ORDER STATISTICS - MEDIAN FINDING, RANK SELECTION

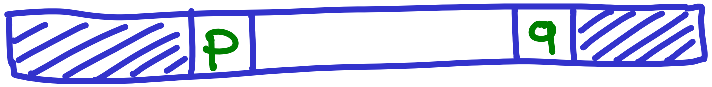
Given n unsorted elements, find the k-th smallest.

↳ easy $O(n)$ if $k = O(1)$ or $n - O(1)$.
In some sense median is hardest

We will assume distinct elements.

RandSelect(p, q, i) // normally written ~(A, p, q, i)
↳ points to Array

?



↳ a randomized algorithm for finding the i-th smallest element between index p and index $q \geq p$ in an array

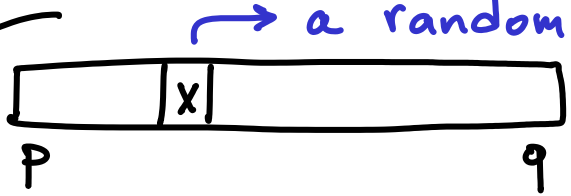
Start out with RandSelect$(1, n, k)$

// Looking for i-th smallest element in [p...q]

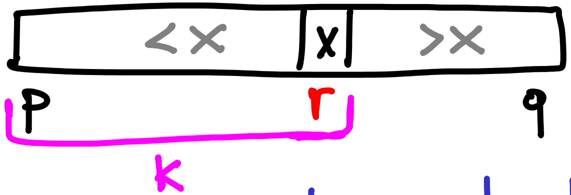
RandSelect(p,q,i)

$r \leftarrow$ RandPartition(p,q)

$k = r - p + 1$ // rank of x in [p...q]



r : the index of x if we place it its sorted position.



// so far we have identified the rank of one random element

if $i = k$, return A[r]

if $i < k$, RandSelect(p, r-1, i) // we know that x has higher rank than i and all elements in left sub-array have smaller rank.

if $i > k$, RandSelect(r+1, q, i-k) // symmetric case. All that changes is that we "reset" the value of the target rank.

[6 10 13 5 8 3 2 11] $n=8$ $i=7$

$r \leftarrow$ RandPartition(1, 8) → Let's just "randomly" pick $x = A[1] = 6$

After $\Theta(n)$ work: [5 3 2 6 10 13 8 11]

↳ $r=4$ $=k$

Recursively call RandSelect(5, 8, 3)   new $i = 7-4$

⇐ We have found the 4-th smallest element, so the 7-th smallest is in [10 13 8 11]

↳ Randpartition(5, 8) → Again pick A[5] = 10 // first parameter of function

Get 5 3 2 6 [8 10 13 11]

↳ $r=6$ $k=2$

Recursively call RandSelect(7, 8, 1)

↳ RandPartition(7, 8) → A[7] = 13 → 5 3 2 6 8 10 [11 13]

↳ $r=8$ $k=2$

trivial end

Analysis of RandSelect

$$T(n) = \underbrace{\Theta(n)}_{\text{RandPartition}} + T(f(n))$$

$f(n)$ = size of subarray
(either left or right of pivot)

"Perfect" pivot : splits array evenly. $T(n) = \Theta(n) + T(\frac{n}{2}) = \Theta(n)$

"Just as perfect" : always split into constant fractions

even if you always recurse on "wrong" side. e.g. $T(n) = \Theta(n) + T(\frac{99n}{100}) = \Theta(n)$

"Unlucky" : split 0 vs n-1 : $T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$

"Just as unlucky" : split $\Theta(1)$ vs $n-\Theta(1)$: $T(n) = \Theta(n) + T(n-c)$

$$\approx n + (n-c) + (n-2c) + (n-3c) + \dots$$

$$\geq \frac{n}{2c} \cdot \frac{n}{2} = \Omega(n^2)$$

Expected time: call a split balanced if pivot ranks in $\left[\frac{n}{4} \dots \frac{3n}{4}\right]$
unbalanced otherwise

Worst case if balanced split: $T(n) \leq T\left(\frac{3n}{4}\right) + dn$

Worst case if unbalanced split: $T(n) \leq T(n-1) + dn < T(n) + dn$

Each split has a 50% chance of being balanced

$$T(n) \leq 0.5\left(\underbrace{T(n)+dn}\right) + 0.5 \cdot \underbrace{\left(T\left(\frac{3n}{4}\right) + dn\right)}$$

$$0.5\, T(n) \leq dn + 0.5 \cdot T\left(\frac{3n}{4}\right)$$

$$T(n) \leq T\left(\frac{3n}{4}\right) + 2dn = \Theta(n)$$

$$2dn \cdot \frac{1}{1-3/4} = 8dn$$

$$\text{Define } X_k \begin{cases} 1 & \text{if RandPartition gives } k \text{ vs } n-k-1 \text{ split.} \\ 0 & \text{otherwise} \end{cases}$$

always assuming worst scenario

$$\underbrace{T(n) \leq \Theta(n) + \text{one of: } \begin{cases} T(\max\{0, n-1\}) \\ T(\max\{1, n-2\}) \\ T(\max\{2, n-3\}) \\ \vdots \\ T(\max\{n-1, 0\}) \end{cases}}_{\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})}$$

$$E[T(n)] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T(\max\{k, n-k-1\})\right]$$

$$= E[\Theta(n)] + \sum_{k=0}^{n-1} E[X_k \cdot T(\max\{k, n-k-1\})]$$ by linearity of expectation

$$= \Theta(n) + \sum E[X_k] \cdot E[T(\max\{k, n-k-1\})]$$ by independence

$$= \Theta(n) + \sum_{k=0}^{n-1} \cdot \frac{1}{n} \cdot E[T(\max\{k, n-k-1\})]$$ by random choice

$$\leq \Theta(n) + \frac{2}{n} \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

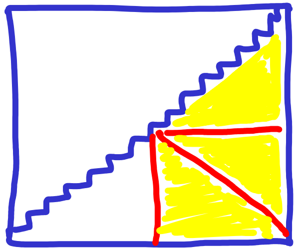
$$E[T(n)] \le \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \le c \cdot n$

assume true for $k<n$

$$\le \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} ck = \Theta(n) + \frac{2c}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k$$

$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k \le \frac{3}{8}n^2$$



$$\le \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \underline{\Theta(n)} + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$= cn - \underbrace{\left(\frac{c}{4}-d\right)}_{\text{positive if } c>4d} n$$

$$E[T(n)] \le 4dn$$