ORDER STATISTICS - MEDIAN FINDING, RANK SELECTION

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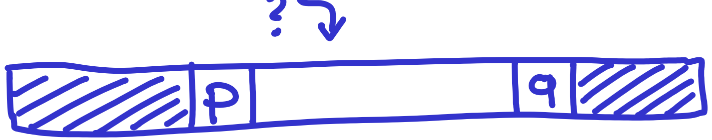
We will assume distinct elements.

RandSelect(p, q, i) // normally written ~(A, p, q, i)

↳ points to Array

?



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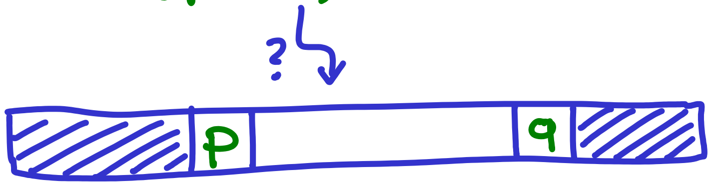

↳ a randomized algorithm for finding the i-th smallest element between index p and index $q \geq p$ in an array

Start out with RandSelect$(1, n, k)$

//Looking for i-th smallest element in [p...q]

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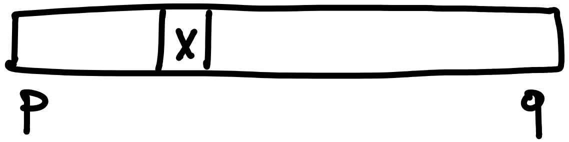
$r \leftarrow$ RandPartition(p,q)

a random element



r : the index of x if we place it its sorted position.

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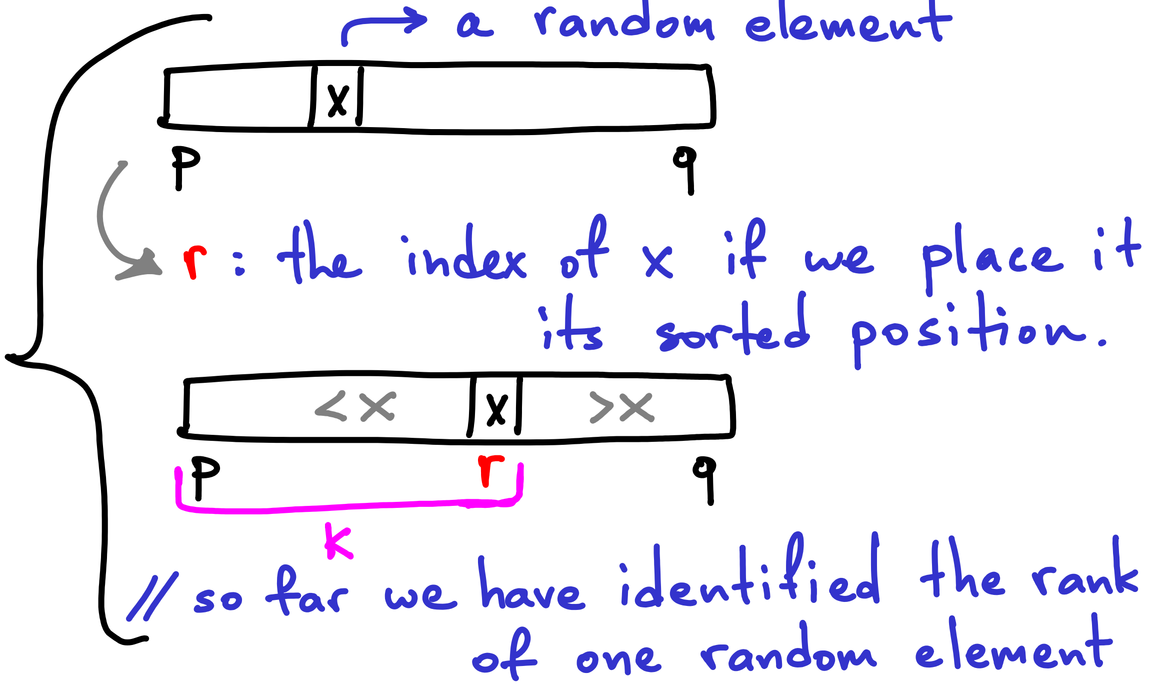
RandSelect(p, q, i)

  r ← RandPartition(p, q)

  k = r - p + 1 // rank of x in [p...q]



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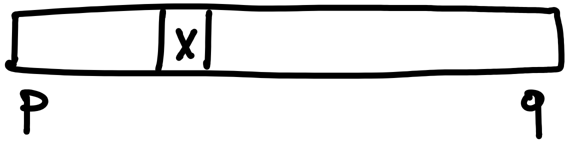

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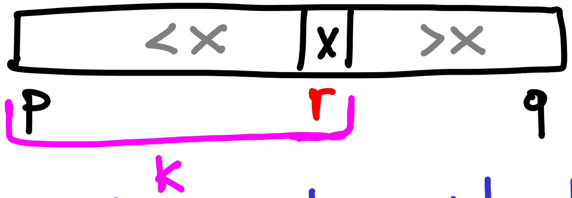

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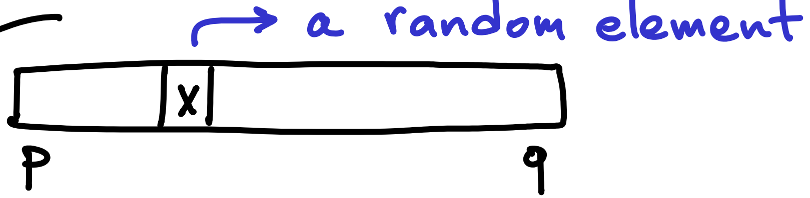

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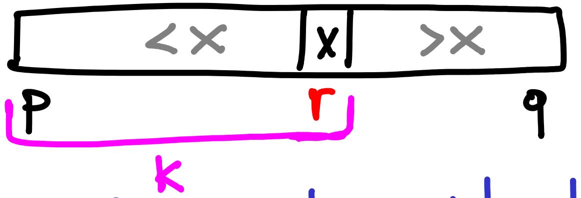

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if $i < k$ ...

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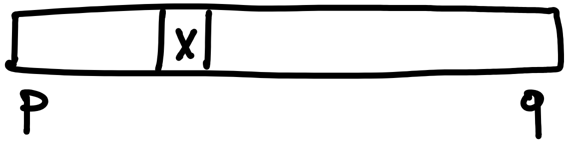

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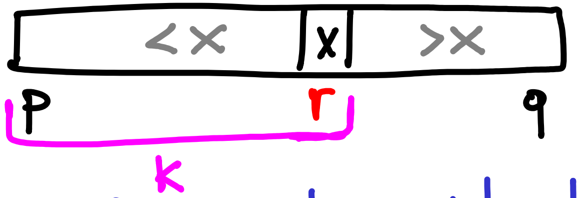

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if $i<k$, RandSelect(p, r-1, i) // we know that x has higher rank than i and all elements in left sub-array have smaller rank.

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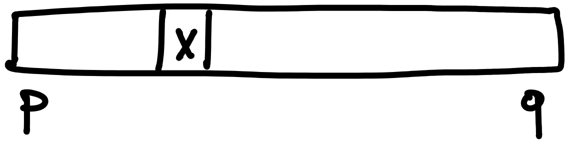

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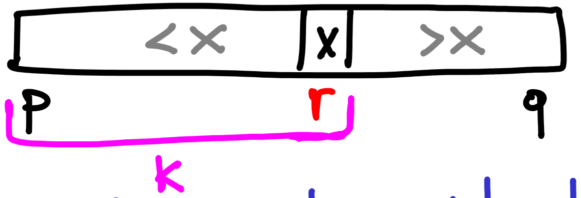

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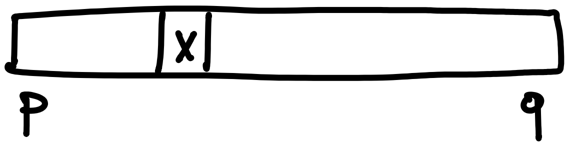

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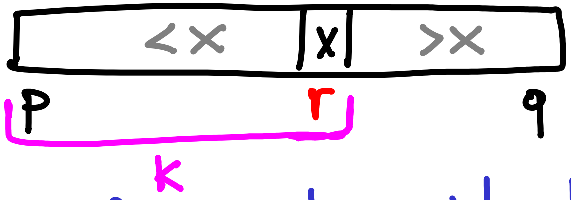

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if $i < k$, RandSelect(p, r-1, i) // we know that x has higher rank than i and all elements in left sub-array have smaller rank.

if $i > k$, RandSelect(r+1, q, i-k) // symmetric case. All that changes is that we "reset" the value of the target rank.

$$\begin{bmatrix} 6 & 10 & 13 & 5 & 8 & 3 & 2 & 11 \end{bmatrix} \qquad n=8 \qquad i=7$$

$[6 \quad 10 \quad 13 \quad 5 \quad 8 \quad 3 \quad 2 \quad 11]$ $n=8$ $i=7$

$r \leftarrow \text{RandPartition}(1, 8)$ $\rightarrow$ Let's just "randomly" pick $x = A[1] = 6$

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Recursively call RandSelect$(5, 8, 3)$ $\Leftarrow$ new $i = 7-4$

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Get 5 3 2 6 [8 10 13 11]

$r = 6$ $k = 2$

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Recursively call RandSelect(7, 8, 1)

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Get 5 3 2 6 [8 10 13 11]

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Recursively call RandSelect(7, 8, 1)

RandPartition(7, 8) $\rightarrow$ $A[7] = 13$ $\rightarrow$ 5 3 2 6 8 10 [11 13]

$r = 8$

$k = 2$

trivial end

Analysis of RandSelect

$$T(n) = \underbrace{\Theta(n)}_{\text{RandPartition}} + T(f(n))$$

$f(n)$ = size of subarray
(either left or right of pivot)

"Perfect" pivot : splits array evenly. $T(n) = \Theta(n) + T(\frac{n}{2}) = \Theta(n)$

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"Just as unlucky" : split $\Theta(1)$ vs $n - \Theta(1)$: $T(n) = \Theta(n) + T(n-c)$

$$\approx n + (n-c) + (n-2c) + (n-3c) + \dots$$

$$\geq \frac{n}{2c} \cdot \frac{n}{2} = \Omega(n^2)$$

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$$2dn \cdot \frac{1}{1-3/4} = 8dn$$

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$$T(n) \leqslant \Theta(n) + \text{one of}: \left.\begin{cases} T(\max\{0, n-1\}) \\ T(\max\{1, n-2\}) \\ T(\max\{2, n-3\}) \\ \vdots \\ T(\max\{n-1, 0\}) \end{cases}\right\} n \text{ possible outcomes}$$

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$$E\left[T(n)\right] \leq E\left[\Theta(n) + \sum_{k=0}^{n-1} X_k \cdot T\left(\max\{k, n-k-1\}\right)\right]$$

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why?

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$$= \Theta(n) + \sum E[X_k] \cdot E[T(\max\{k, n-k-1\})]$$ why?

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$$E[T(n)] \leqslant \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \leqslant c \cdot n$

assume true for $k < n$

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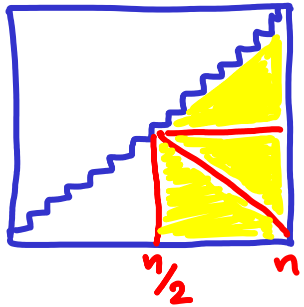
$$\sum_{k=\lfloor\frac{n}{2}\rfloor}^{n-1} k \leq \frac{3}{8}n^2$$



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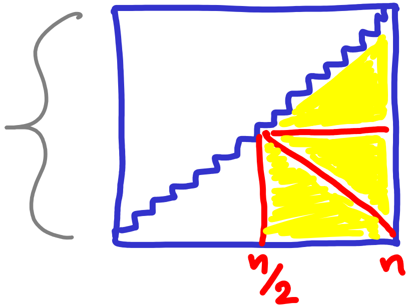
$$\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} k \leq \frac{3}{8}n^2$$



$$\leq \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \Theta(n) + \frac{3cn}{4}$$

$$E[T(n)] \leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \leq c \cdot n$

assume true for $k<n$

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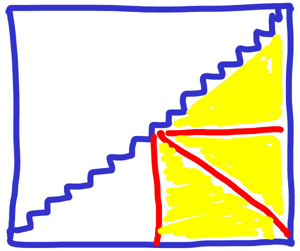

$$\leq \Theta(n) + \frac{2c}{n} \cdot \frac{3}{8}n^2 = \Theta(n) + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$E[T(n)] \le \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \le c \cdot n$

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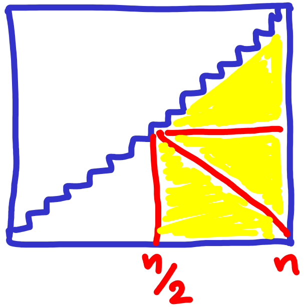

$$\le \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \underline{\Theta(n)} + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$= cn - \underbrace{\left(\frac{c}{4} - d\right)}_{\text{positive if } c>4d} n$$

$$E[T(n)] \leq \Theta(n) + \frac{2}{n}\sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)]$$

Guess $E[T(n)] \leq c \cdot n$

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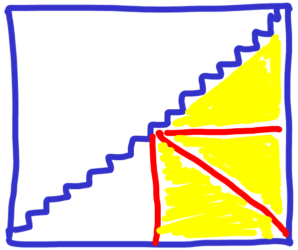

$$\leq \Theta(n) + \frac{2c}{n}\cdot\frac{3}{8}n^2 = \underline{\Theta(n)} + \frac{3cn}{4} = cn - \frac{1}{4}cn + dn$$

$$= cn - \underbrace{\left(\frac{c}{4} - d\right)}_{\text{positive if } c>4d} n$$

$$E[T(n)] \leq 4dn$$