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e.g., suppose you must access all data → $\Omega(n)$ lower bound

(and iterating doesn't work)

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Works for any polynomial $f(n)$

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$$T(n) = \Theta(n) + T\left(\frac{n}{x}\right) \ldots$$

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$$T(n) = T\left(\frac{n}{x}\right) + T\left(\frac{n}{b}\right) + pn \qquad \text{if } \frac{1}{x} + \frac{1}{b} = \frac{1}{c} \quad \text{for } c > 1$$

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$$\leq dn \qquad \text{if } d \geq p \cdot \frac{c}{c-1}$$