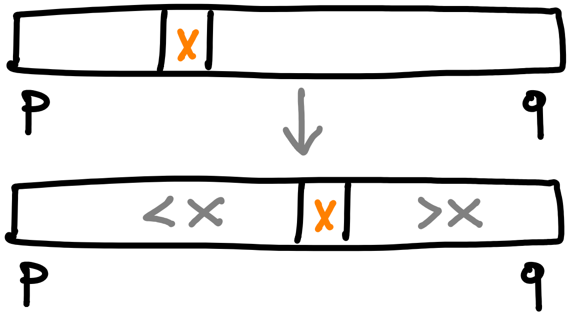

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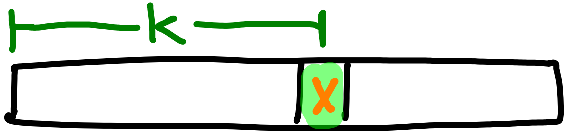
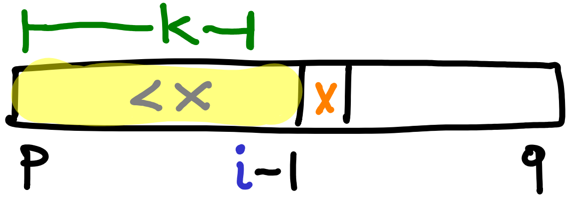


if k < rank(x), RandSelect(k, p, i−1)



// Find k-th smallest within [p,q]

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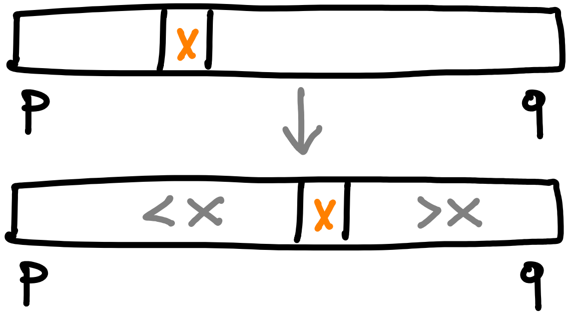

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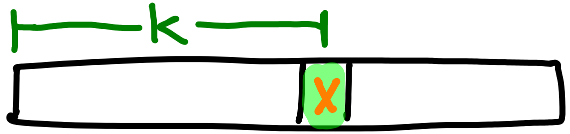

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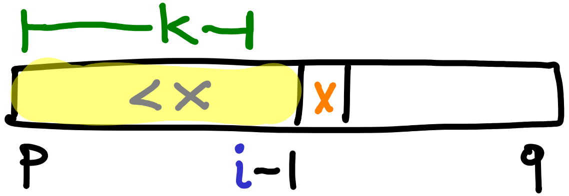
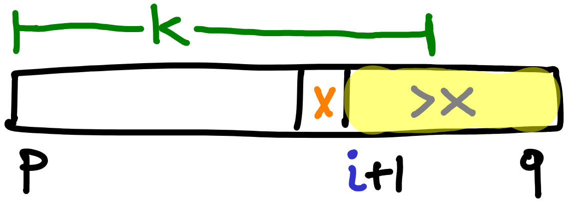


if $\text{rank}(x) < k$, RandSelect(?, ?, ?)



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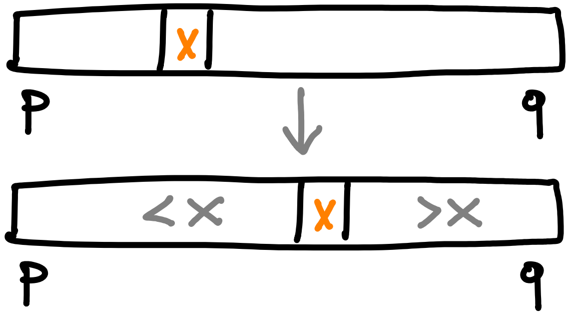

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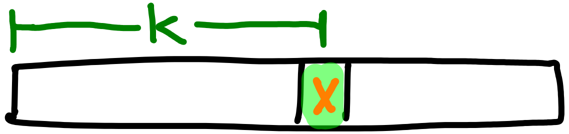

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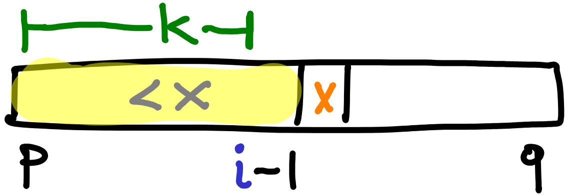
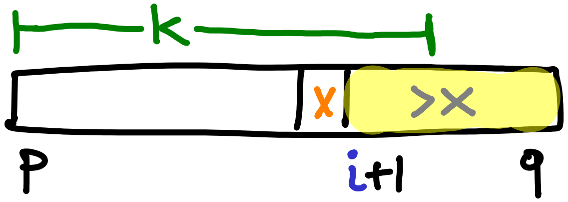


if rank(x) < k, RandSelect(k − rank(x), i+1, q)



Example: Find 7th smallest

11	10	8	13	9	3	2	6	5	1	12	15
1	2	3	4	5	6	7	8	9	10	11	12

Example: Find 7th smallest

$k = r = 7, \quad p = 1, \quad q = n = 12$

1	2	3	4	5	6	7	8	9	10	11	12
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RandSelect(7, 1, 12)

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↳ Partition: pivot $= x = 11$

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Rank(11) = 9

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↳ Partition: pivot $= x = 11$

$7 = k < \text{Rank}(11) = 9$

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RandSelect(7, 1, 8)

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RandSelect$(7, 1, 12)$ →

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↳ Partition: $x = 5$

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$7 = k < \text{Rank}(11) = 9$

RandSelect(7, 1, 8) →

↳ Partition: $x = 5$

1	2	3	5	9	8	10	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

$4 = \text{Rank}(5) < k = 7$

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$k = r = 7, \quad p = 1, \quad q = n = 12$

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RandSelect(7, 1, 8) →

1	2	3	5	9	8	10	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: x = 5

$4 = \text{Rank}(5) < k = 7$

RandSelect(3, 5, 8)

$k - \text{Rank}(5) = 3$

Example: Find 7th smallest

$k = r = 7, \quad p = 1, \quad q = n = 12$

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RandSelect(7, 1, 12) →

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↳ Partition: pivot $= x = 11$

$7 = k < \text{Rank}(11) = 9$

RandSelect(7, 1, 8) →

1	2	3	5	9	8	10	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: $x = 5$

$4 = \text{Rank}(5) < k = 7$

RandSelect(3, 5, 8)

↳ Partition: $x = 9$

Example: Find 7th smallest

$k = r = 7, \quad p = 1, \quad q = n = 12$

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RandSelect(7, 1, 12) →

5	10	8	1	9	3	2	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: pivot = x = 11

$7 = k < \text{Rank}(11) = 9$

RandSelect(7, 1, 8) →

1	2	3	5	9	8	10	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: $x = 5$

$4 = \text{Rank}(5) < k = 7$

RandSelect(3, 5, 8) →

1	2	3	5	9	8	10	6	11	13	12	15
1	2	3	4	5	6	7	8	9	10	11	12

↳ Partition: $x = 9$

Example: Find 7th smallest

$k = r = 7, \quad p = 1, \quad q = n = 12$

11	10	8	13	9	3	2	6	5	1	12	15
1	2	3	4	5	6	7	8	9	10	11	12

RandSelect(7, 1, 12) →

5	10	8	1	9	3	2	6	11	13	12	15
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↳ Partition: $x = 5$

$4 = \text{Rank}(5) < k = 7$

RandSelect(3, 5, 8) →

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RandSelect(3, 5, 8) →

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1	2	3	5	6	8	9	10	11	13	12	15
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$\text{Rank}(9) = 3$

Example: Find 7th smallest

$k = r = 7, \quad p = 1, \quad q = n = 12$

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↳ Partition: $x = 5$

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return 9

1	2	3	5	6	8	9	10	11	13	12	15
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RandSelect recap

RANDSELECT recap

- If necessary shuffle data to make random order.

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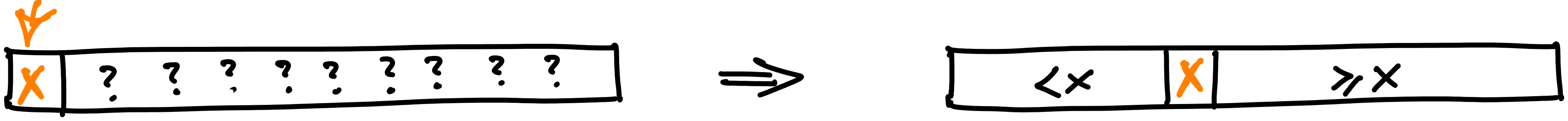

RandSelect recap

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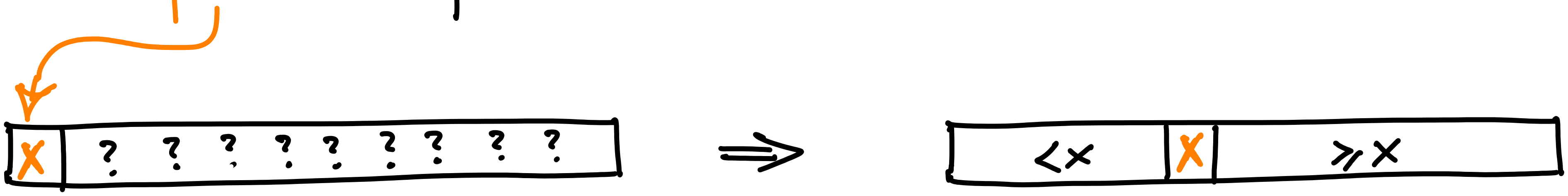

- in the worst case, RandSelect the larger side.

RANDSELECT recap

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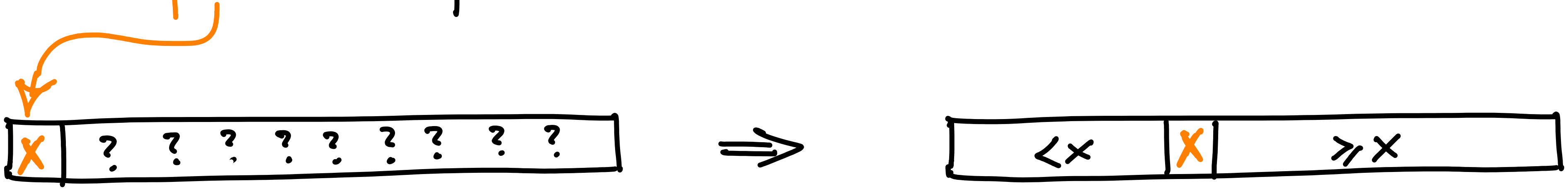

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$T(n) = ?$

RandSelect recap

- If necessary shuffle data to make random order.

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$$T(n) = \underline{\Theta(n)} + \ ?$$

RandSelect recap

- If necessary shuffle data to make random order.
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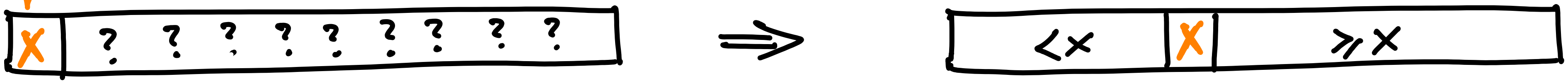

position j

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RandSelect recap

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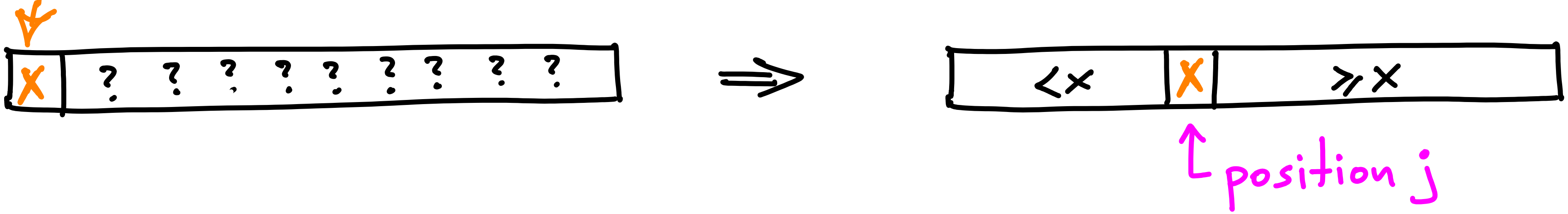

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$$T(n) = \Theta(n) + \max\{T(j-1), T(n-j)\}$$

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↳ already sorted input, reverse-sorted, nearly sorted...

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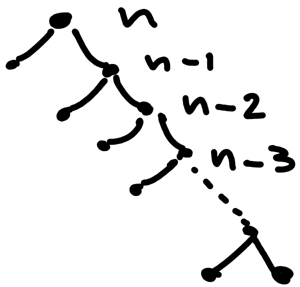

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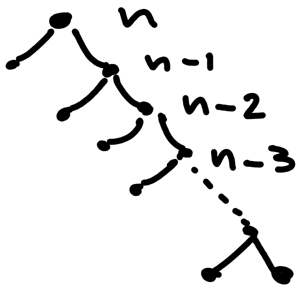

What would be ideal? (assuming we must actually recurse)

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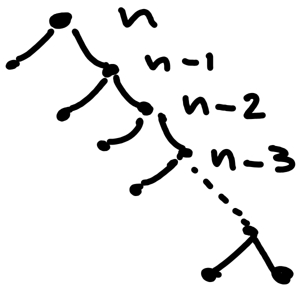

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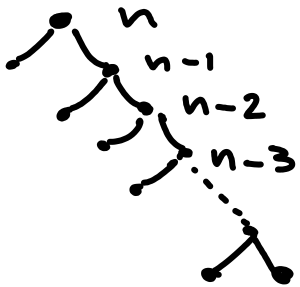

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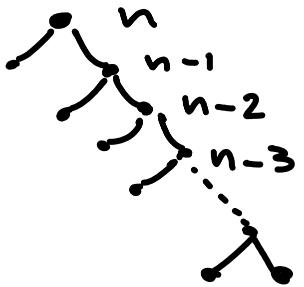

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e.g., $T(n) = T\left(\frac{9n}{10}\right) + \Theta(n) \quad = ?$

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Expected time

Expected time: call a split balanced if pivot ranks in $[\frac{n}{4} \dots \frac{3n}{4}]$
unbalanced otherwise

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Worst case if balanced split: $T(n) \leq T\left(\frac{3n}{4}\right) + dn$

Worst case if unbalanced split: $T(n) \leq \dots$

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$$T(n) \leq 0.5\underbrace{\left(T(n)+dn\right)} + 0.5 \cdot \underbrace{\left(T(\tfrac{3n}{4}) + dn\right)}$$

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$$T(n) \leq 0.5\left(T(n) + dn\right) + 0.5 \cdot \left(T\left(\frac{3n}{4}\right) + dn\right)$$

$$0.5\,T(n) \leq dn + 0.5 \cdot T\left(\frac{3n}{4}\right)$$

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$$T(n) \leq T\left(\frac{3n}{4}\right) + 2dn \quad = \Theta(n)$$

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$$T(n) \leq T\left(\frac{3n}{4}\right) + 2dn = \Theta(n)$$

$$2dn \cdot \frac{1}{1 - 3/4} = 8dn$$

Expected time complexity of RandSelect

More refined calculation

always assuming worst scenario

$$T(n) \leq \Theta(n) + \text{one of}: \left.\begin{cases} T(\max\{0, n-1\}) \\ T(\max\{1, n-2\}) \\ T(\max\{2, n-3\}) \\ \vdots \\ T(\max\{n-1, 0\}) \end{cases}\right\} n \text{ possible outcomes}$$

Define $X_k \begin{cases} 1 & \text{if RandPartition gives } k \text{ vs } n-k-1 \text{ split.} \\ 0 & \text{otherwise} \end{cases}$

always assuming worst scenario

$$T(n) \leq \Theta(n) + \text{one of: } \begin{cases} T(\max\{0, n-1\}) & k = 0 \\ T(\max\{1, n-2\}) & \\ T(\max\{2, n-3\}) & \\ \vdots & \\ T(\max\{n-1, 0\}) & k = n-1 \end{cases}$$

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$$\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})$$

$$E\left[T(n)\right] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T\left(\max\{k, n-k-1\}\right)\right]$$

$$E\left[T(n)\right] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})\right]$$

$$= E\left[\Theta(n)\right] + \sum_{k=0}^{n-1} E\left[X_k \cdot T(\max\{k, n-k-1\})\right] \qquad \text{why?}$$

$$E\left[T(n)\right] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})\right]$$

$$= E\left[\Theta(n)\right] + \sum_{k=0}^{n-1} E\left[X_k \cdot T(\max\{k, n-k-1\})\right]$$ by linearity of expectation

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$$= \Theta(n) + \sum \underbrace{E[X_k]} \cdot \underbrace{E[T(\max\{k, n-k-1\})]}$$ why?

$$E\left[T(n)\right] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})\right]$$

$$= E\left[\Theta(n)\right] + \sum_{k=0}^{n-1} E\left[X_k \cdot T(\max\{k, n-k-1\})\right]$$ by linearity of expectation

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$$\leq \Theta(n) + \frac{2}{n} \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E\left[T(k)\right]$$

$$E\left[T(n)\right] \leqslant \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E\left[T(k)\right]$$

$$E[T(n)] \leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \leq c \cdot n$

↳ assume true for $k < n$

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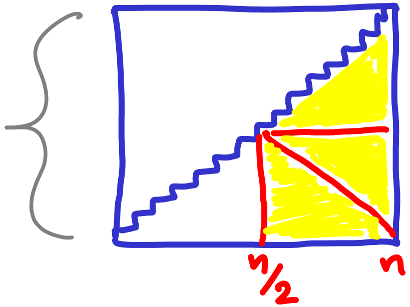
$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k \leq \frac{3}{8}n^2$$



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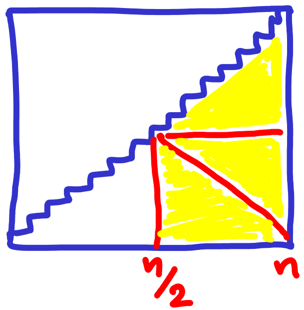
Guess $E[T(n)] \leq c \cdot n$

assume true for $k<n$

$$\leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} ck = \Theta(n) + \frac{2c}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k$$

$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k \leq \frac{3}{8}n^2$$



$$\leq \Theta(n) + \frac{2c}{n} \cdot \frac{3}{8}n^2 = \Theta(n) + \frac{3cn}{4}$$

$$E[T(n)] \leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor\frac{n}{2}\rfloor}^{n-1} E[T(k)]$$

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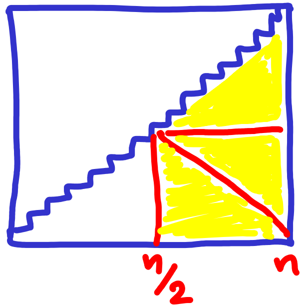

$$\leq \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \Theta(n) + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$E[T(n)] \le \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \le c \cdot n$

assume true for $k<n$

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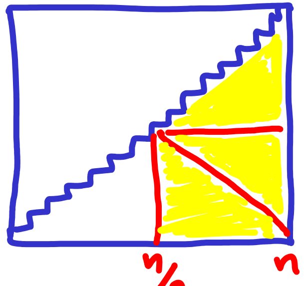

$$\le \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \underline{\Theta(n)} + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$= cn - \underbrace{\left(\frac{c}{4} - d\right)}_{\text{positive if } c>4d} n$$

$$E[T(n)] \leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \leq c \cdot n$

↳ assume true for $k<n$

$$\leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} ck = \Theta(n) + \frac{2c}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k$$

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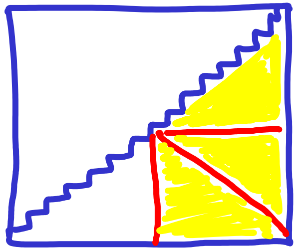

$$\leq \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \underline{\Theta(n)} + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$= cn - \underbrace{\left(\frac{c}{4} - d\right)}_{\text{positive if } c>4d} n$$

$$\boxed{E[T(n)] \leq 4dn}$$