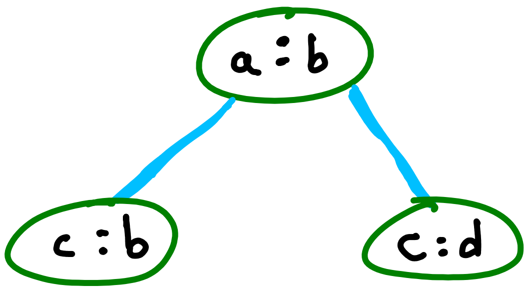
ALGORITHMS REPRESENTED AS DECISION TREES



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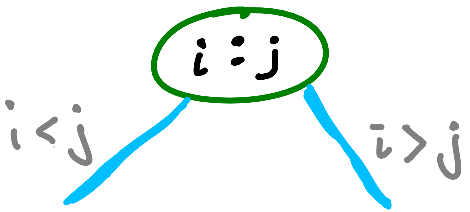
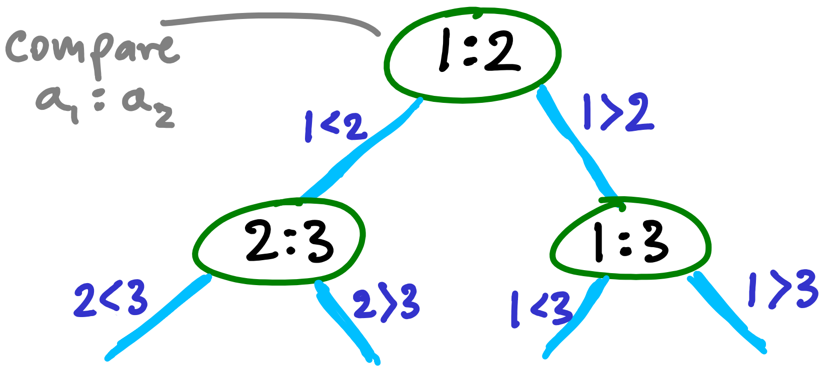


internal nodes represent comparison of two elements i,j

branches represent outcome of comparison

Left : $i<j$
Right $i>j$

Algorithms represented as Decision Trees



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Example: sort $a_1 a_2 a_3$

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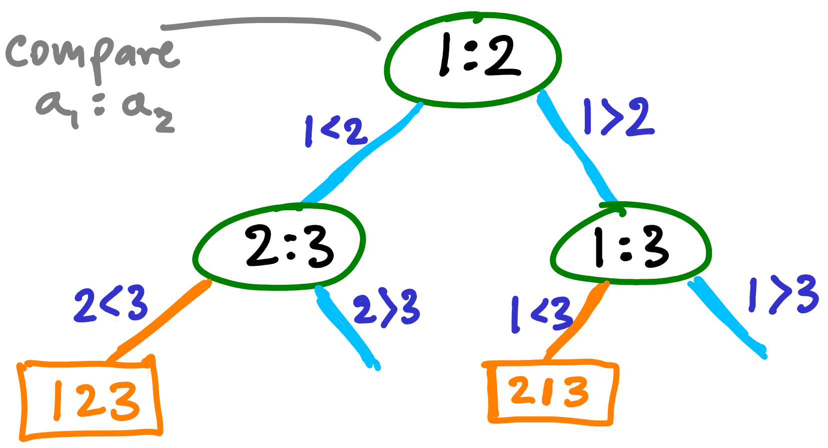

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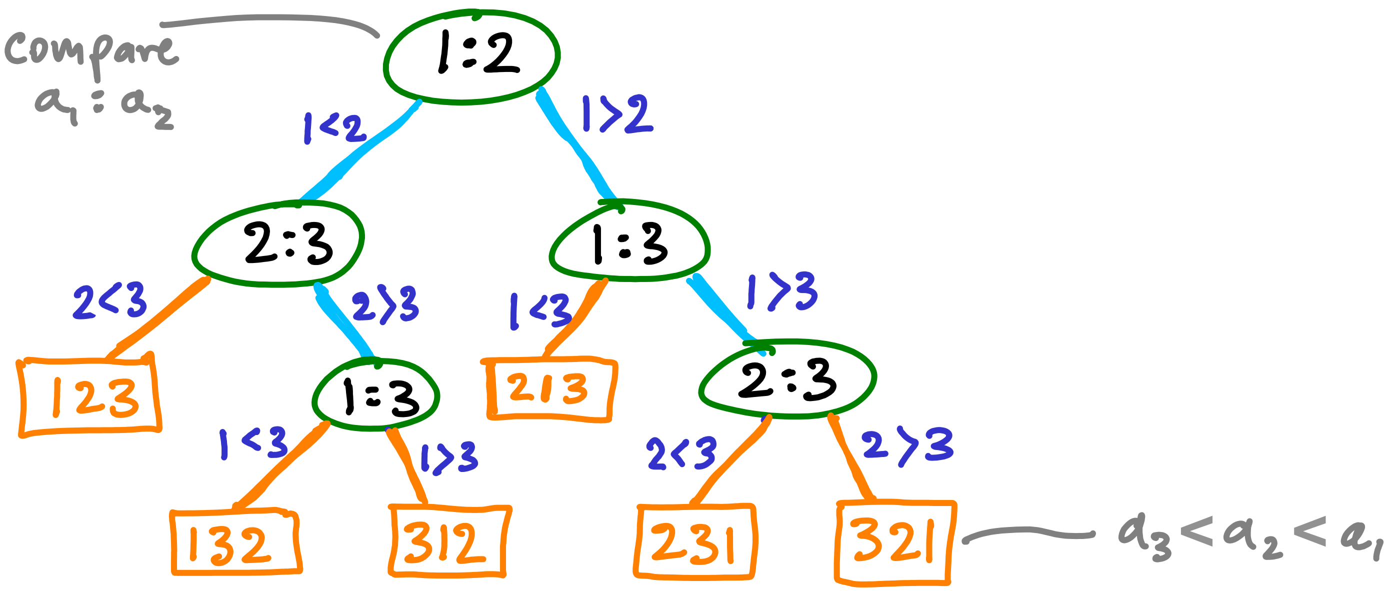

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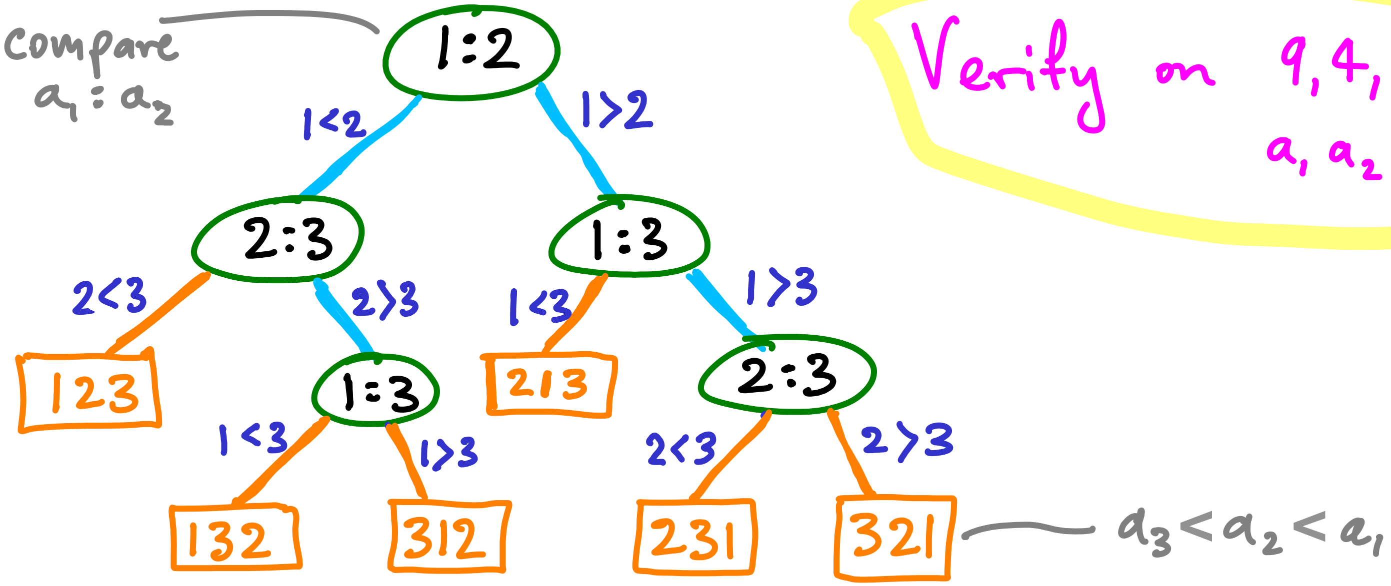

Verify on 9, 4, 6
$a_1\ a_2\ a_3$

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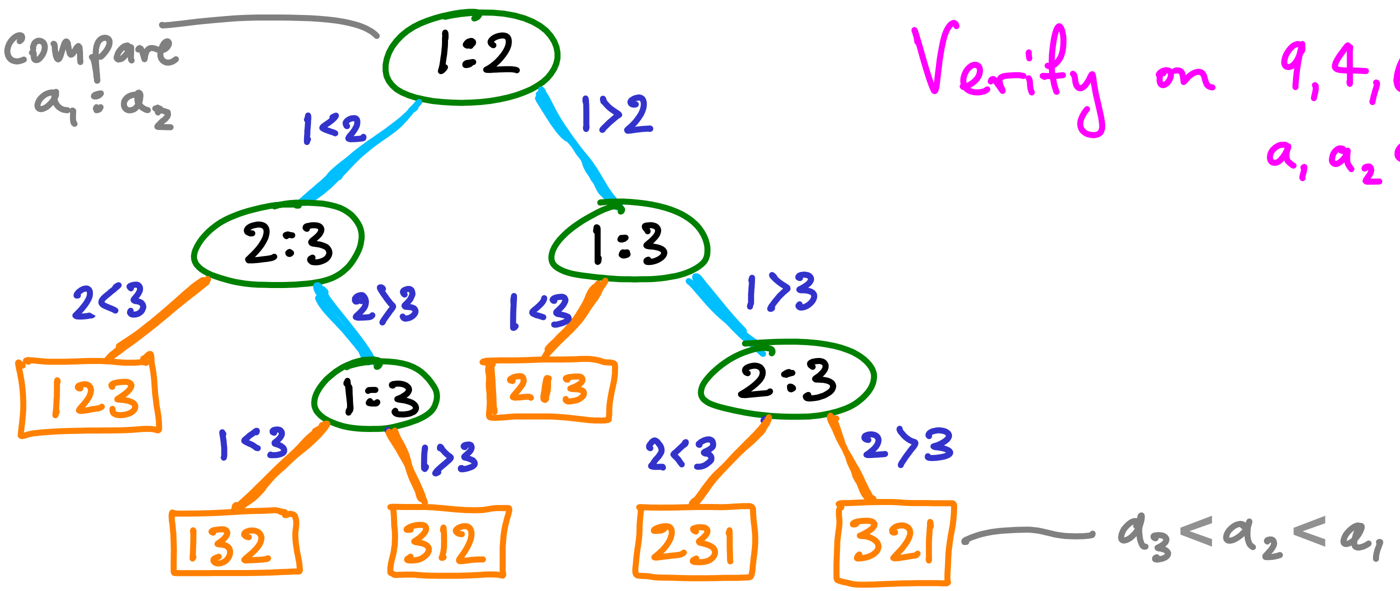

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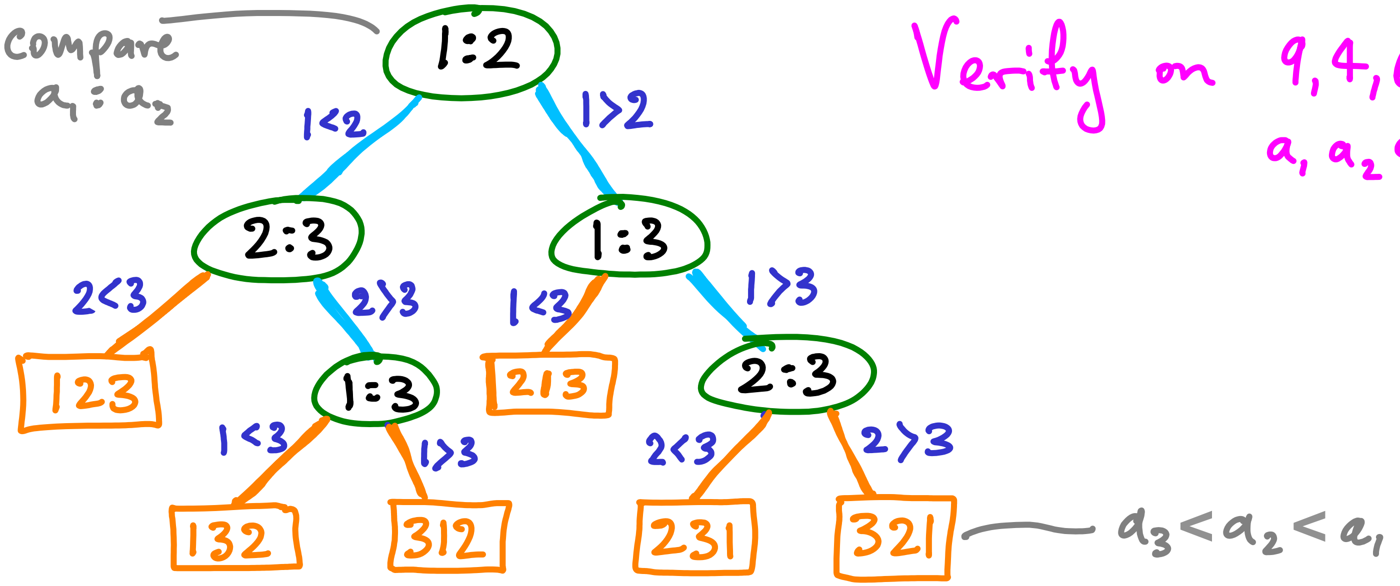

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$\boxed{\text{Any decision-based algorithm can be encoded as a decision tree.}}$

If you are designing a decision tree
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↳ so max path length $= \binom{n}{2}$

for a good algo

for sorting

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What is the shortest possible tree for comparison-sort?

A correct decision tree for sorting must have every possible output represented at a leaf node.

$\hookrightarrow$ # leaves $\geq$?

A correct decision tree for sorting must have every permutation of the input represented at a leaf node.

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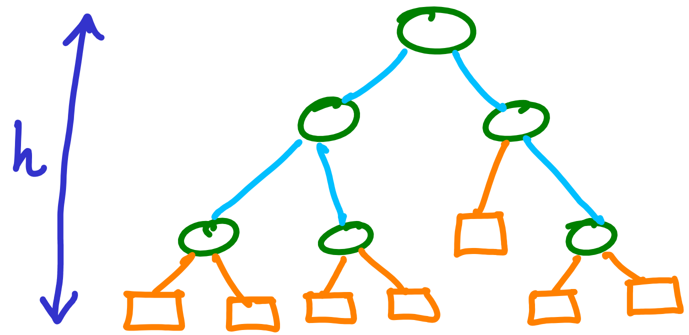

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height of tree = worst case time = h ⇝ # leaves $\leq 2^h$

[binary tree; every node has 2 children]

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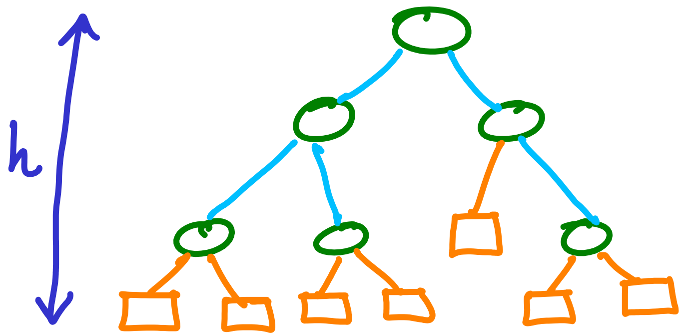

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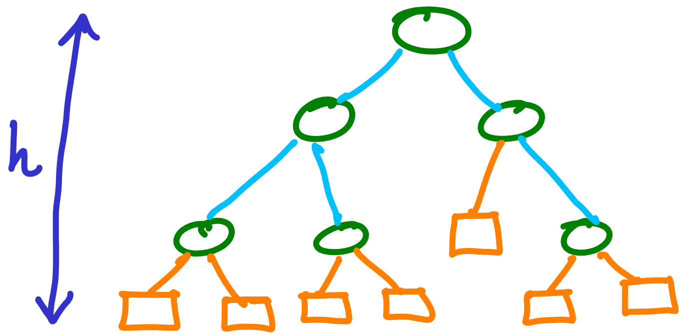

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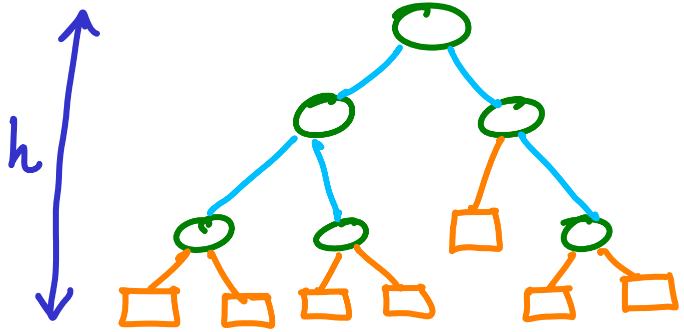

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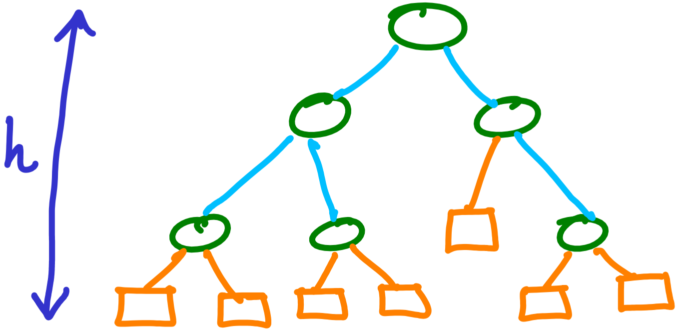

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Stirling's formula: $n! \geq \left(\frac{n}{e}\right)^n$

$h \geq \log\left(\frac{n}{e}\right)^n$

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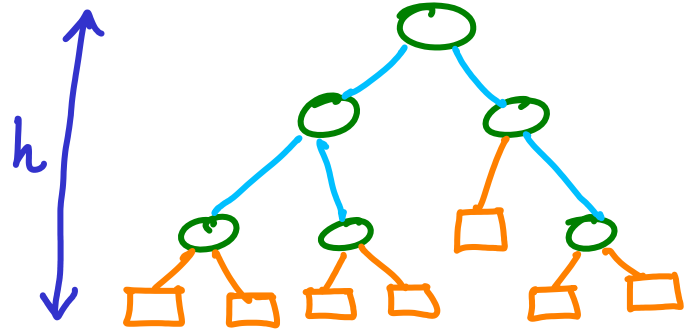

↳ # leaves $\geqslant n!$

height of tree = worst case time = h ⇝ # leaves $\leqslant 2^h$

[binary tree; every node has 2 children]

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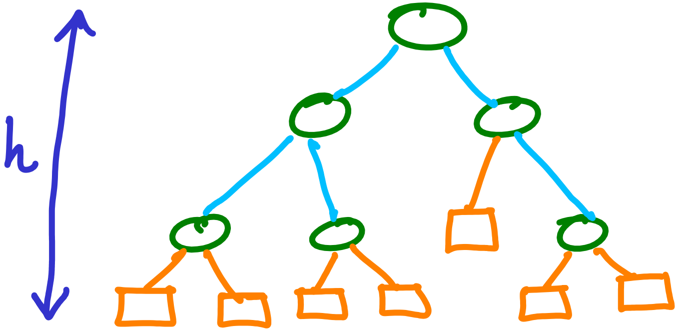

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Stirling's formula: $n! \geq \left(\frac{n}{e}\right)^n$

$$h \geq \log\left(\frac{n}{e}\right)^n = n \cdot \log \frac{n}{e}$$
$$= n\log n - n\log e$$
$$= n\log n - \Theta(n)$$

$$h = \Omega(n\log n)$$

extra analysis of logn! follows

$\log(n!) = O(?)$

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$$\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots \cdots 3 \cdot 2 \cdot 1)$$

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$$= \log(n \cdot 1 \cdot (n-1) \cdot 2 \cdot (n-2) \cdot 3 \cdot (n-3) \cdot 4 \cdots \cdots \sim (n-\tfrac{n}{2}) \cdot (n-\tfrac{n}{2}))$$

exactly if n: even

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$$= \log\left(\underbrace{n \cdot 1} \cdot \underbrace{(n-1) \cdot 2} \cdot \underbrace{(n-2) \cdot 3} \cdot \underbrace{(n-3) \cdot 4} \cdots \cdots \sim \underbrace{\left(n - \tfrac{n}{2}\right) \cdot \left(n - \tfrac{n}{2}\right)}\right)$$

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$$\Rightarrow \log(n!) \geqslant \frac{n}{2}\log n$$

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---

so $\frac{1}{2}n\log n \leq \log(n!) \leq n\log n$