QUICKSORT

An in-place divide & conquer algorithm

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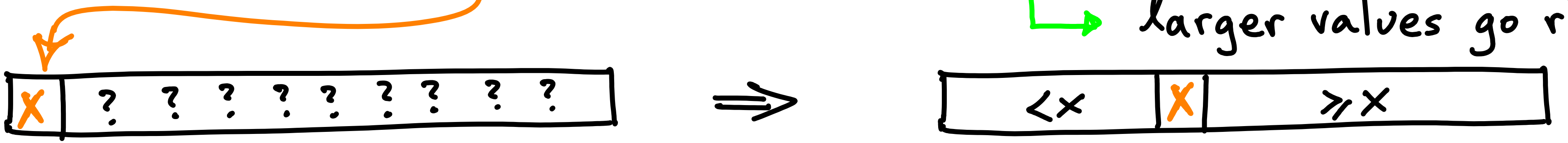
- DIVIDE: choose a pivot & partition → smaller values go left
  → larger values go right



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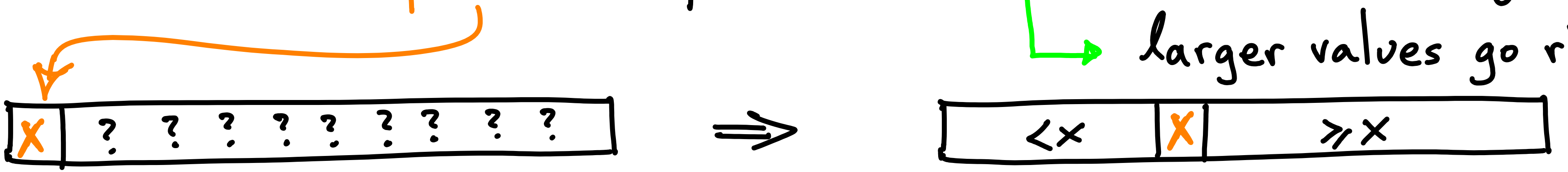

- CONQUER: Quicksort each side

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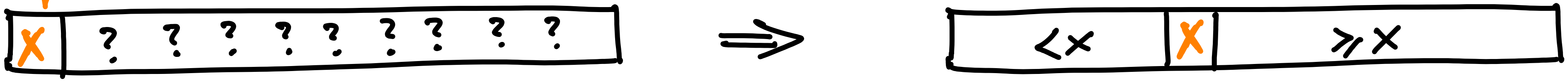

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Unlike Mergesort, there is no Combine phase.

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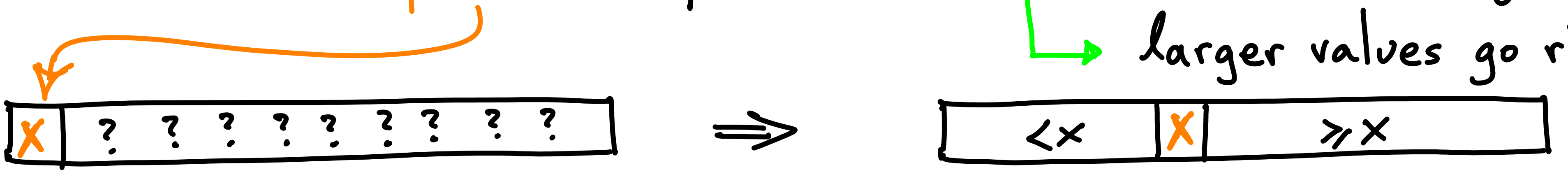

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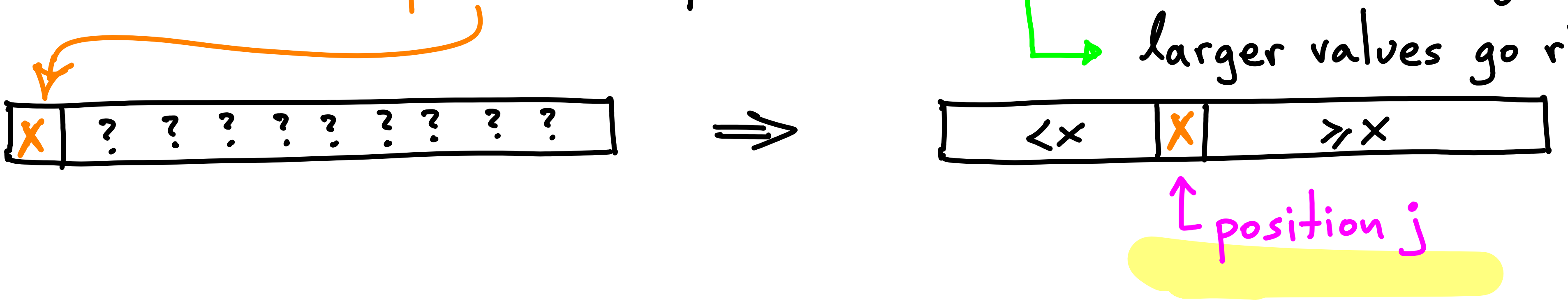

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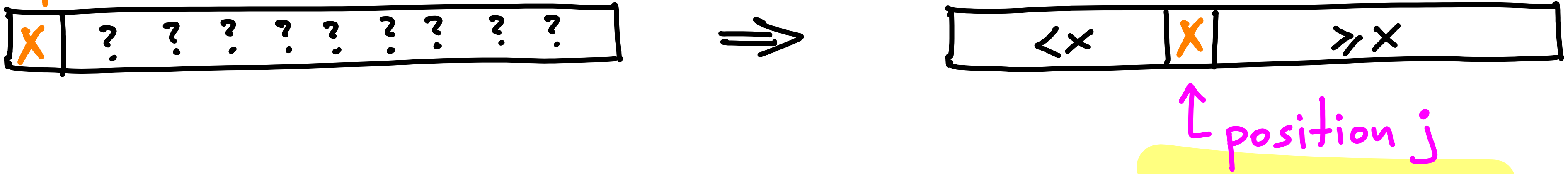

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$$T(n) = \Theta(n) + T(j-1) + T(n-j)$$

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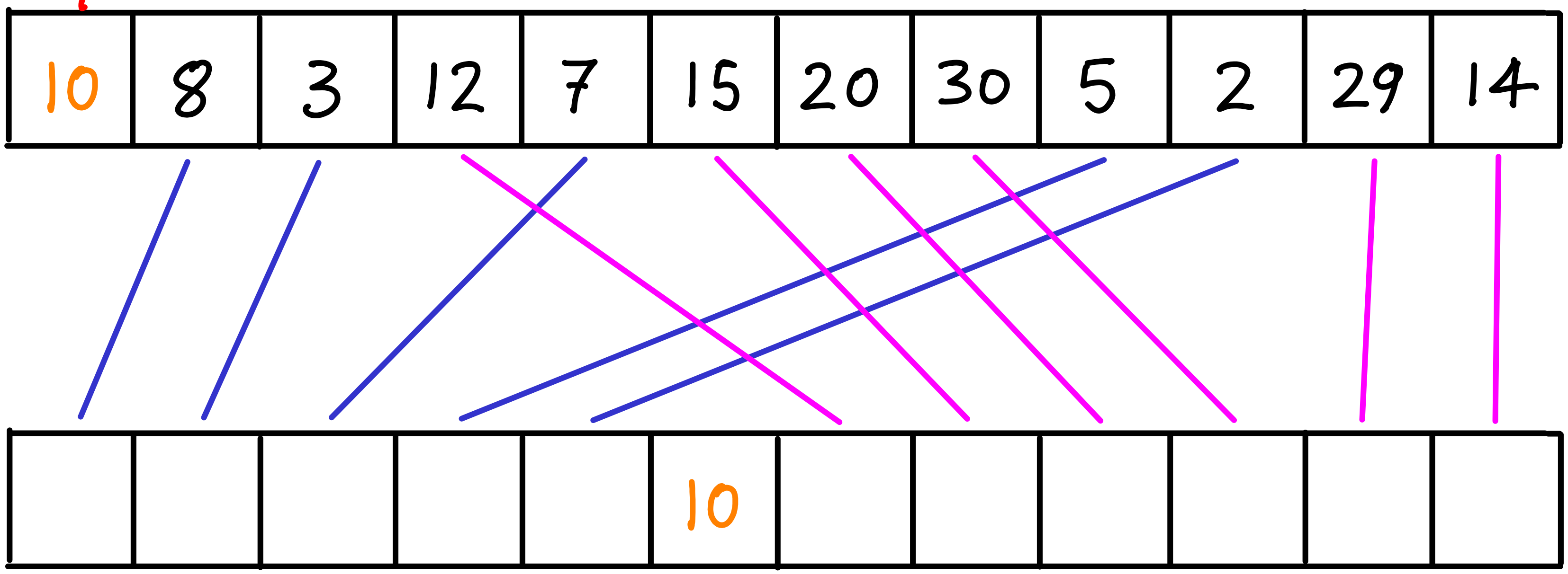

There are stable versions as well

What is the worst-case time complexity, and why?

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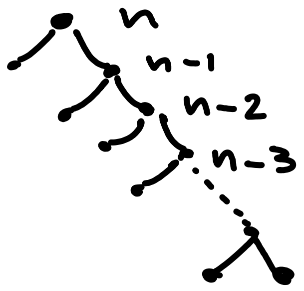

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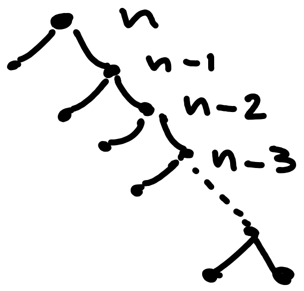

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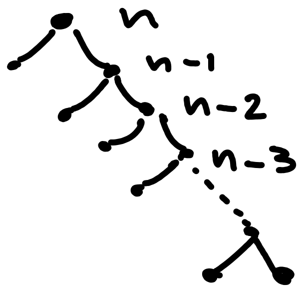

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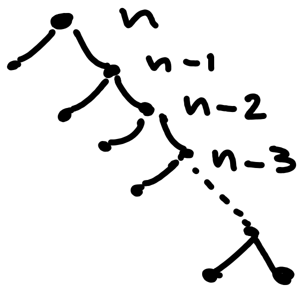

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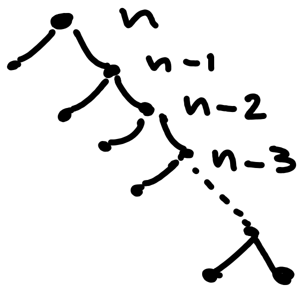

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Why use Quicksort? We expect $\Theta(n \log n)$ ... with a small constant

(& it's in-place, and stable)

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e.g., $T(n) = T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) + c \cdot n$

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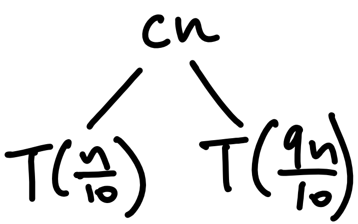

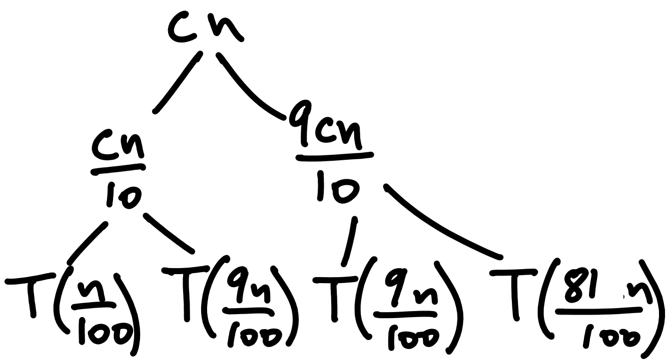

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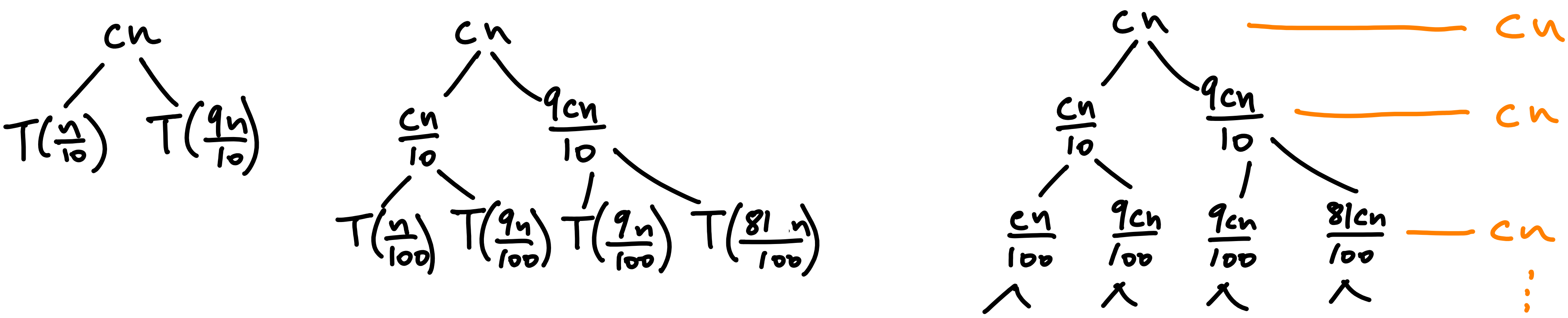

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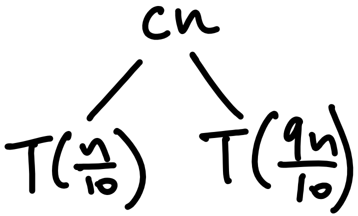

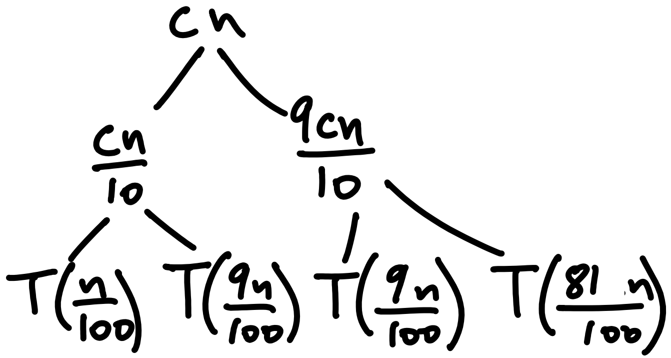

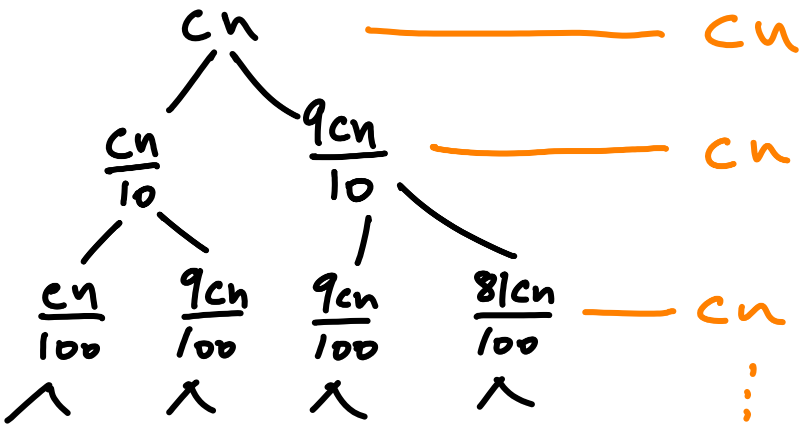

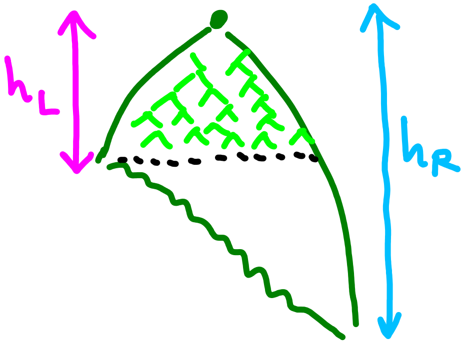

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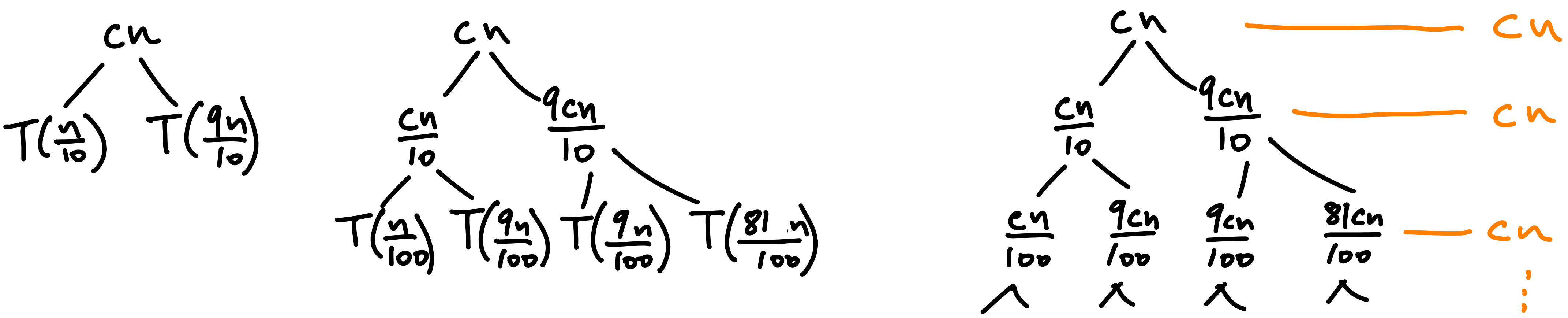

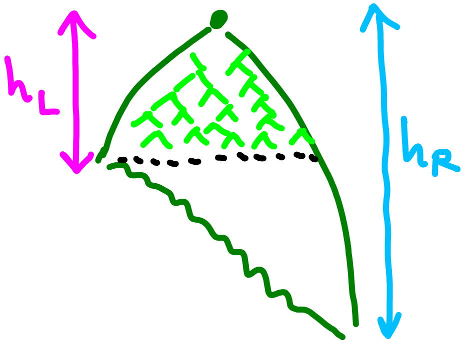

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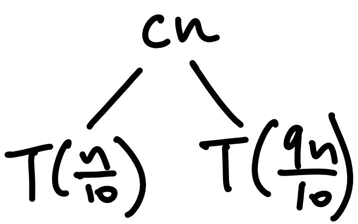

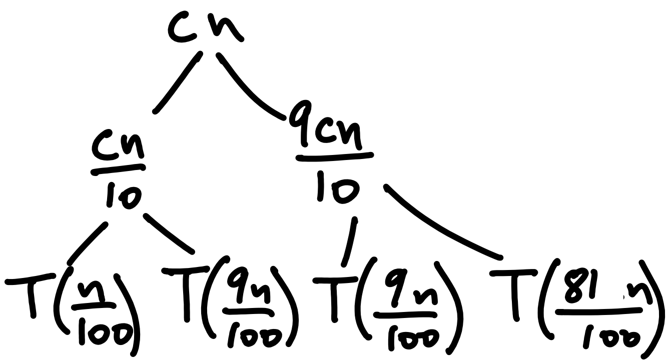

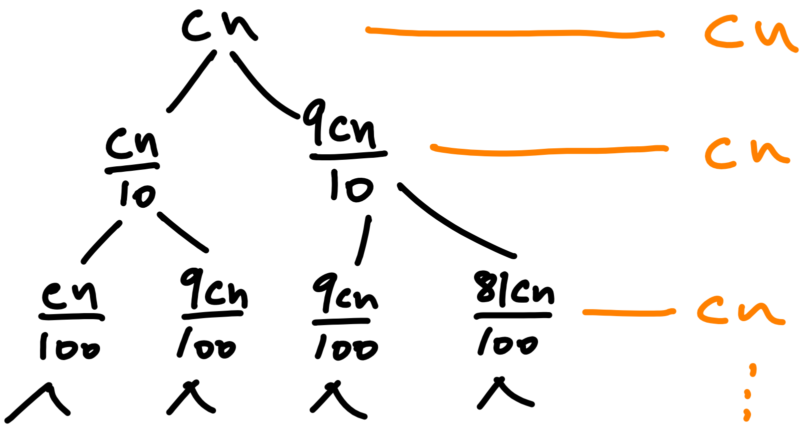

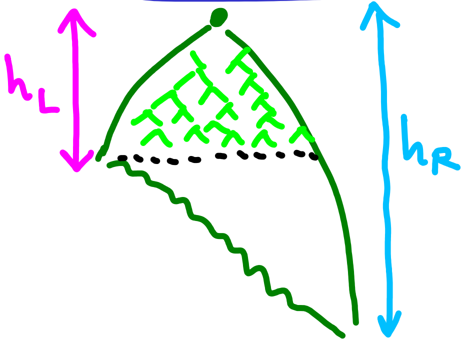

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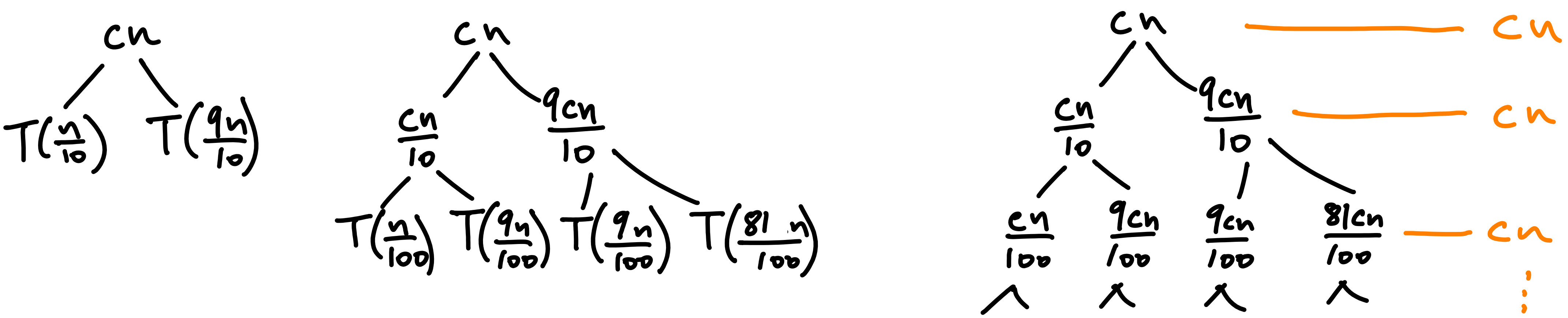

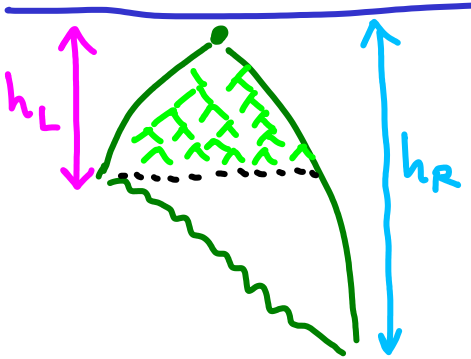

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Any constant-fraction-split will give $\Theta(n \log n)$

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Worst case if unbalanced split: $T(n) \leq T(n-1) + T(0) + dn \leq \underline{T(n) + dn}$

Each split has a 50% chance of being balanced

$$T(n) \leq 0.5\left(T(n)+dn\right) + 0.5\cdot\left(T\left(\tfrac{3n}{4}\right) + T\left(\tfrac{n}{4}\right) + dn\right)$$

Expected time: call a split balanced if pivot ranks in $\left[\frac{n}{4} \dots \frac{3n}{4}\right]$

unbalanced otherwise

Worst case if balanced split: $T(n) \leq T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + dn$

Worst case if unbalanced split: $T(n) \leq T(n) + dn$

Each split has a 50% chance of being balanced

$$T(n) \leq 0.5\left(\underline{T(n)+dn}\right) + 0.5 \cdot \left(\underline{T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + dn}\right)$$

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Each split has a 50% chance of being balanced

$$T(n) \leq 0.5\left(T(n) + dn\right) + 0.5 \cdot \left(T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + dn\right)$$

$$0.5\,T(n) \leq dn + 0.5\left(T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right)\right)$$

Expected time: call a split balanced if pivot ranks in $\left[\frac{n}{4} \ldots \frac{3n}{4}\right]$

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Worst case if balanced split: $T(n) \leq T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + dn$

Worst case if unbalanced split: $T(n) \leq T(n) + dn$

Each split has a 50% chance of being balanced

$$T(n) \leq 0.5\left(T(n) + dn\right) + 0.5 \cdot \left(T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + dn\right)$$

$$0.5\,T(n) \leq dn + 0.5\left(T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right)\right)$$

$$T(n) \leq T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + 2dn \quad = \Theta(n \log n)$$