Formal Analysis of Quicksort: Expected Time Complexity

define X_k : $\begin{cases} 1 & \text{if a pivot partitions array into } \underbrace{k}_{\text{left}} \text{ \& } \underbrace{n-k-1}_{\text{right}} \\ 0 & \text{otherwise} \end{cases}$

for $k = 0 \ldots n-1$

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$=$ probability of generating a "k-split" $= \frac{1}{n}$ } all pivots/splits are equally likely

(= Prob. picking the k^{th} smallest)

$$T(n) = \Theta(n) + \underbrace{\left\{ \begin{array}{l} T(0) + T(n-1) \\ T(1) + T(n-2) \\ T(2) + T(n-3) \\ \quad\vdots \\ T(n-1) + T(0) \end{array} \right\}}_{}\ n \text{ cases}$$

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$$E[T(n)] = E[\Sigma]$$

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$$= \Theta(n) + \ldots$$

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by independence of random variables

$$= \Theta(n) + \sum E[X_k] \cdot E\big[T(k) + T(n-k-1)\big]$$

There are random variables (X_j) in recursive calls, but independent of X_k

$$\Theta(n) + \sum E[X_k] \cdot E\left[T(k) + T(n-k-1)\right]$$

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$$= \Theta(n) + \underbrace{\frac{1}{n} \cdot \sum E\left[T(k)\right] + \frac{1}{n} \sum E\left[T(n-k-1)\right]}_{\text{symmetric: equal}}$$

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symmetric : equal

$$= \Theta(n) + \frac{2}{n} \sum_{k=0}^{n-1} E[T(k)]$$

$$\Theta(n) + \sum E[X_k] \cdot E\left[T(k) + T(n-k-1)\right]$$

$$= \Theta(n) + \sum_{k=0}^{n-1} \frac{1}{n} \cdot E\left[T(k) + T(n-k-1)\right]$$

$$= \Theta(n) + \underbrace{\frac{1}{n} \cdot \sum E[T(k)]}_{\text{symmetric}} + \underbrace{\frac{1}{n} \sum E[T(n-k-1)]}_{\text{equal}}$$

symmetric : equal

$$= \Theta(n) + \frac{2}{n} \sum_{k=0}^{n-1} E[T(k)]$$

$$= \Theta(n) + \frac{2}{n}\Big(\underbrace{E[T(0)] + E[T(1)]}_{\text{to make math easier later}}\Big) + \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)]$$

to make math
easier later

$$\Theta(n) + \sum E[X_k] \cdot E\left[T(k) + T(n-k-1)\right]$$

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guess $E\left[T(n)\right] \leq a \cdot n \log n$

assume true for $k < n$

$$\Theta(n) + \sum E[X_k] \cdot E[T(k) + T(n-k-1)]$$

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$$\le \Theta(n) + \frac{2}{n} \sum_{k=2}^{n-1} a \cdot k \cdot \log k$$

Use: $\sum_{k=2}^{n-1} k \log k \le \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$ [see CLRS]

$$\Theta(n) + \sum E[X_k] \cdot E\left[T(k) + T(n-k-1)\right]$$

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guess $E[T(n)] \leq a \cdot n \log n$

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$$E\left[T(n)\right] = \Theta(n) + \frac{2}{n} \sum_{k=2}^{n-1} E\left[T(k)\right]$$

$$\leq \Theta(n) + \frac{2}{n} \sum_{k=2}^{n-1} a \cdot k \cdot \log k$$

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$$\leq \Theta(n) + \frac{2a}{n} \cdot \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2\right)$$

$$= a n \log n - \frac{an}{4} + \Theta(n)$$

$$\Theta(n) + \sum E[X_k] \cdot E[T(k) + T(n-k-1)]$$

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$$\leq \Theta(n) + \frac{2a}{n} \cdot \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2\right)$$

$$= an\log n - \frac{an}{4} + \Theta(n)$$

$$= an\log n - \left(\frac{an}{4} - c \cdot n\right)$$

$$\Theta(n) + \sum E[X_k] \cdot E\left[T(k) + T(n-k-1)\right]$$

$$= \Theta(n) + \sum_{k=0}^{n-1} \frac{1}{n} \cdot E\left[T(k) + T(n-k-1)\right]$$

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$$= \Theta(n) + \frac{2}{n} \sum_{k=0}^{n-1} E[T(k)]$$

$$= \underbrace{\Theta(n) + \frac{2}{n}\underbrace{\left(E[T(0)] + E[T(1)]\right)}_{\text{to make math easier later}}}_{\Theta(n)} + \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)]$$

guess $E[T(n)] \le a \cdot n\log n$

assume true for $k < n$

$$E[T(n)] = \Theta(n) + \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)]$$

$$\le \Theta(n) + \frac{2}{n} \sum_{k=2}^{n-1} a \cdot k \cdot \log k$$

Use: $\sum_{k=2}^{n-1} k \log k \le \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$ [see CLRS]

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$$= an\log n - \frac{an}{4} + \Theta(n)$$

$$= an\log n - \underbrace{\left(\frac{an}{4} - c \cdot n\right)}_{\text{positive if } a > 4c}$$