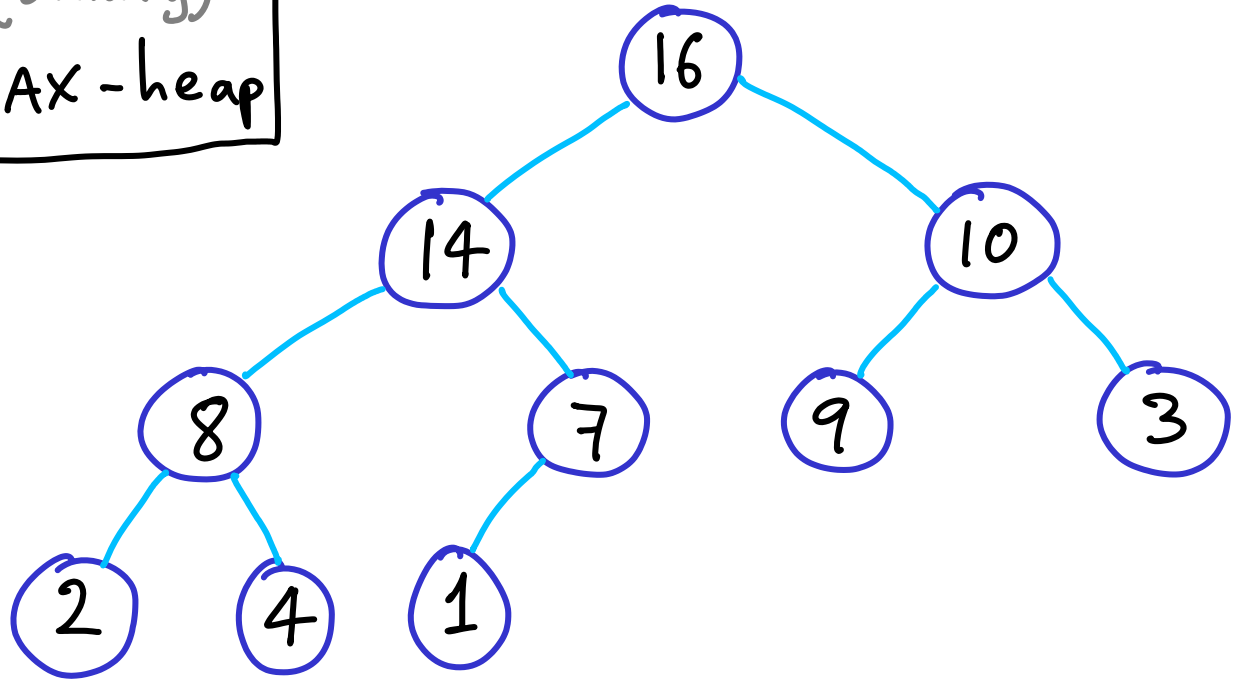
(binary)
MAX-heap



Rules:

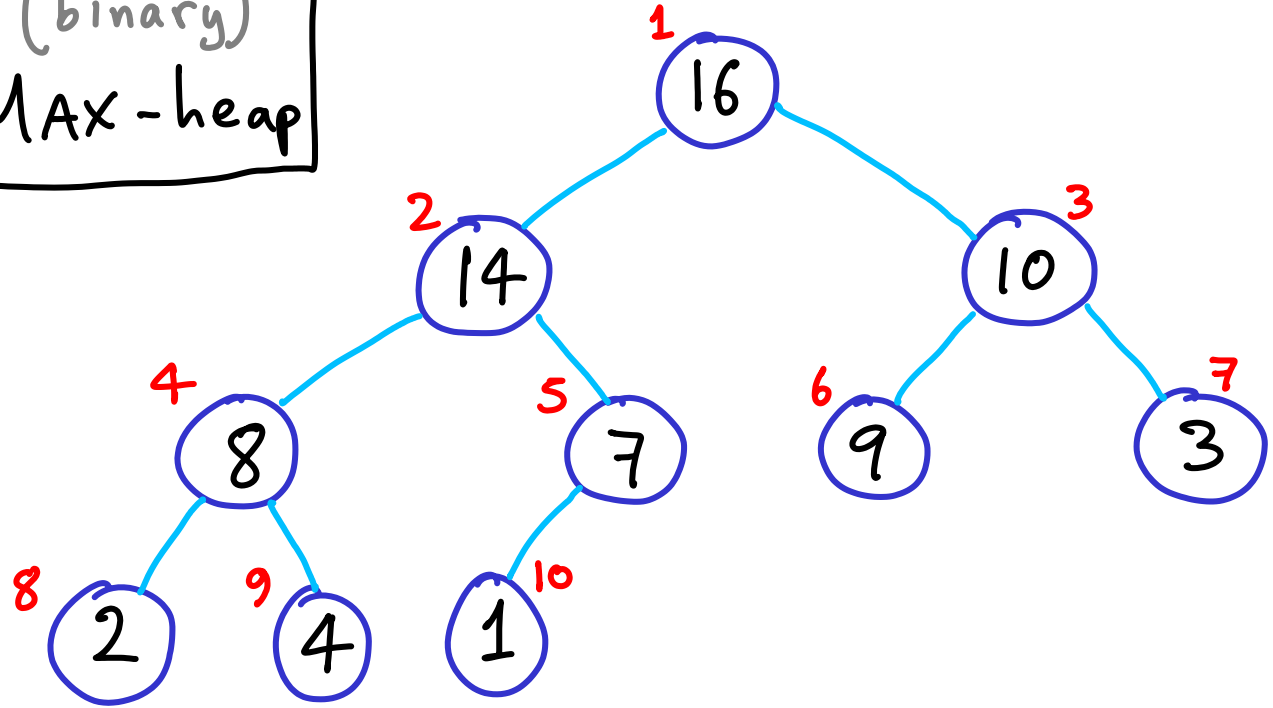
1. • last row filled from left
   • other rows full
2. • parent > children

Rule 1 (structure) is not always required in all applications

E.g. “delete only”

We will assume it’s required.

(binary)
MAX-heap



[Notice every subtree is also a heap]

We can use an array to store a heap. No pointers.

1	2	3	4	5	6	7	8	9	10	
16	14	10	8	7	9	3	2	4	1	.....

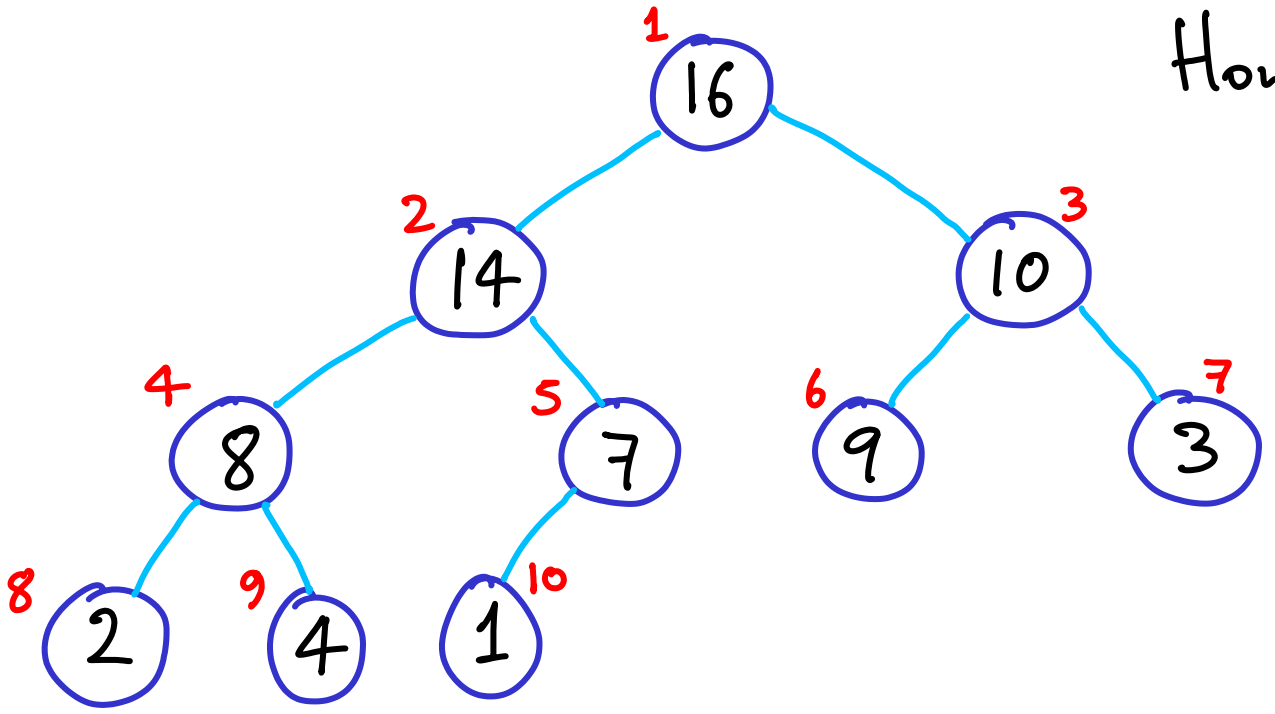
Rules:

- last row filled from left
- other rows full
- parent > children

$parent(i) = \lfloor i/2 \rfloor$

$left\text{-}child(i) = 2i$

$right\text{-}child(i) = 2i+1$



How does this relate to sorting?

Largest element is on top.

2nd largest is in level 2.

3rd largest is

↳ in level 2

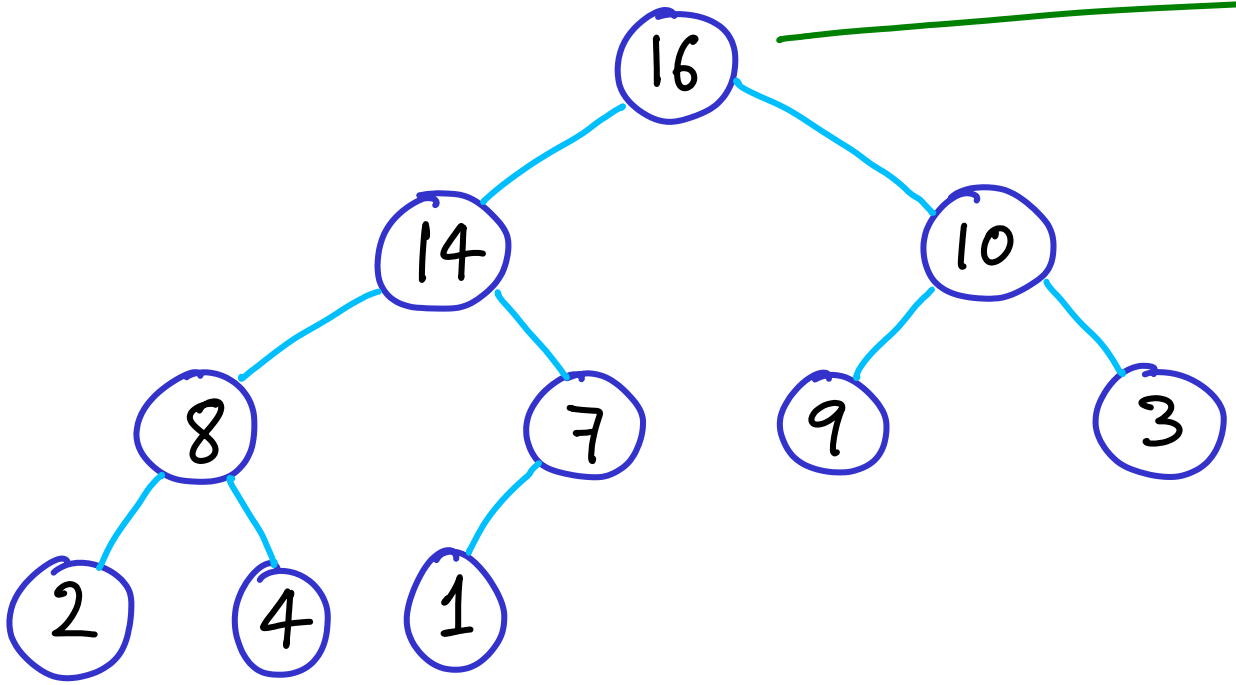
OR

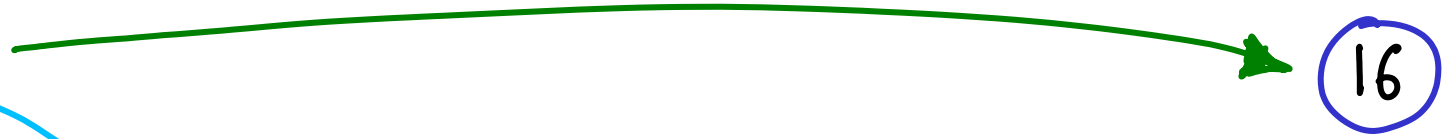
↳ in level 3

& child of 2nd

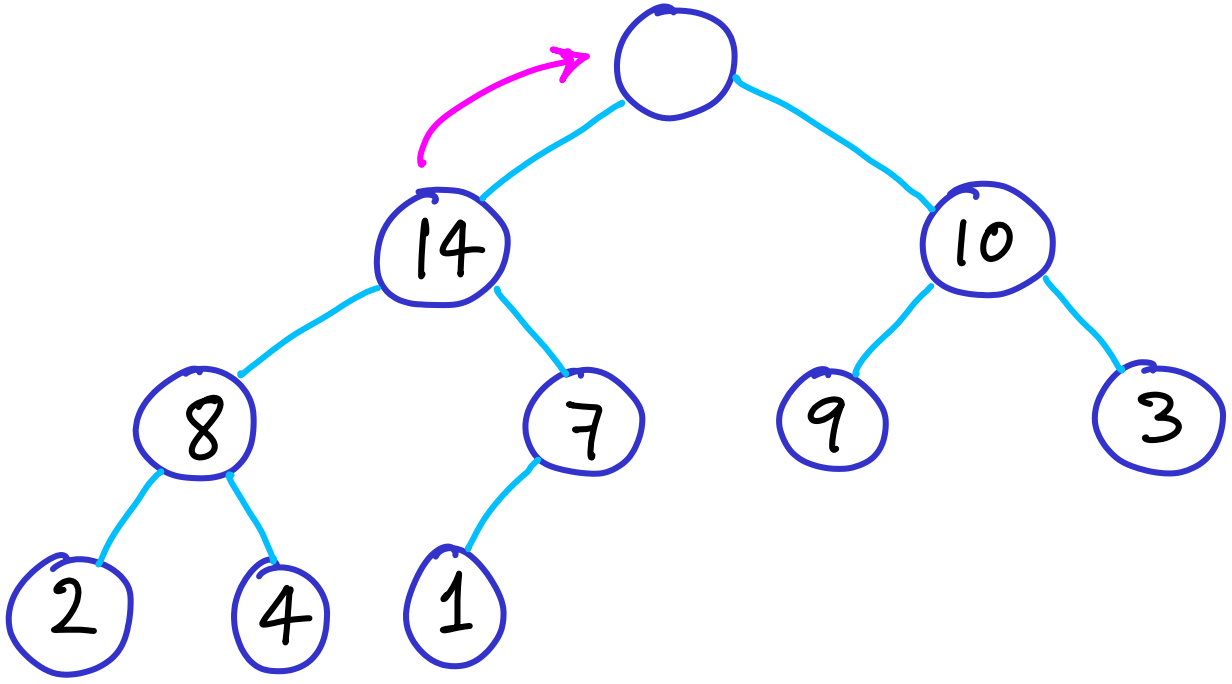
⋮

getting messy

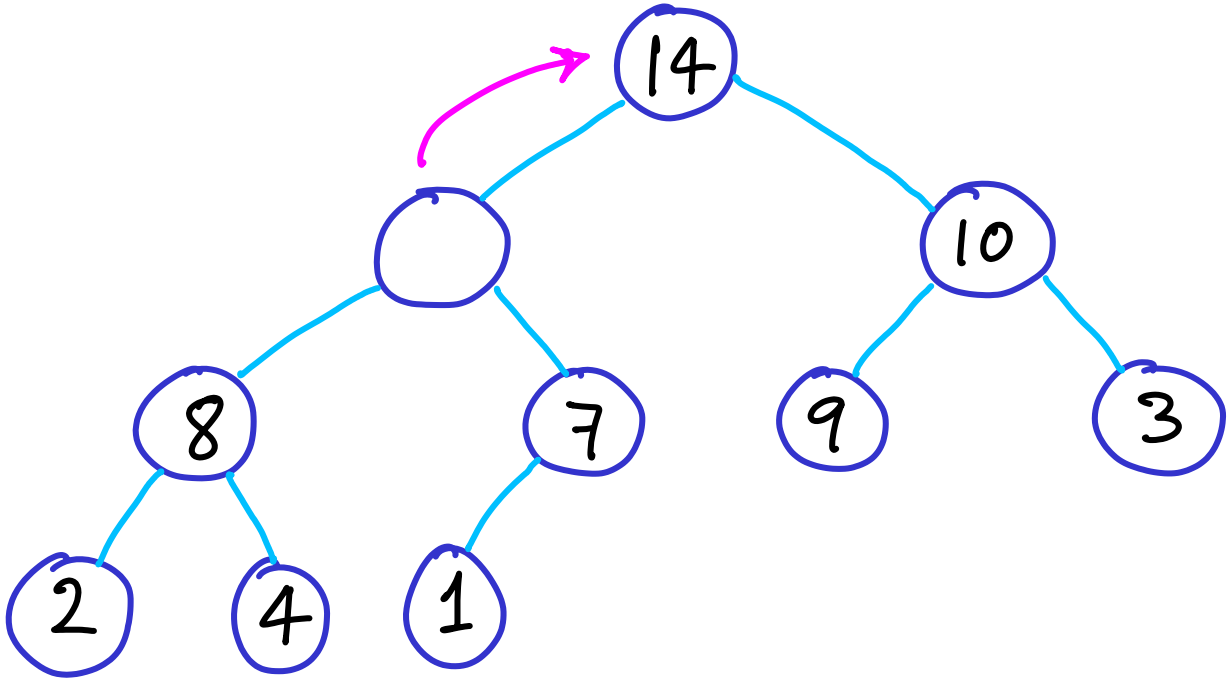




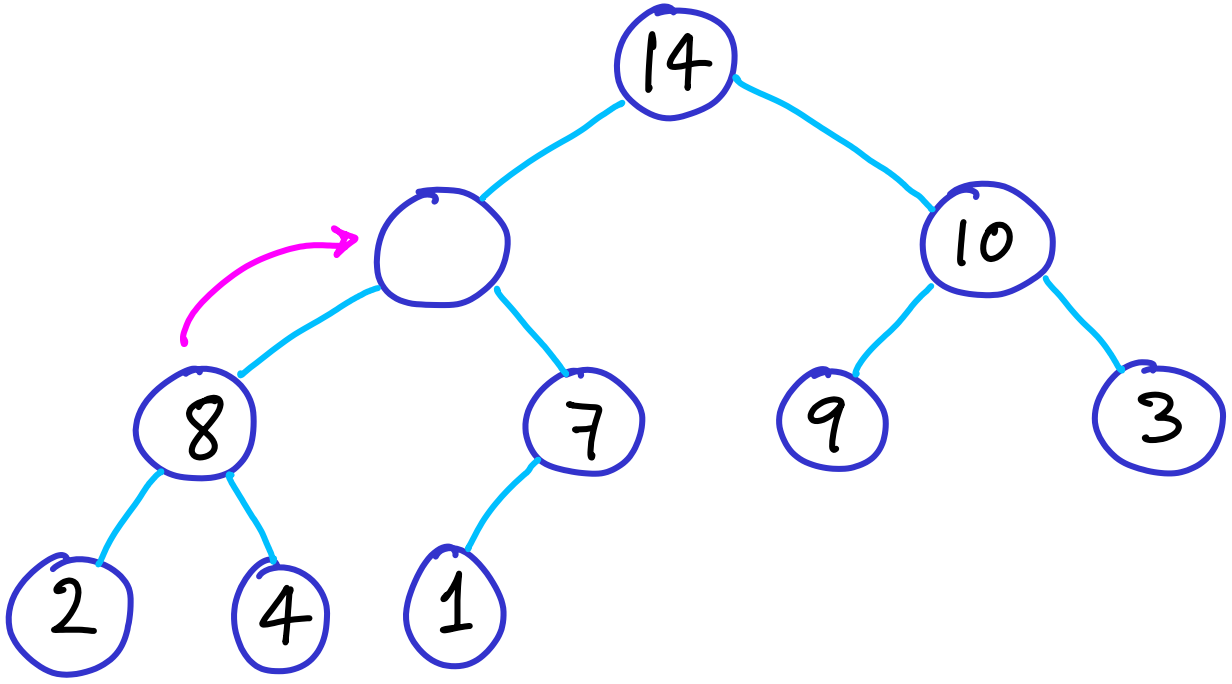
extract MAX



attempt to extract MAX & restore heap

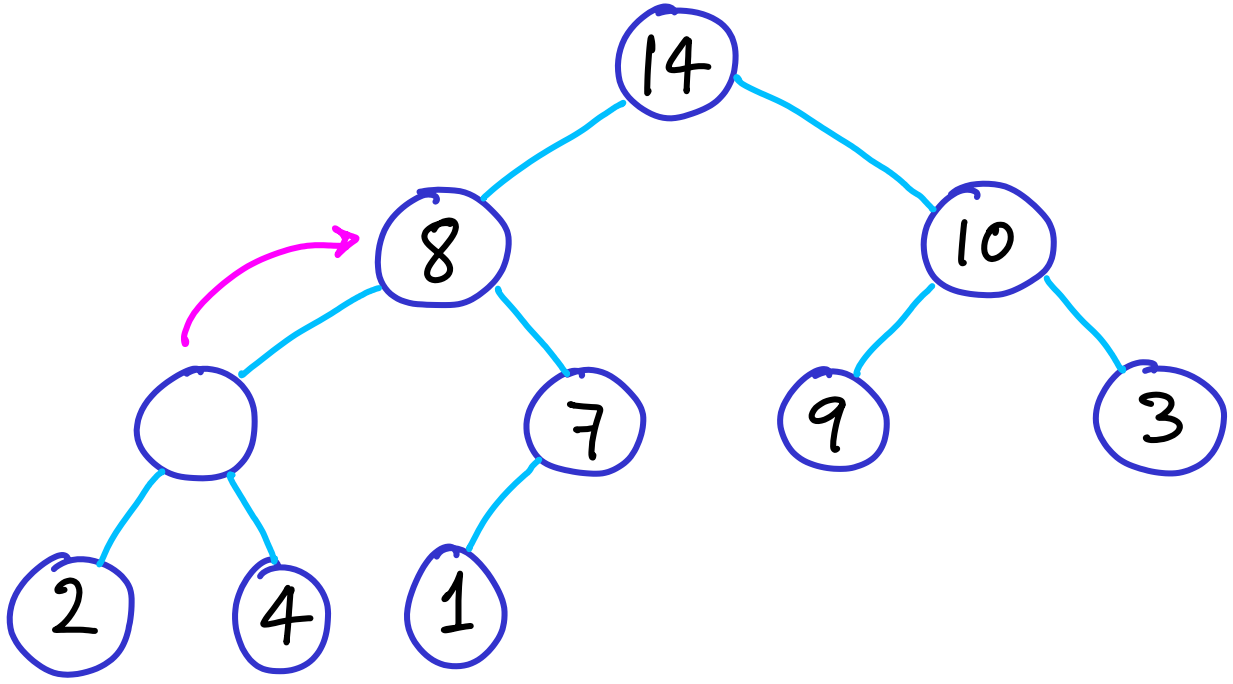


attempt to extract MAX & restore heap



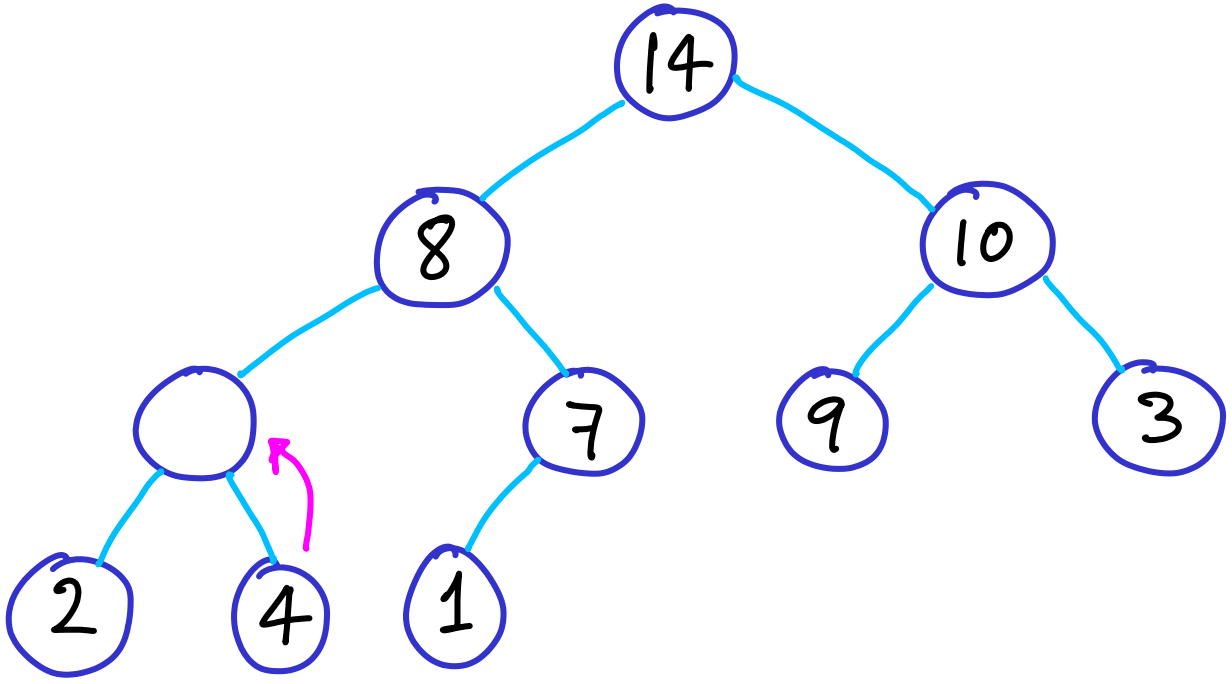
16

attempt to extract MAX & restore heap



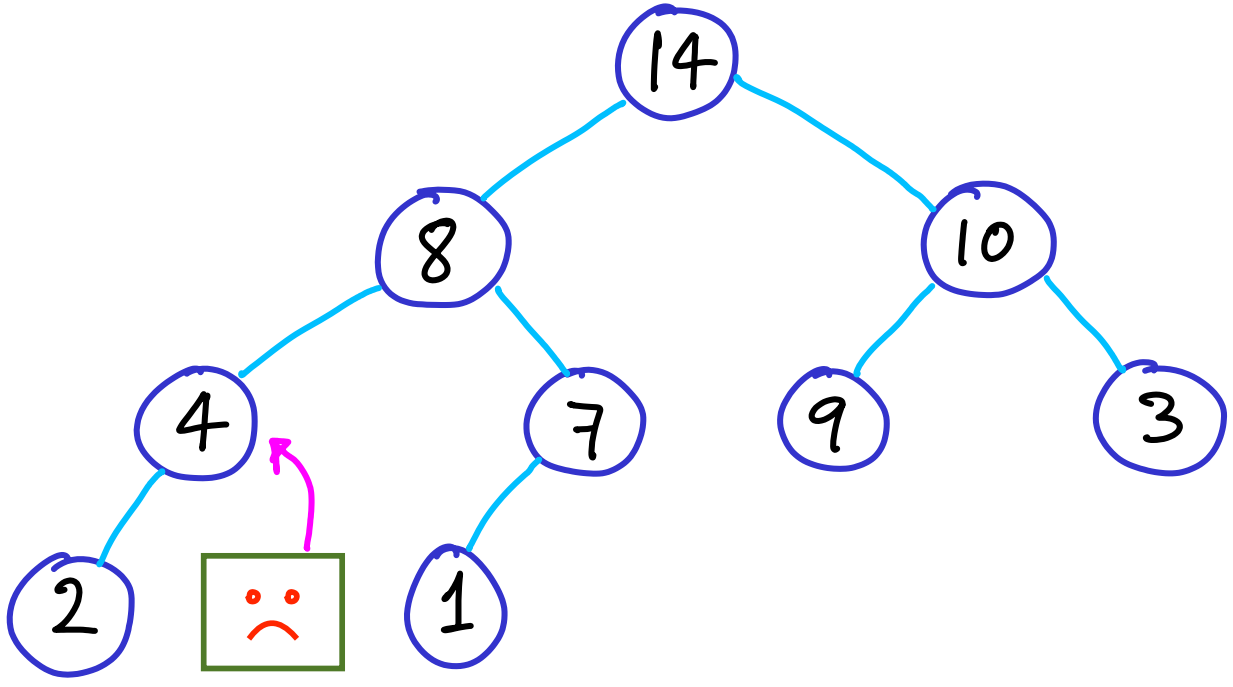
16

attempt to extract MAX & restore heap



16

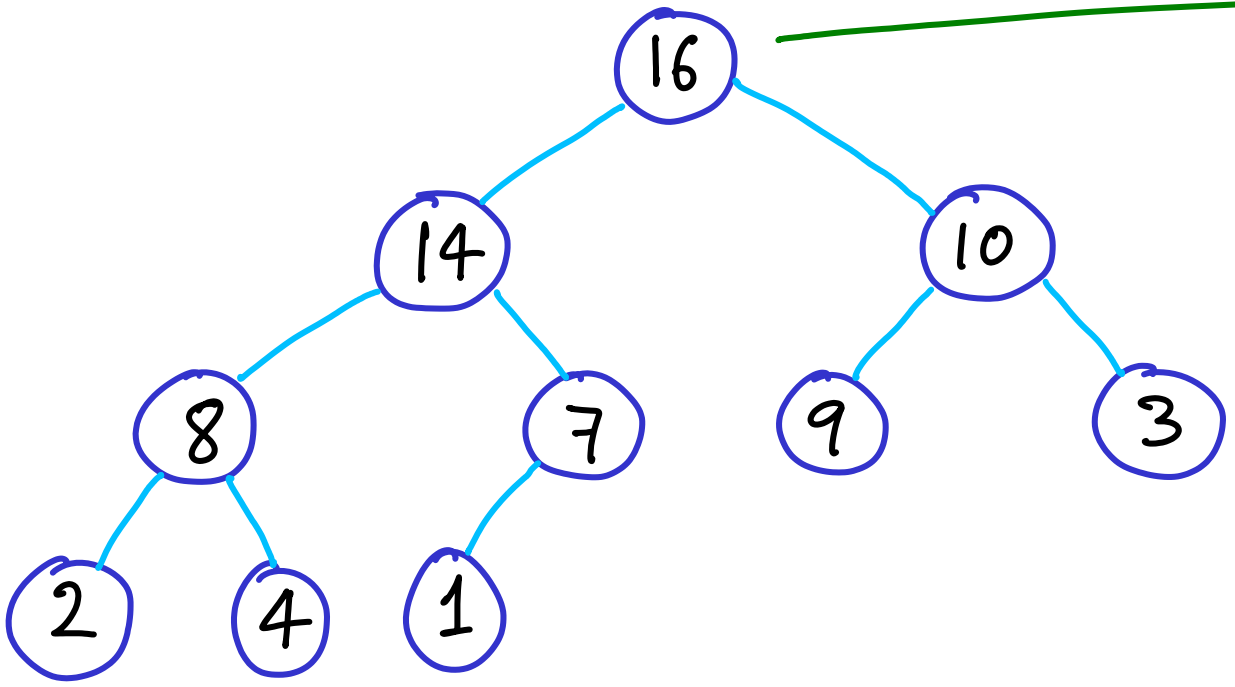
attempt to extract MAX & restore heap

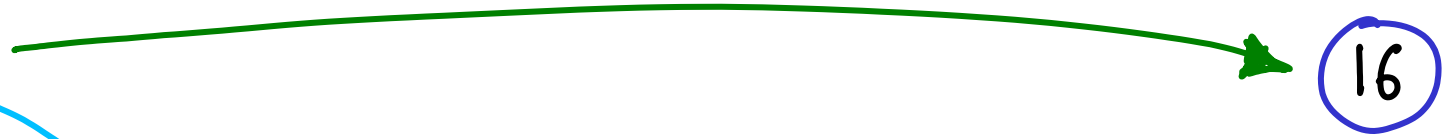


attempt to extract MAX & restore heap

↳ now what? .... failed

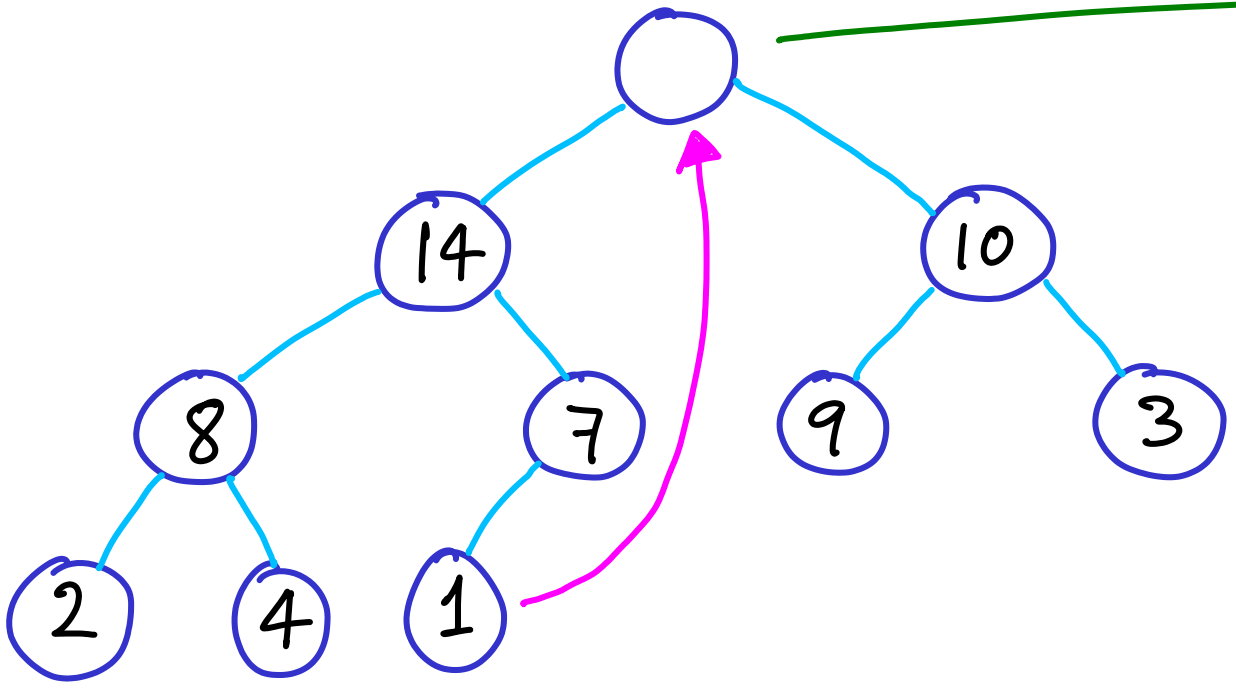
(we want to preserve the structure)





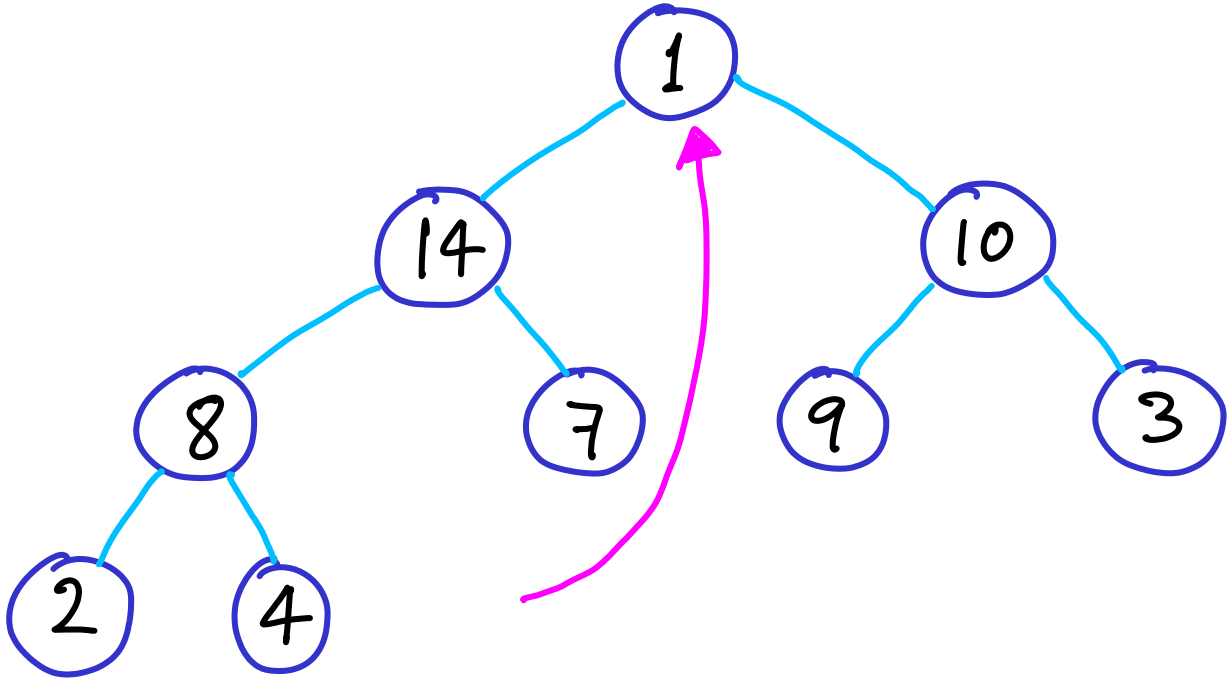
- extract MAX

Try again:
Modify extraction



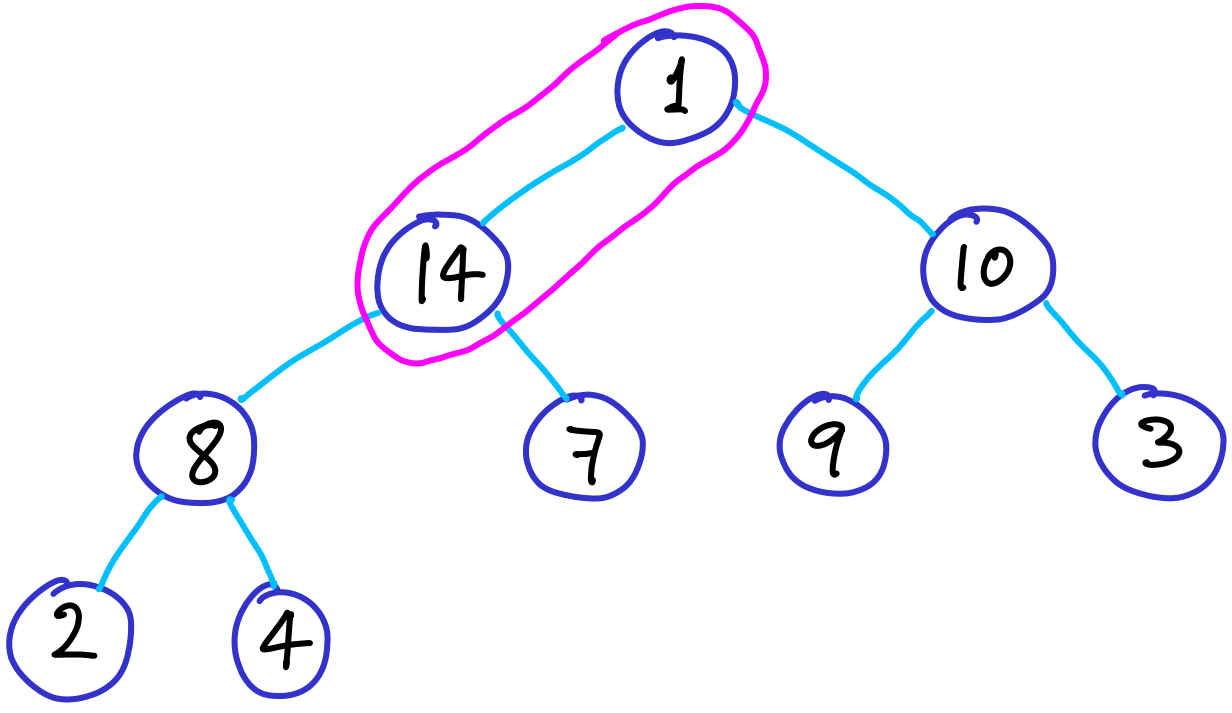
- extract MAX
- place rightmost leaf at top

(strange move... it's small!)



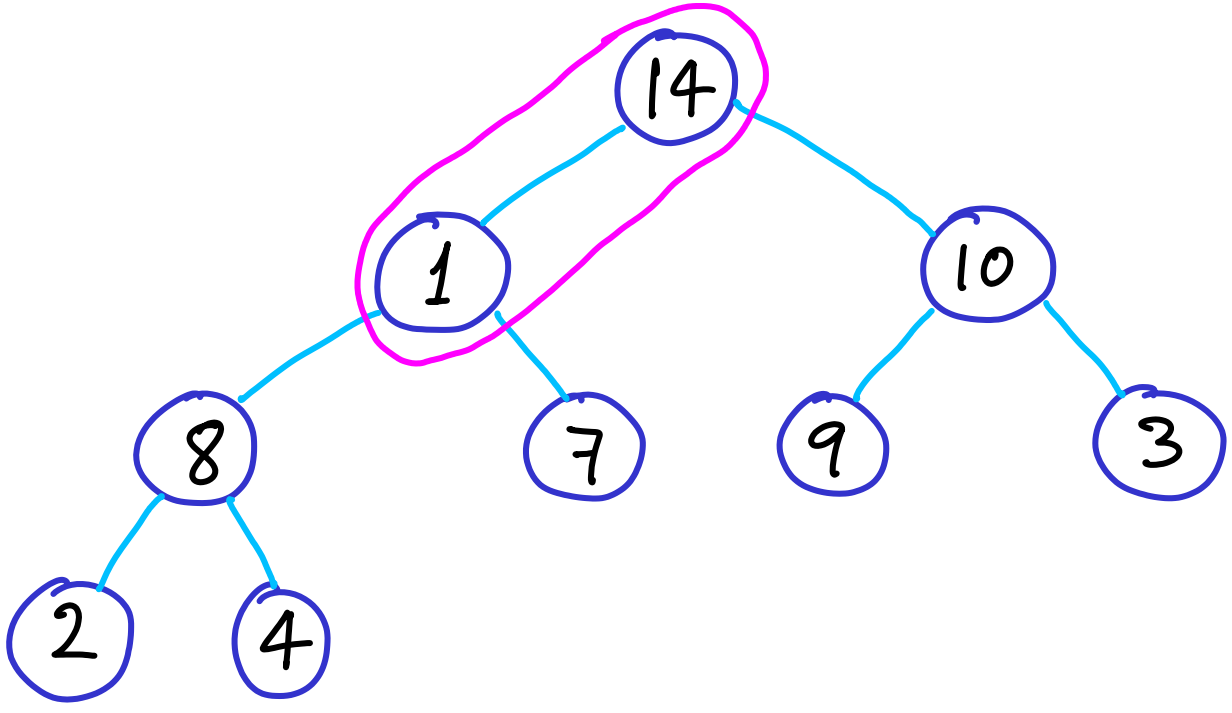
- extract MAX
- place rightmost leaf at top

(strange move... it's small!)



16

- extract MAX
- place rightmost leaf at top
  (strange move... it's small!)
- swap top w/ largest child
  (locally restore parent > child)



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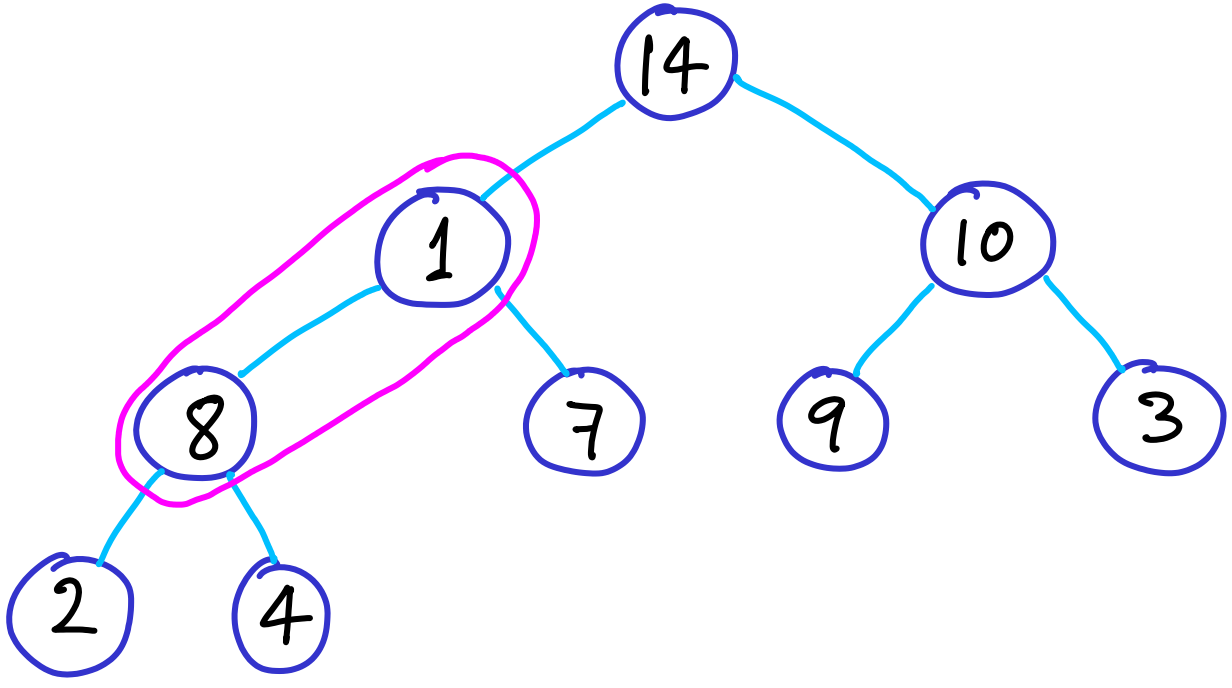
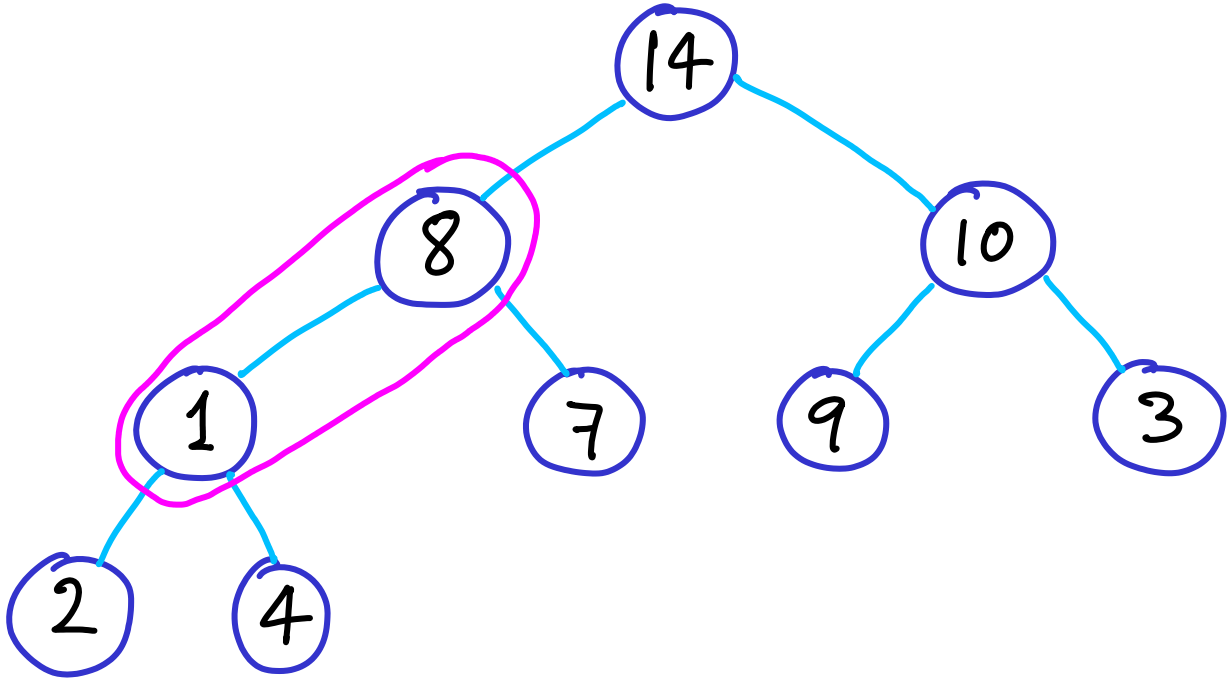


16

- extract MAX
- place rightmost leaf at top
  (strange move... it's small!)
- swap top w/ largest child
  (locally restore parent > child)
- repeat downward while needed



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- place rightmost leaf at top
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- swap top w/ largest child
  (locally restore parent > child)
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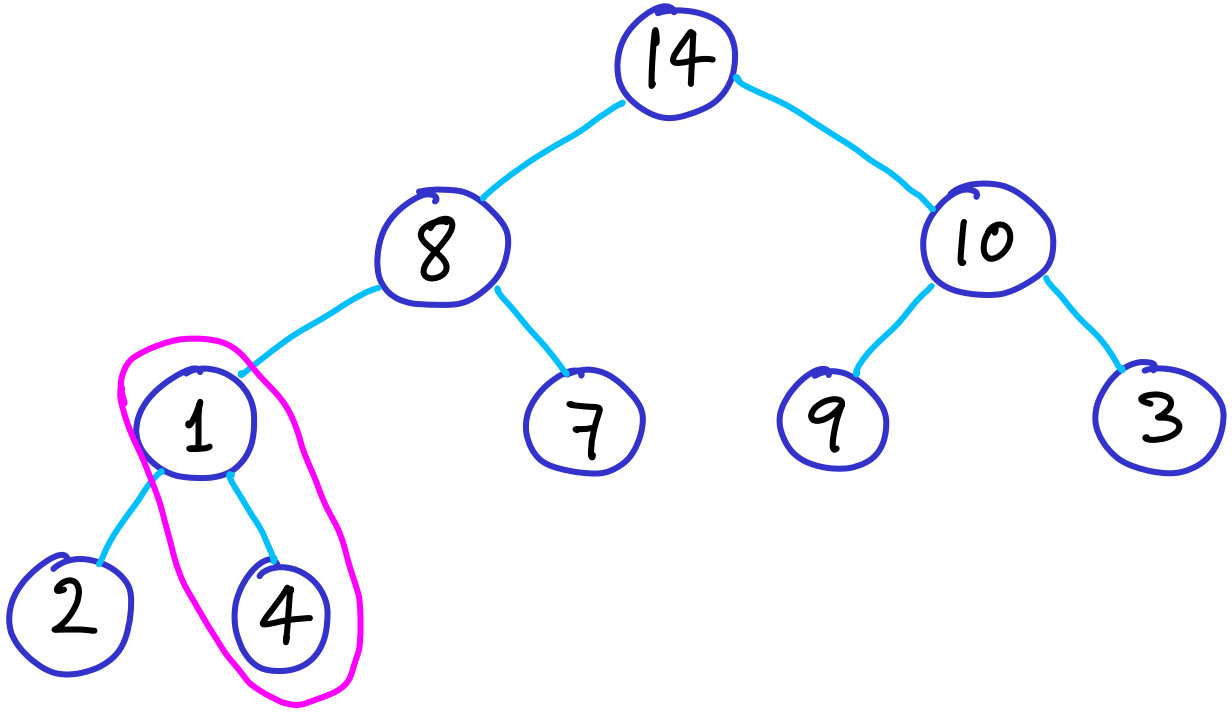

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- place rightmost leaf at top
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- swap top w/ largest child
  (locally restore parent > child)
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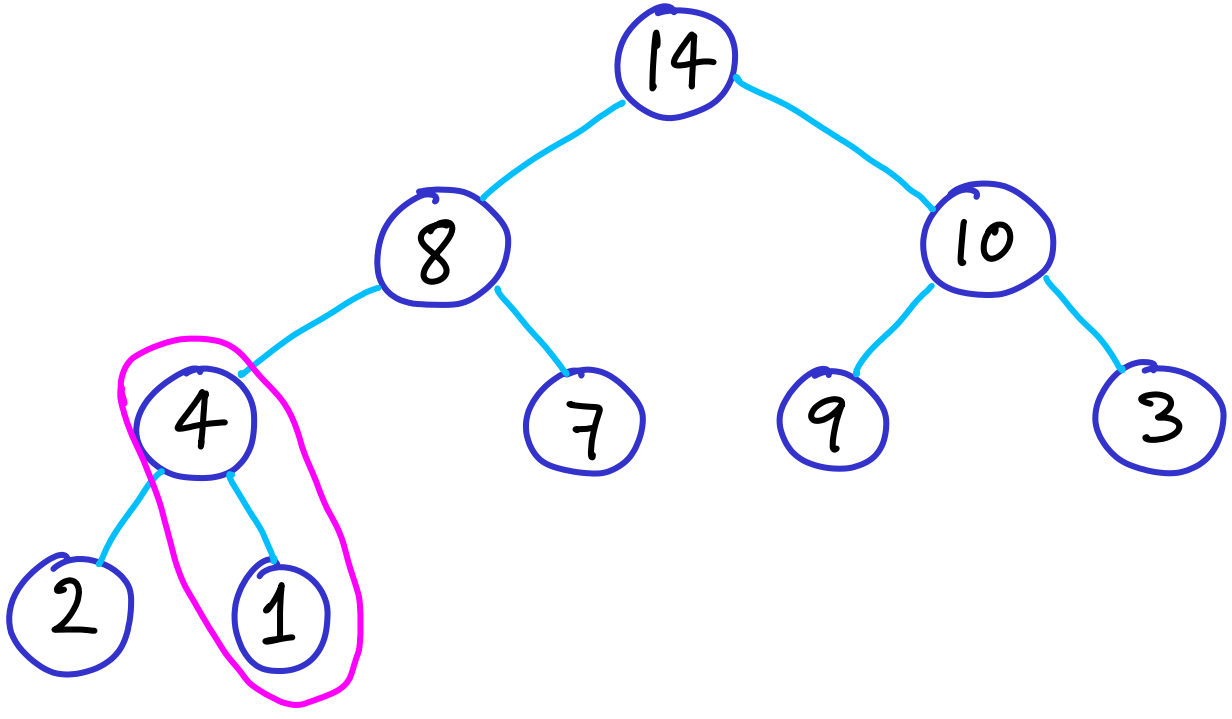

16

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  (locally restore parent > child)
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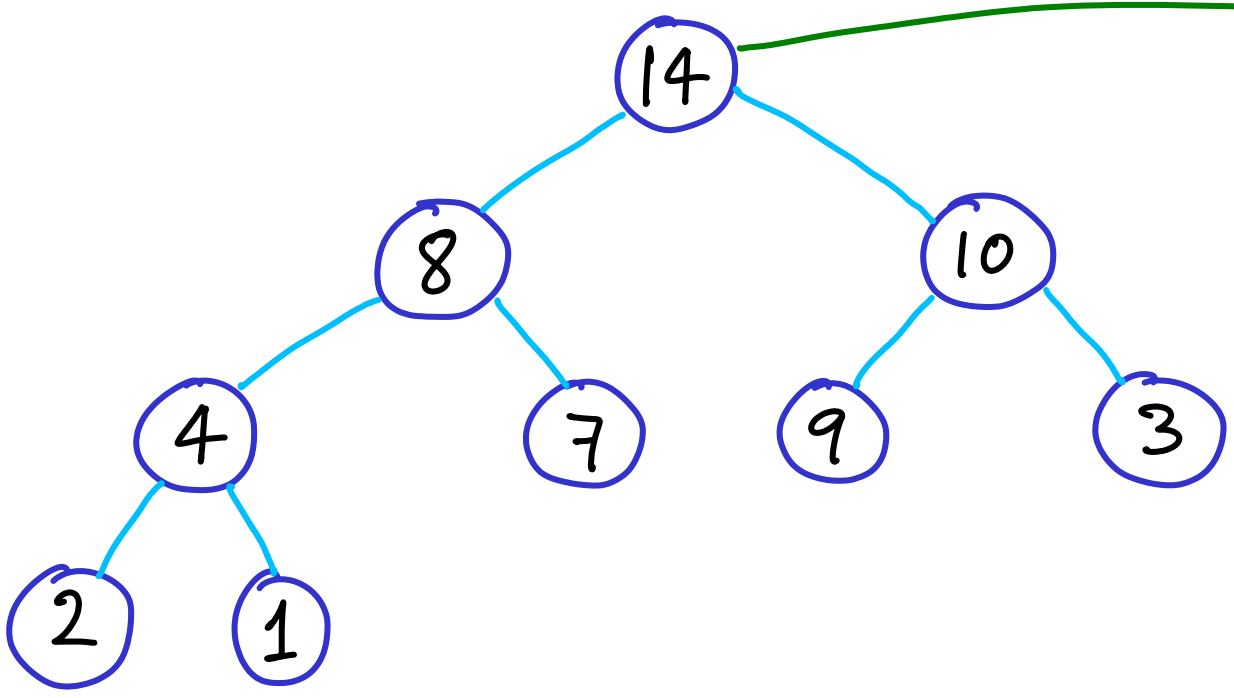
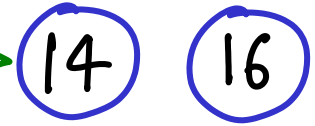


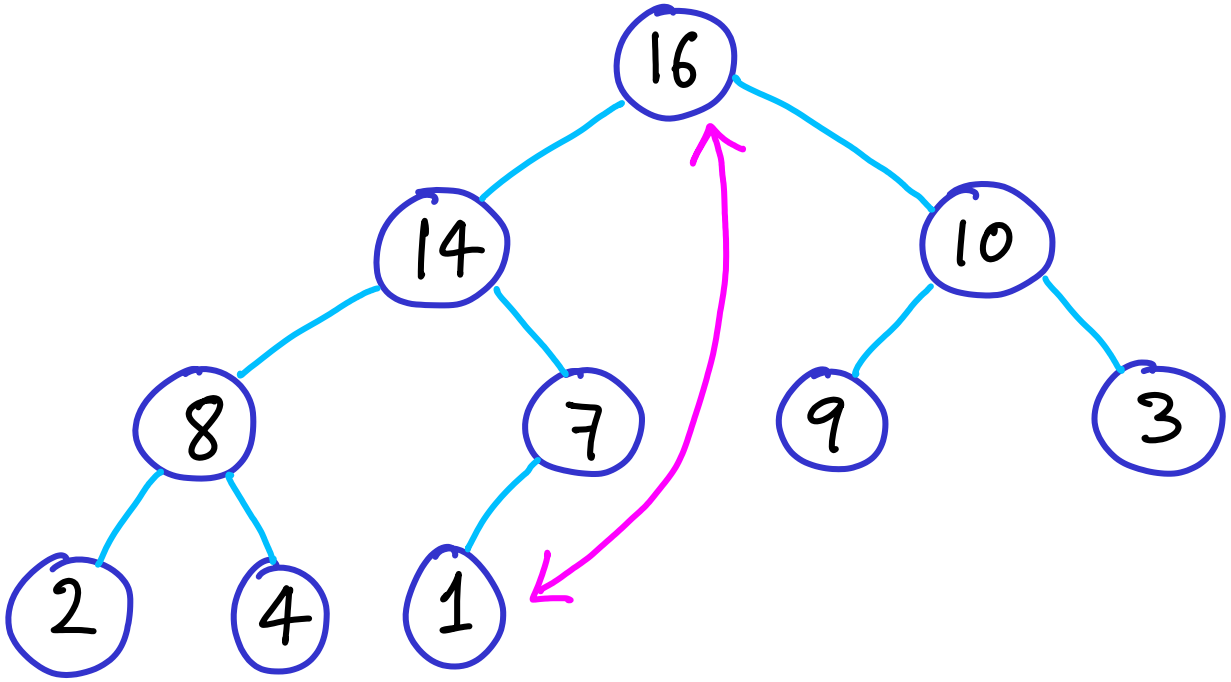


- extract MAX
- place rightmost leaf at top
  (strange move... it's small!)
- swap top w/ largest child
  (locally restore parent > child)
- repeat downward while needed

time?

height of heap? → $\Theta(\log n)$

Ready to extract new MAX

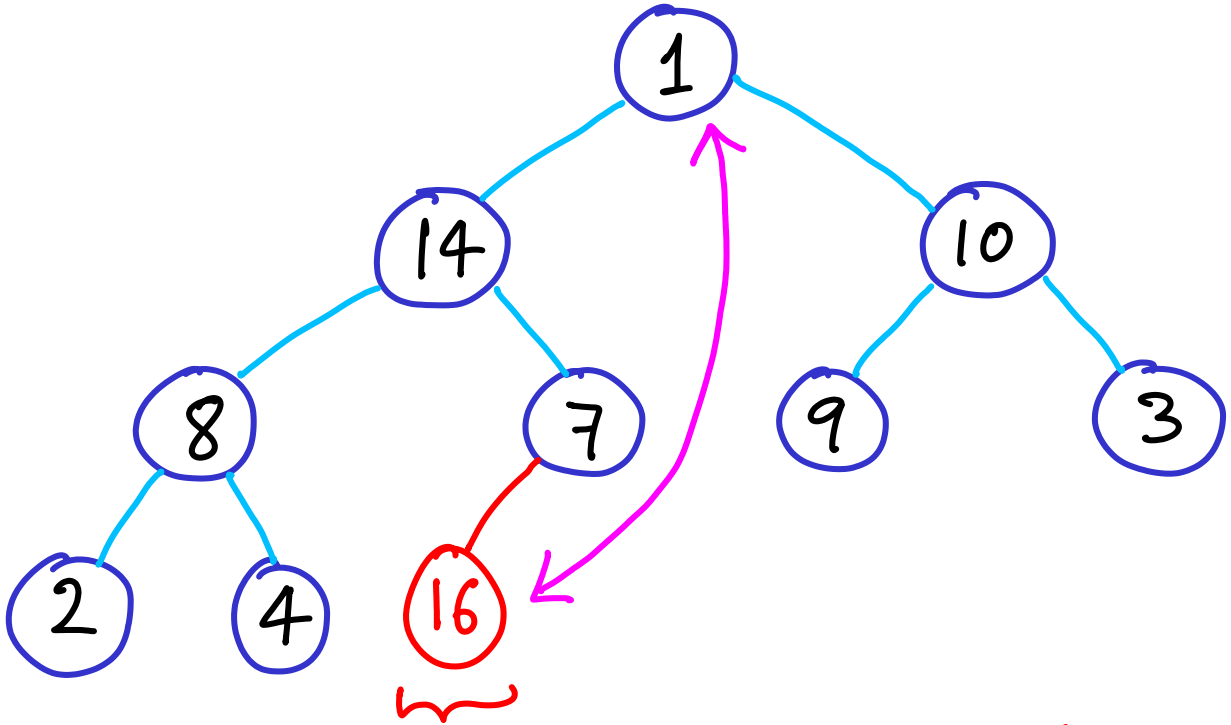


To work in-place

when extracting MAX

we can swap it w/ leaf.

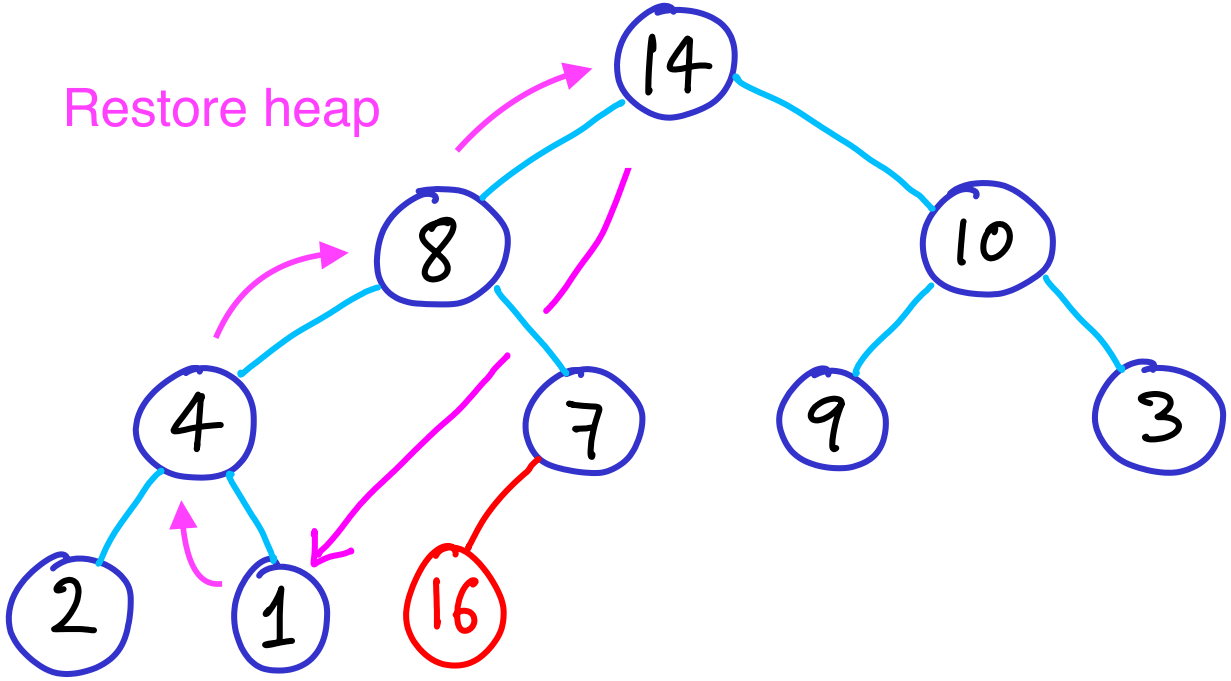
One more time:
More space-efficient



To work in-place
when extracting MAX
we can swap it w/ leaf.

instead of deleting this node, just ignore it.

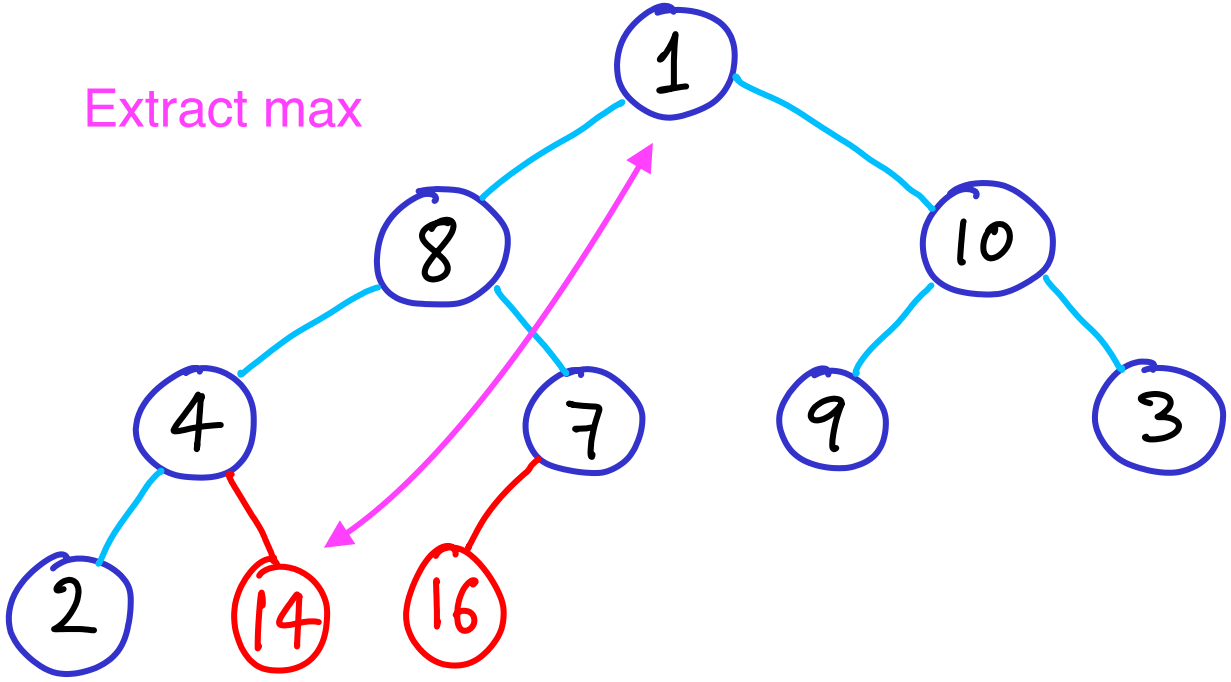
Notice MAX is stored at the max index of our array.



To work in-place

when extracting MAX

we can swap it w/ leaf.

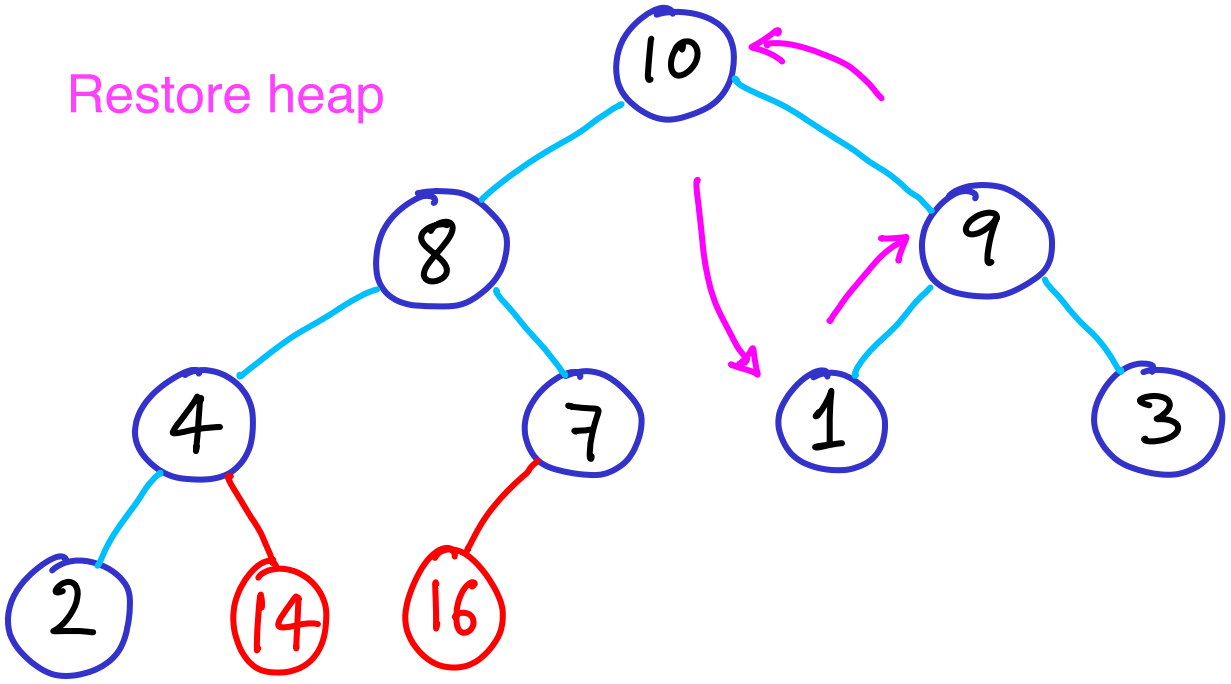


To work in-place

when extracting MAX

we can swap it w/ leaf.

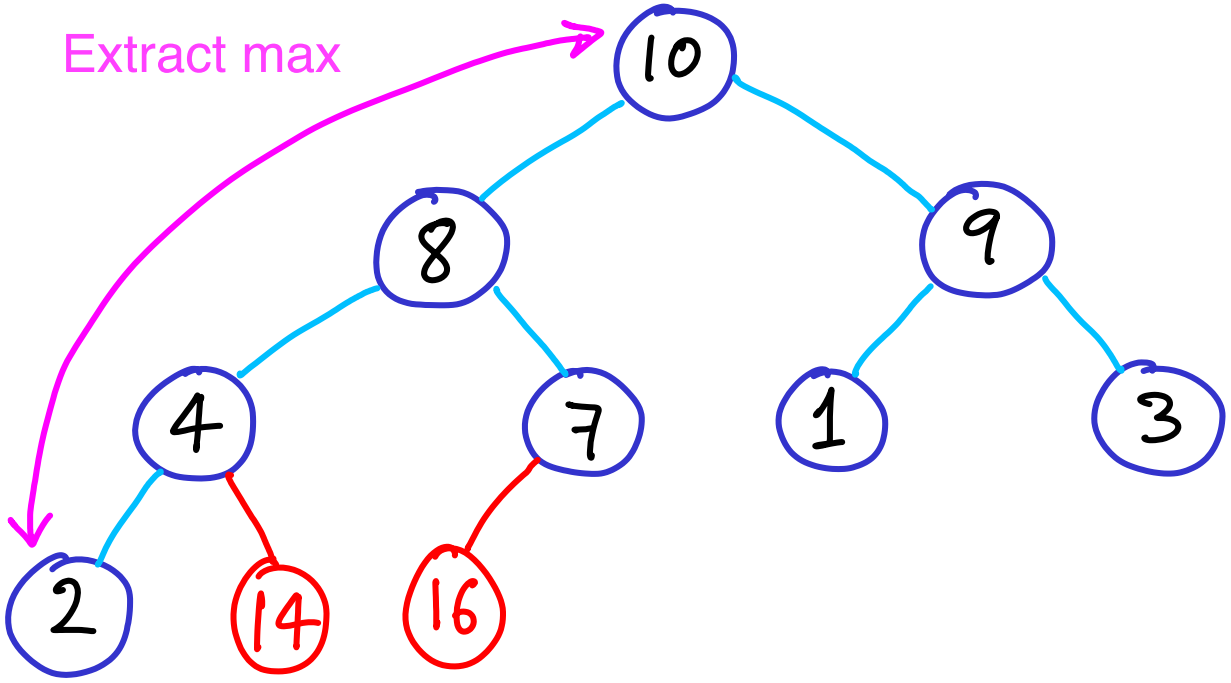
Restore heap



To work in-place

when extracting MAX

we can swap it w/ leaf.



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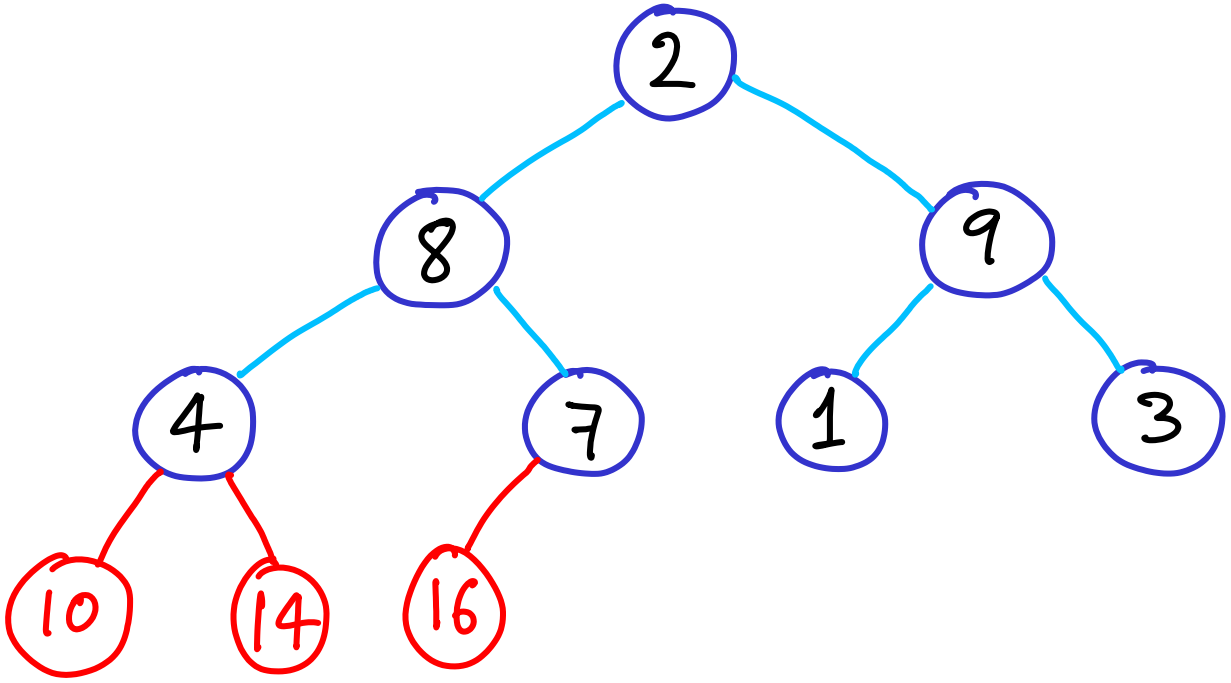

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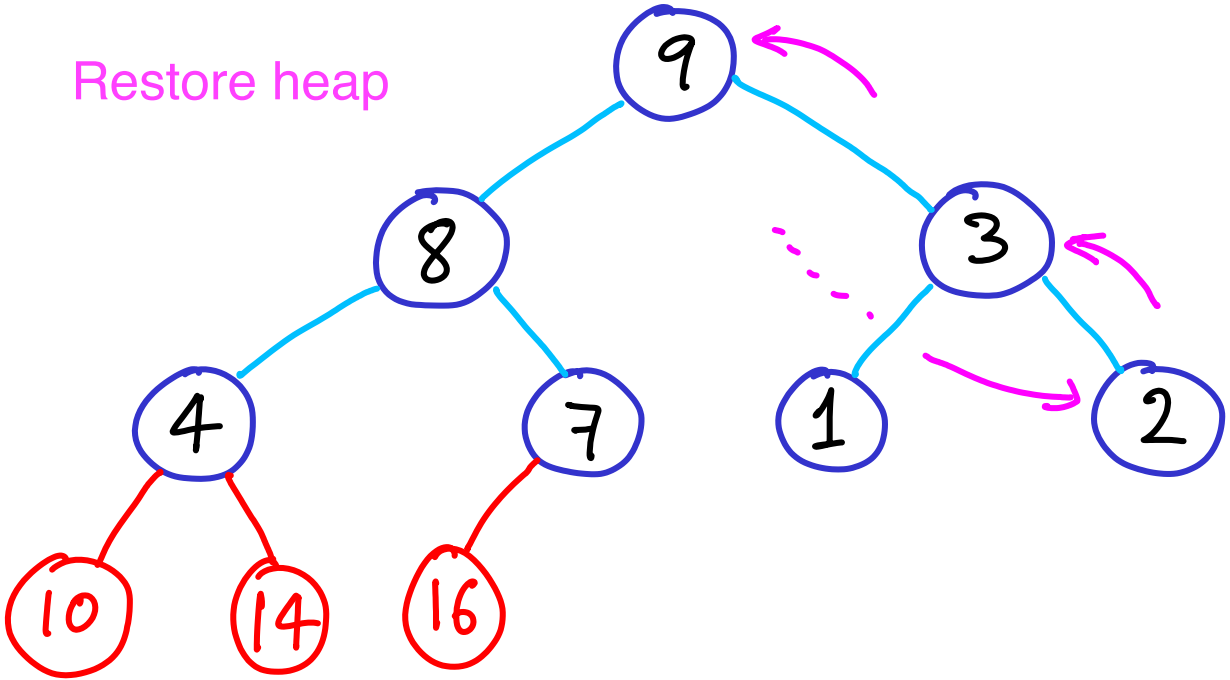

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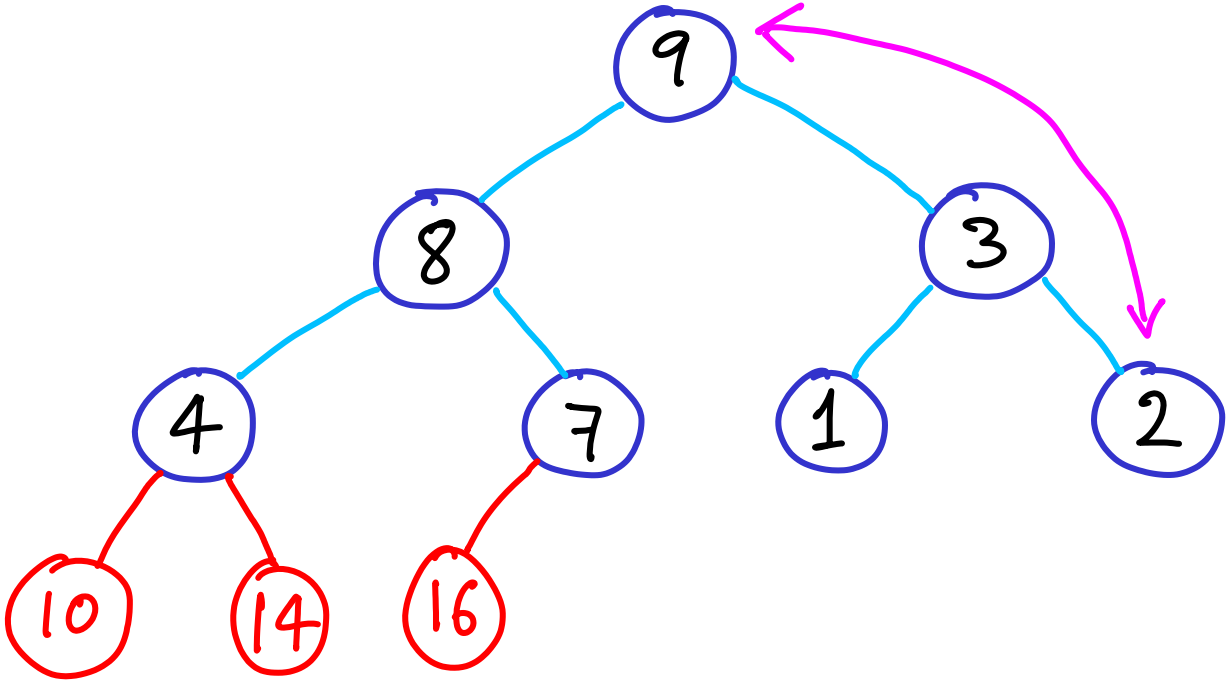

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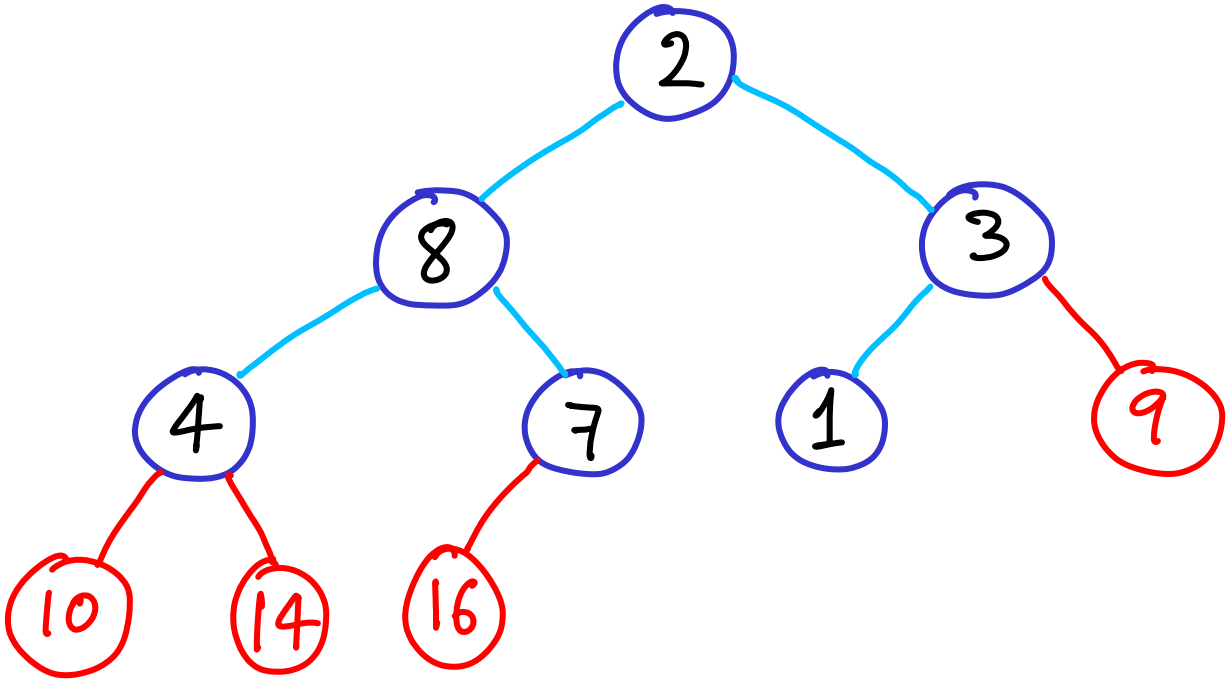

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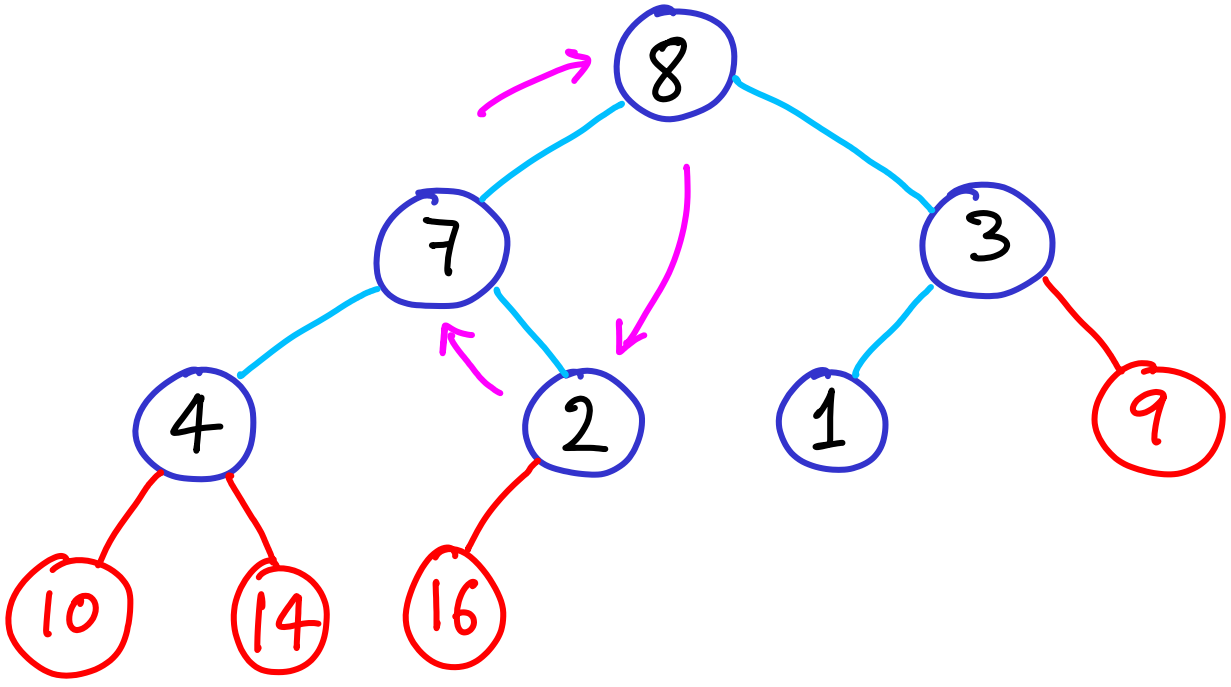

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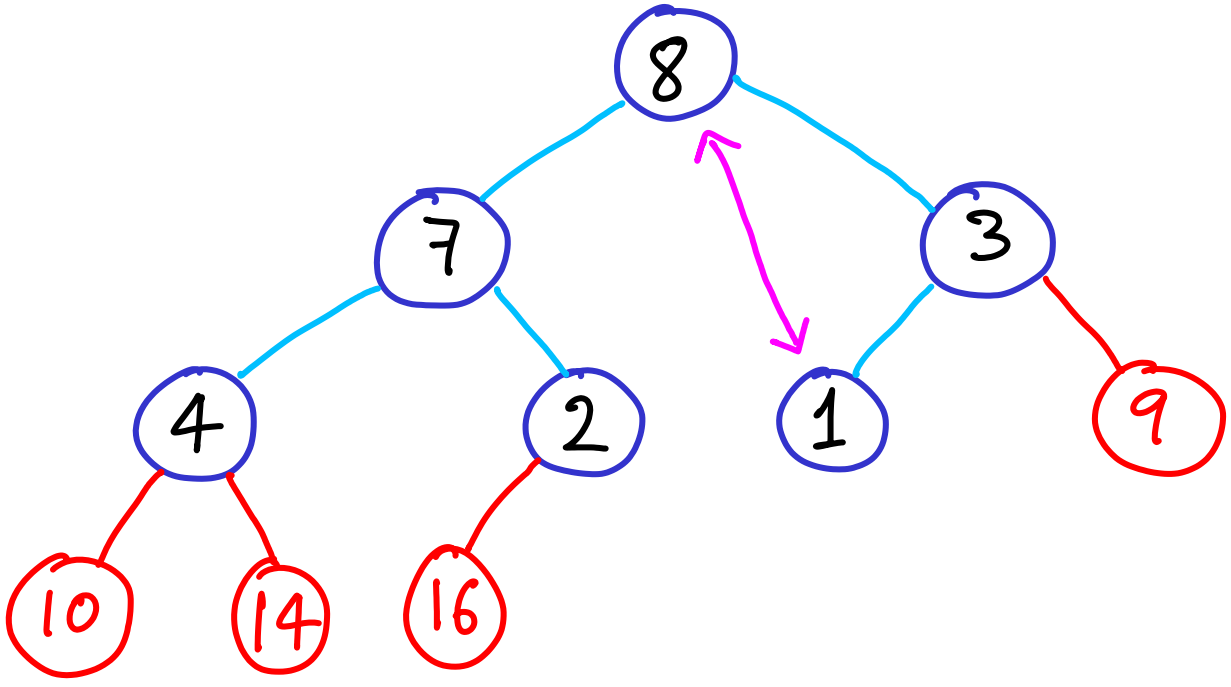

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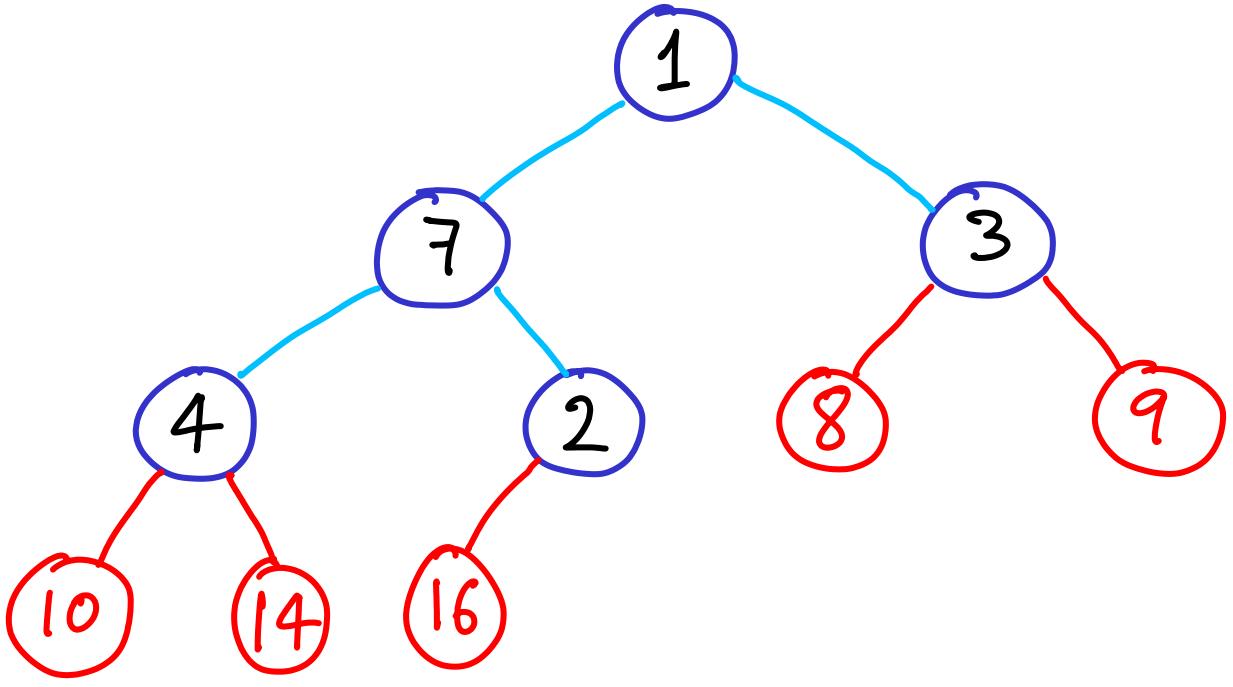

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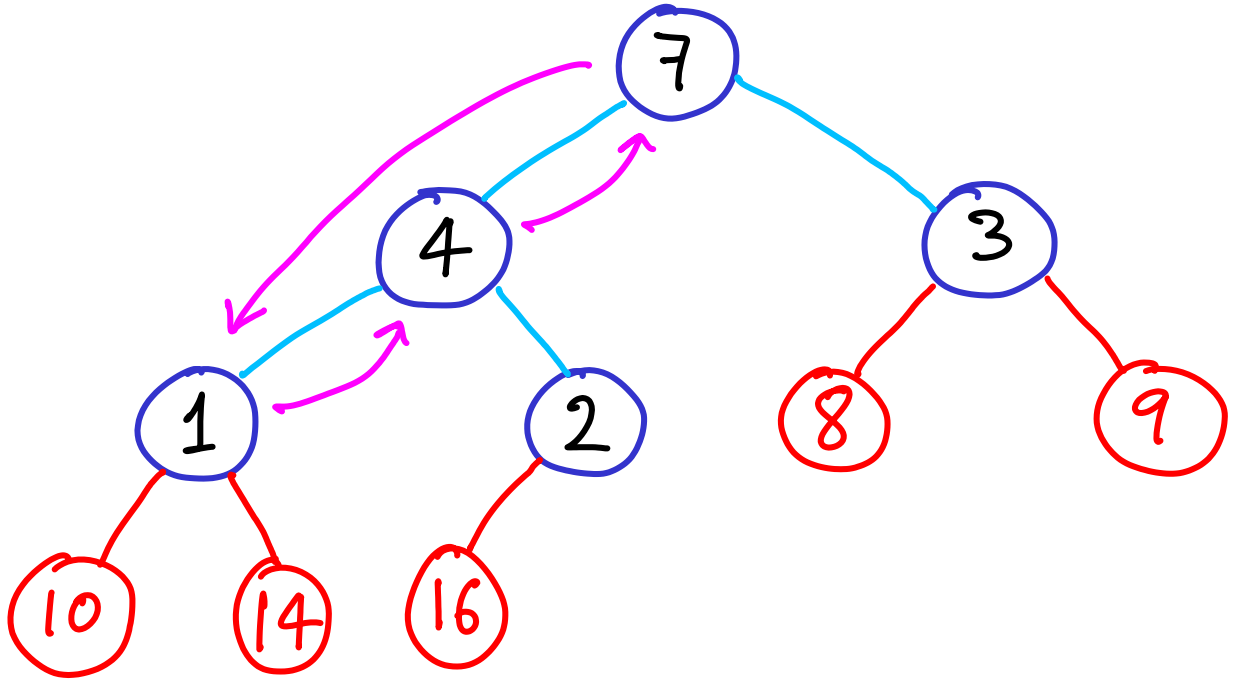

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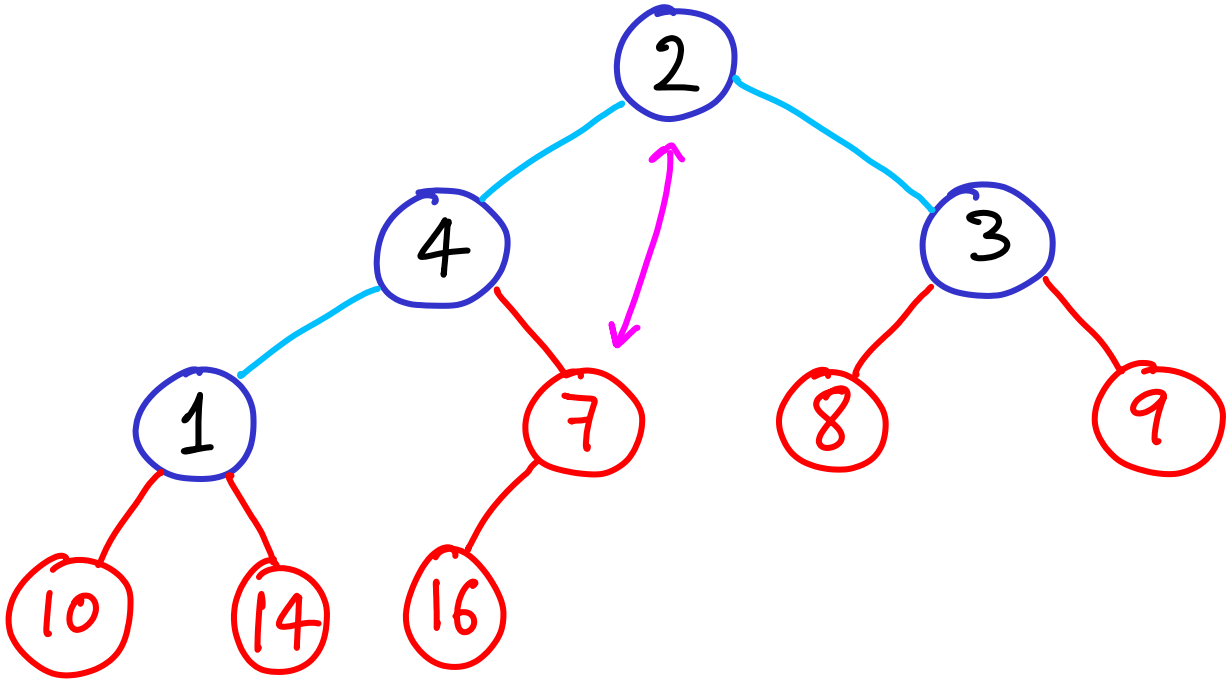

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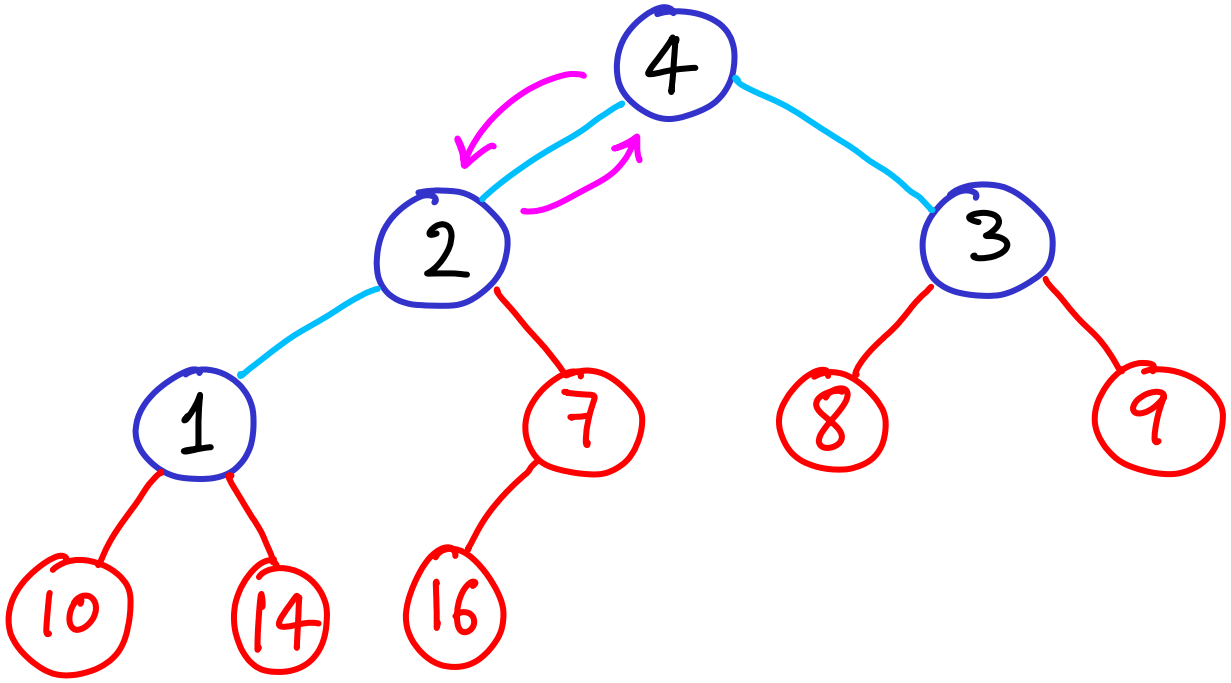

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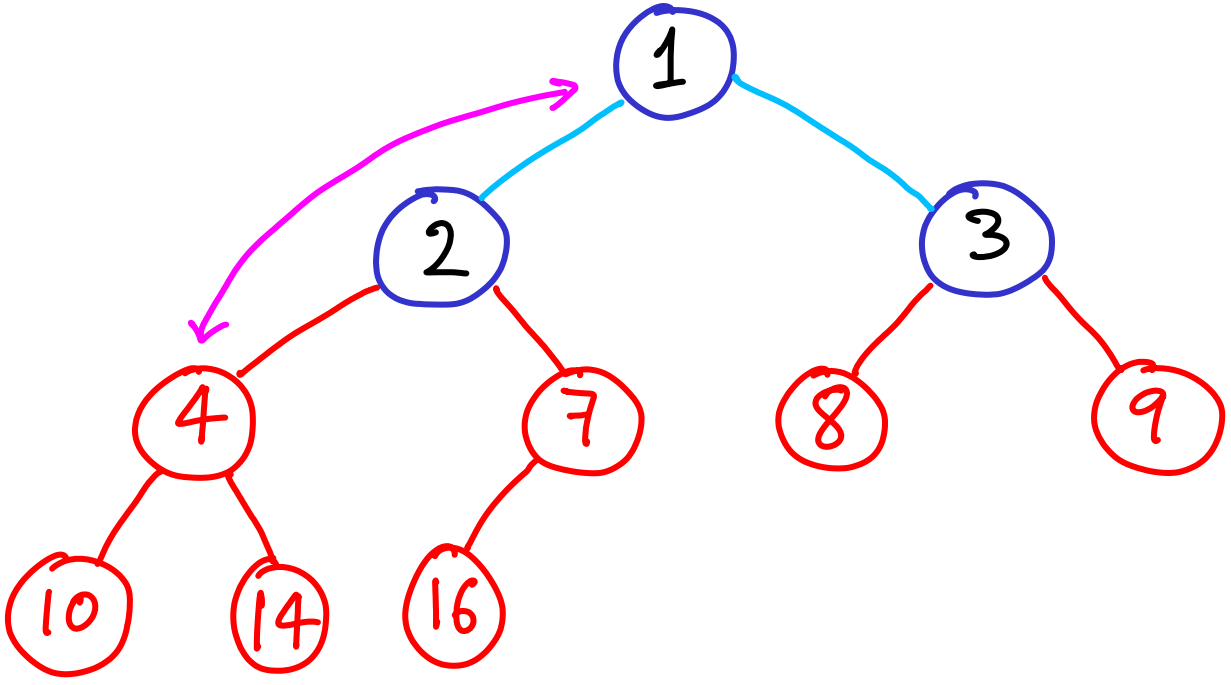

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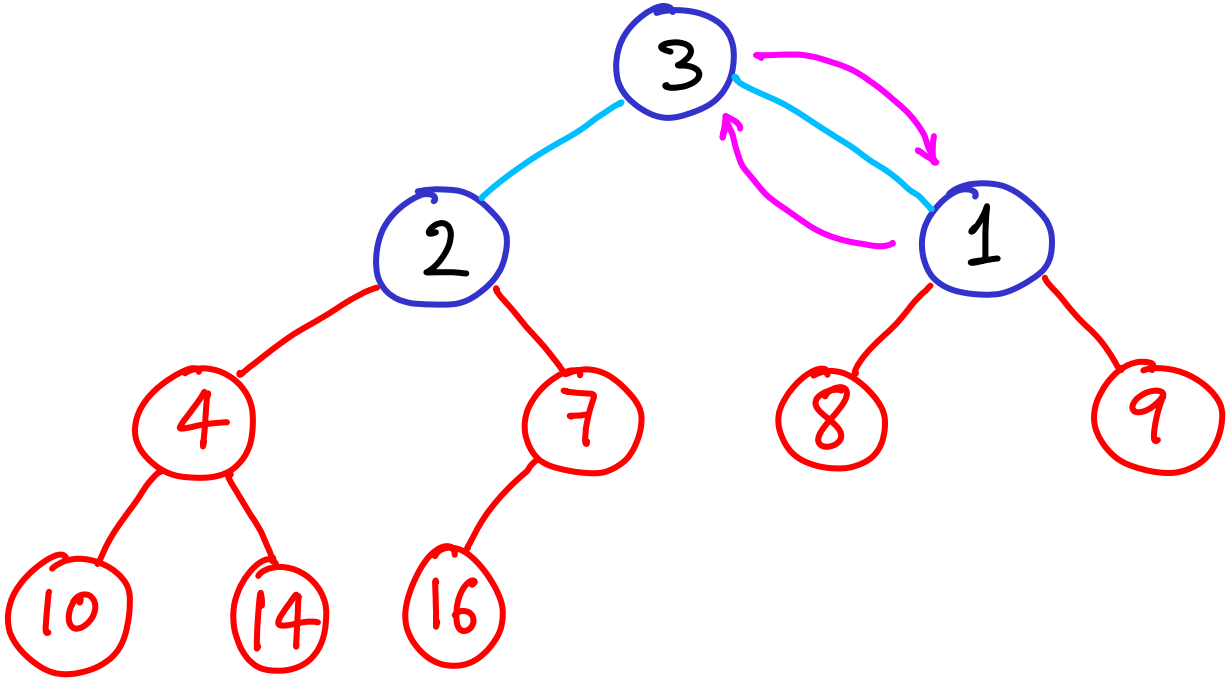

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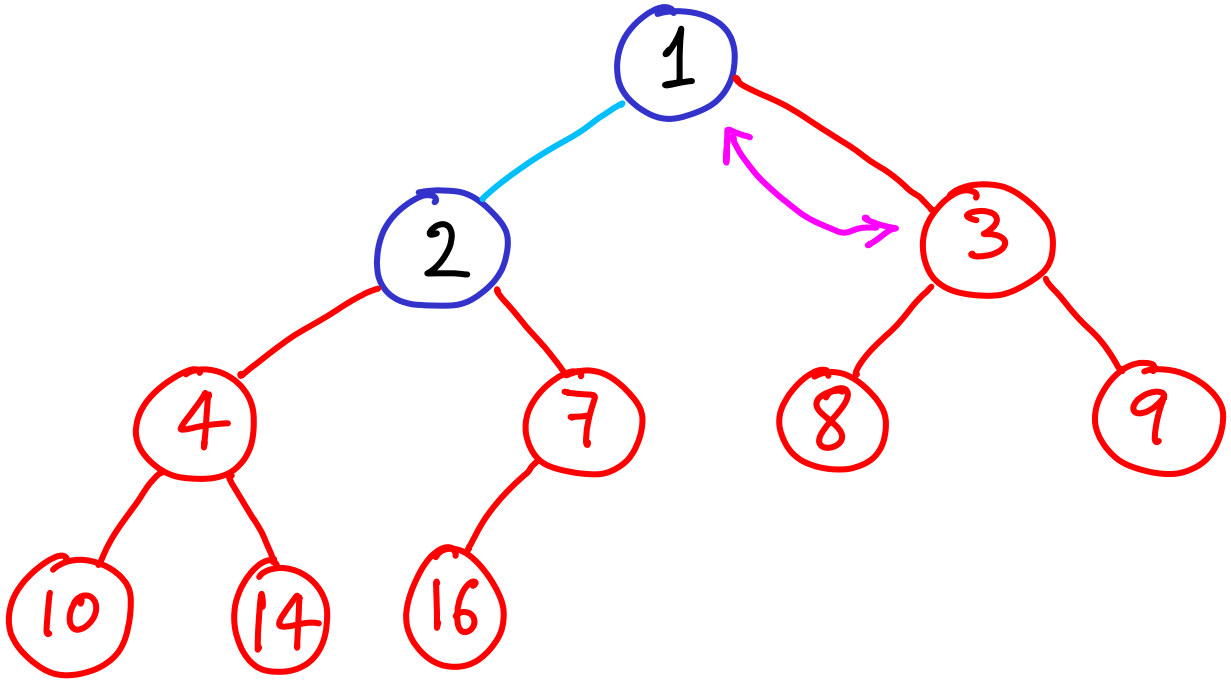
when extracting MAX

we can swap it w/ leaf.



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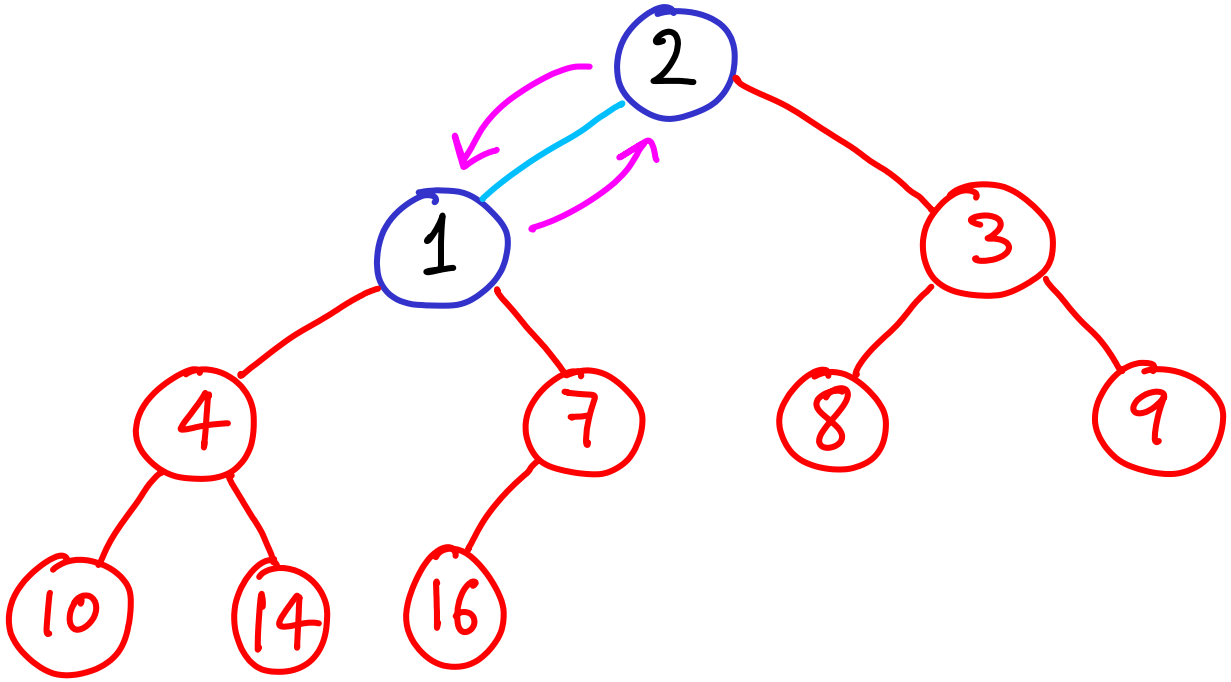

To work in-place

when extracting MAX

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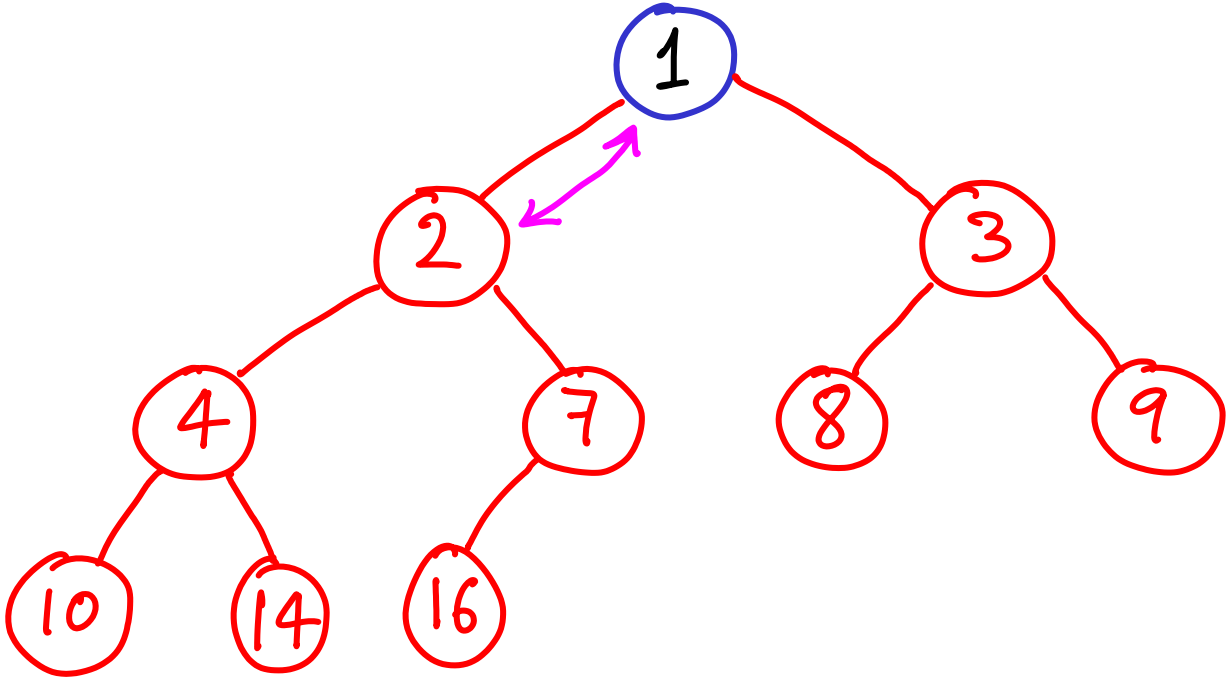
DONE



To work in-place

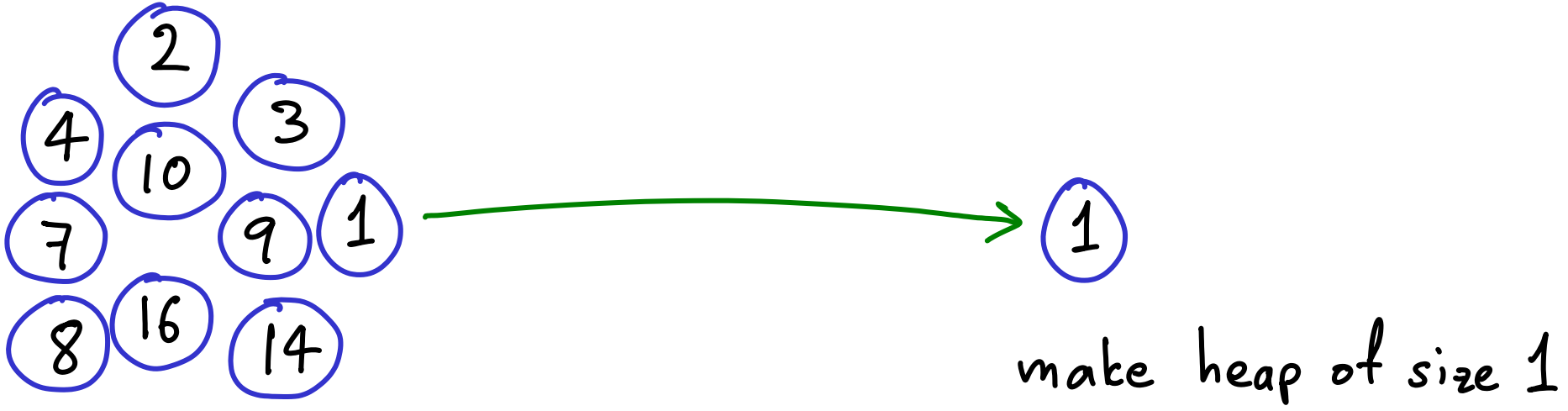
when extracting MAX

we can swap it w/ leaf.

So we can extract MAX & maintain a heap, in $O(\log n)$ time.

↳ do this n times → $O(n \log n)$

But how did we have a heap in the first place?



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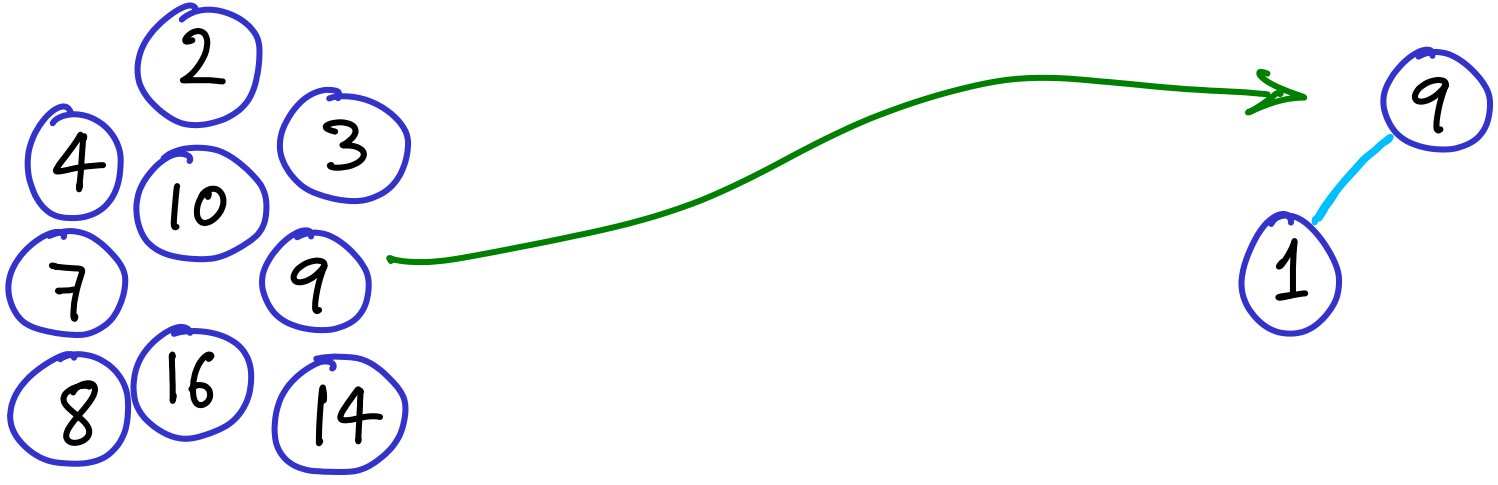

trivially get heap of size 2, possibly w/ a swap

So we can extract MAX & maintain a heap, in $O(\log n)$ time.

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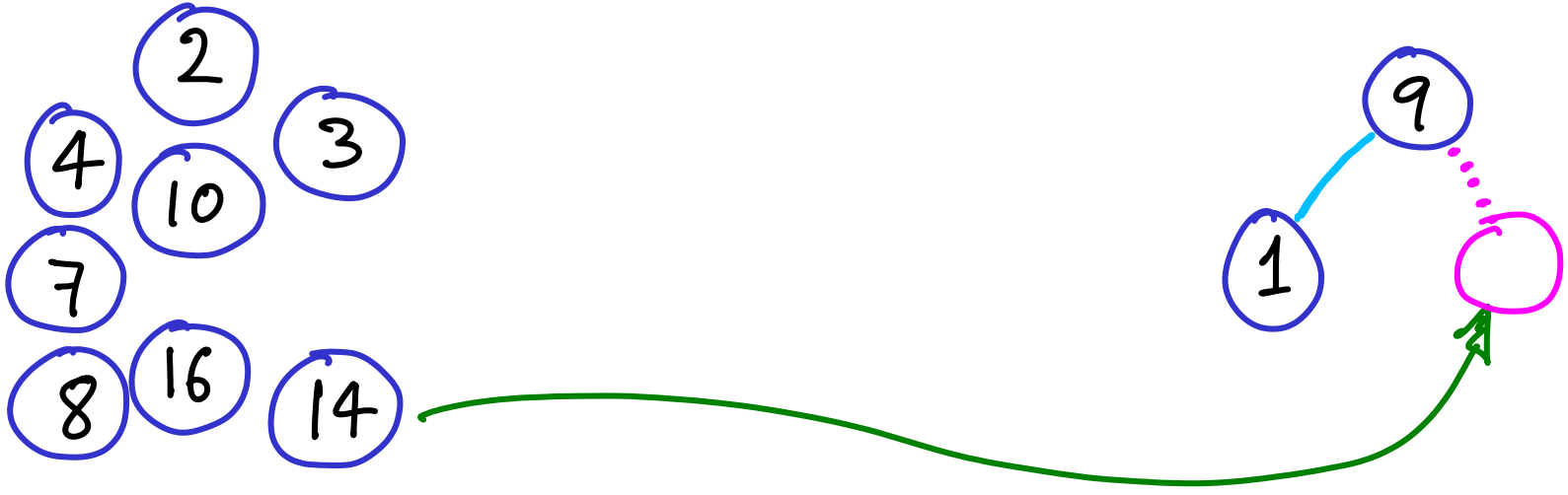
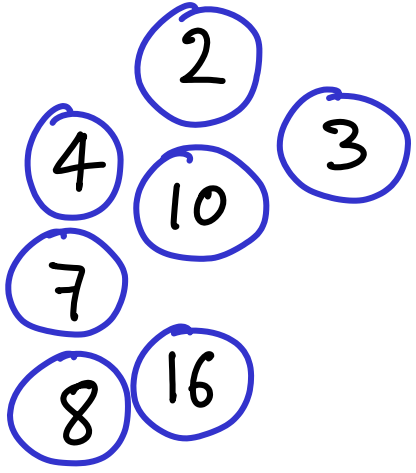


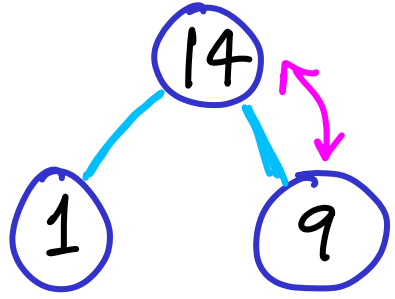
generally, insert new element as rightmost leaf in lowest level

So we can extract MAX & maintain a heap, in $O(\log n)$ time.

↳ do this n times $\rightarrow$ $O(n \log n)$

But how did we have a heap in the first place?



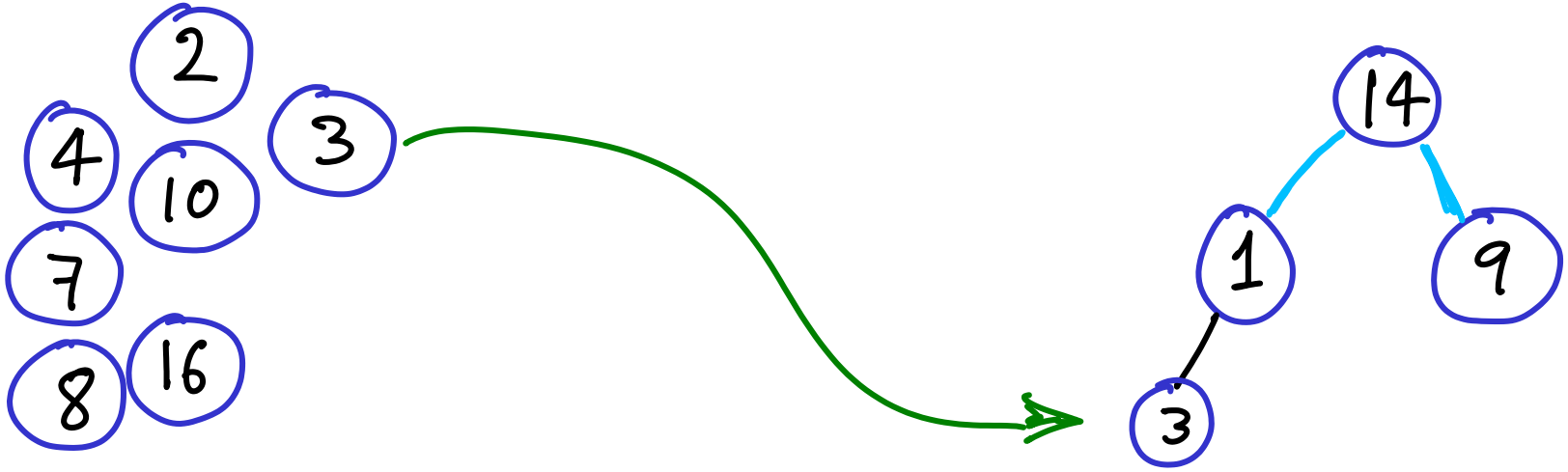


then repeat swapping while required

So we can extract MAX & maintain a heap, in $O(\log n)$ time.

↳ do this n times → $O(n \log n)$

But how did we have a heap in the first place?



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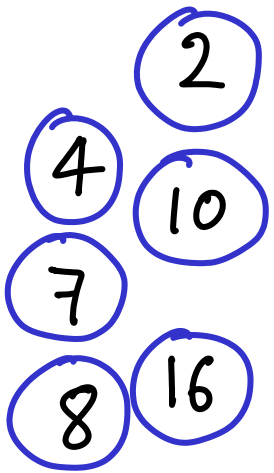
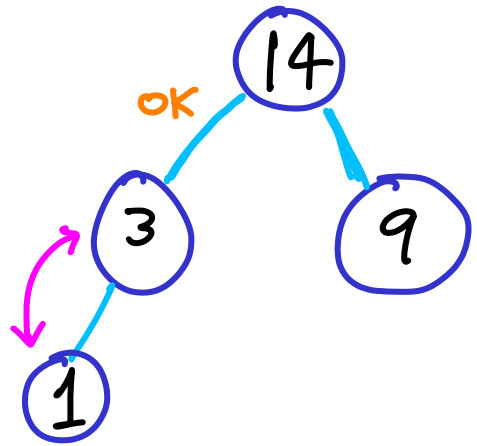
↳ do this n times → O(n log n)

But how did we have a heap in the first place?





So we can extract MAX & maintain a heap, in $O(\log n)$ time.

↳ do this n times $\rightarrow$ $O(n \log n)$

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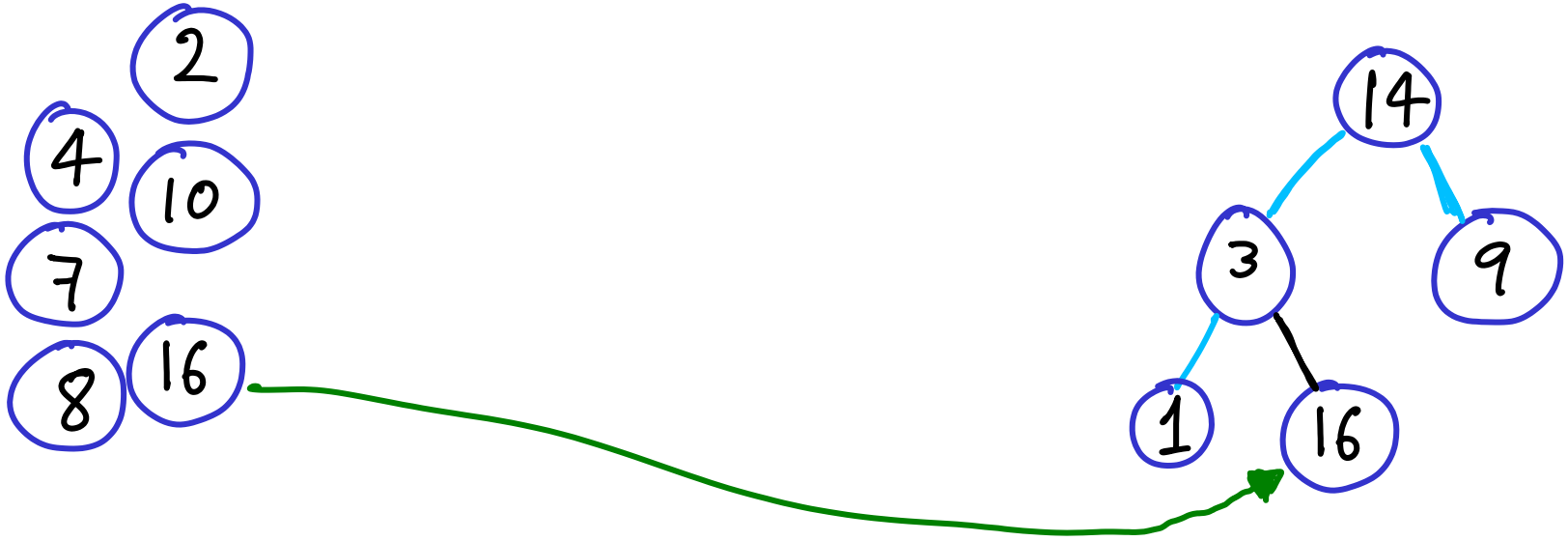

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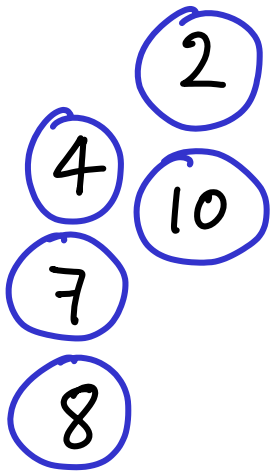

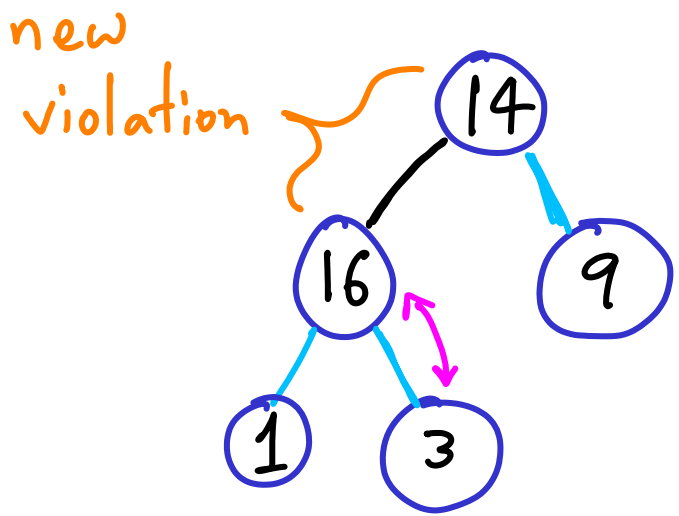

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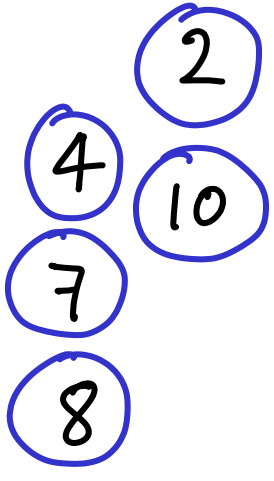
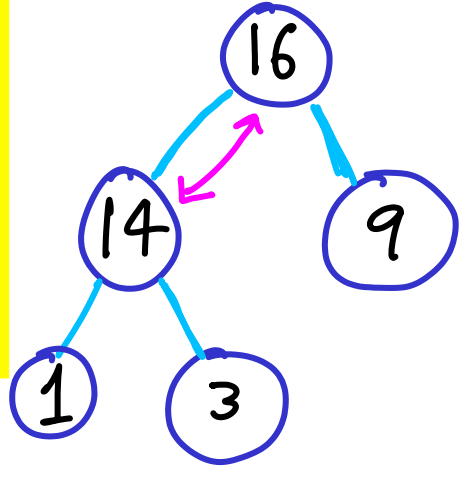


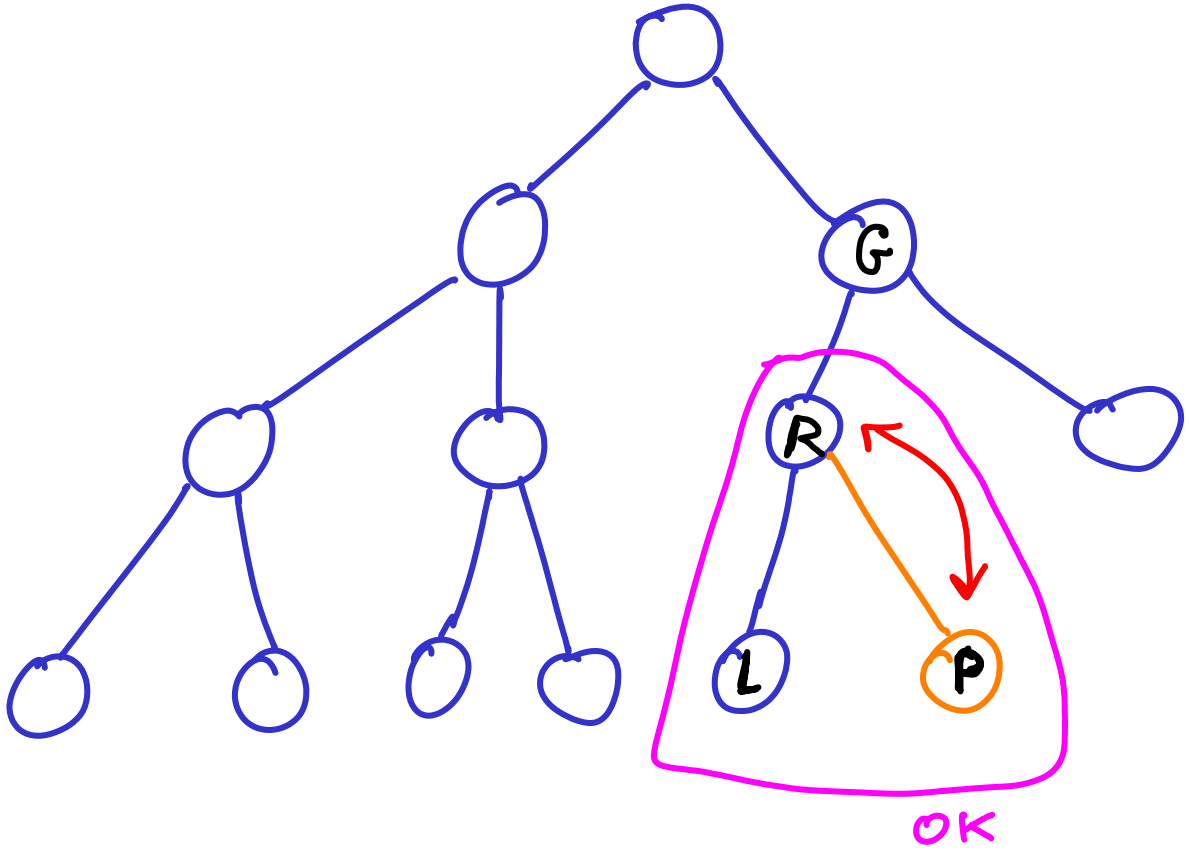
"Forward" method

read input array left to right

Also works "online":
(streaming data)



etc



When inserting a new leaf

(wlog R)

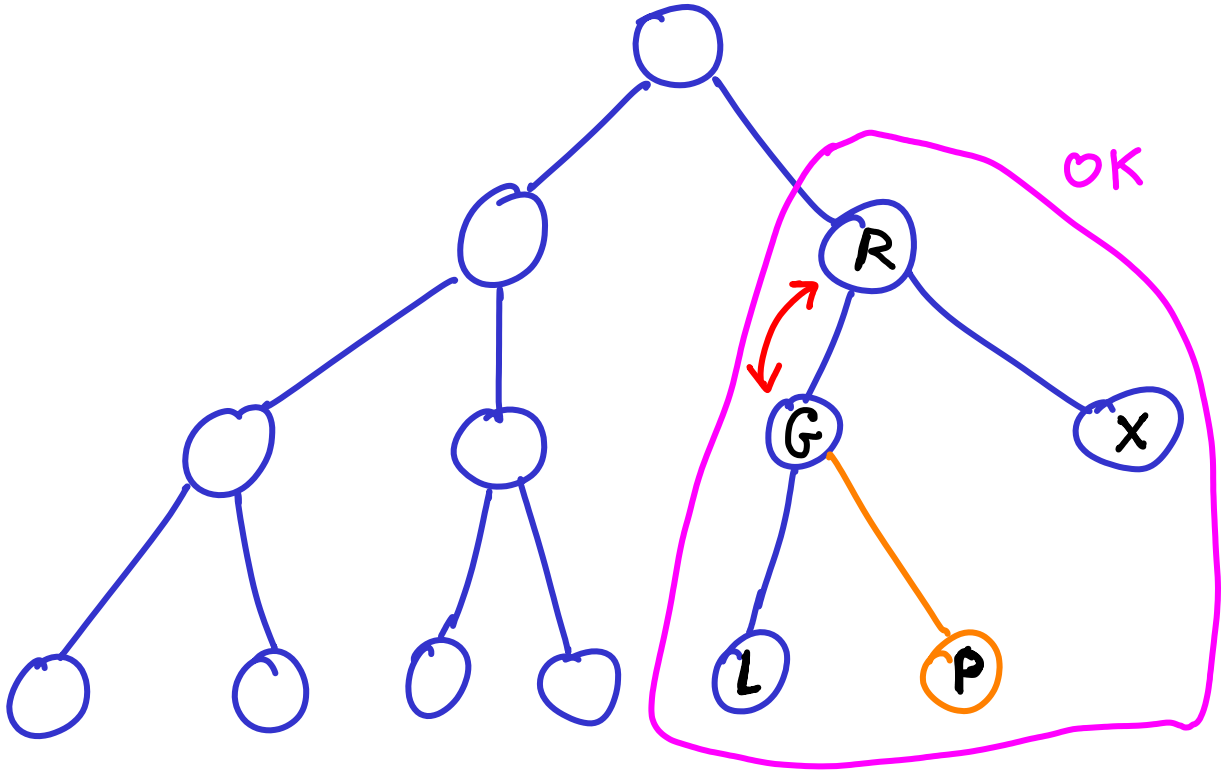
there is a problem iff

$R > P$

By swapping $R \leftrightarrow P$

we have a heap in subtree(R)

$(R > P > L)$



When inserting a new leaf

(wlog R)

there is a problem iff

$R > P$

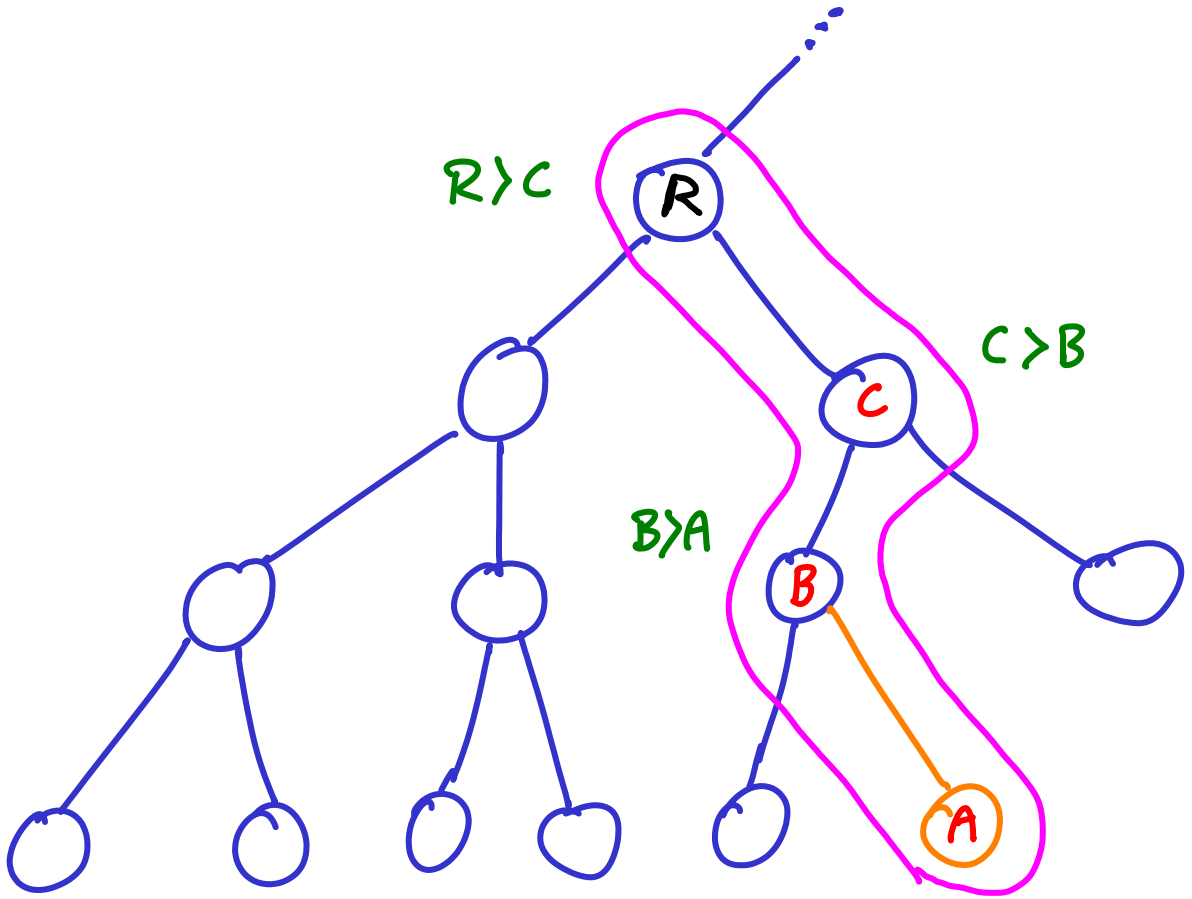
By swapping $R \leftrightarrow P$

we have a heap in subtree(R)

$(R > P > L)$

...but we may have a new problem iff $R > G$

then $R \leftrightarrow G$. $G > P > L$: OK & $R > G > X$: OK



R will move up some path until smaller than node above.

That path will "shift down" so every subtree has a larger root.

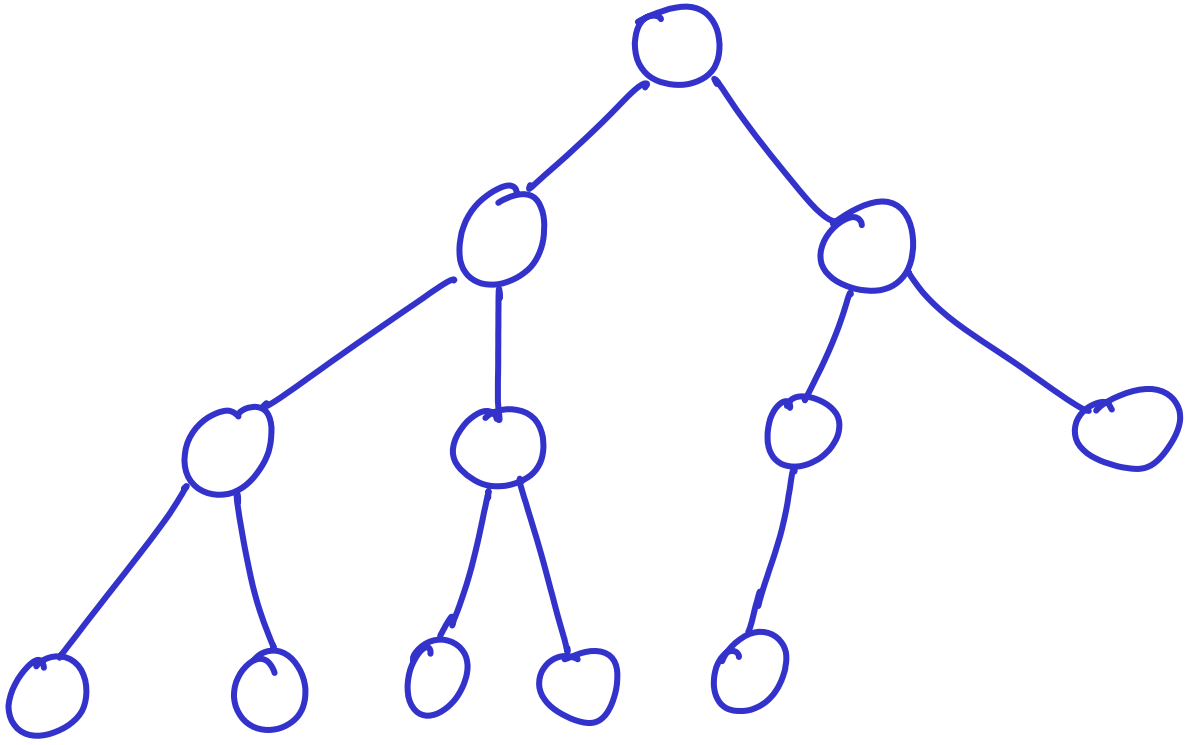
$O(n \log n)$ time

per node

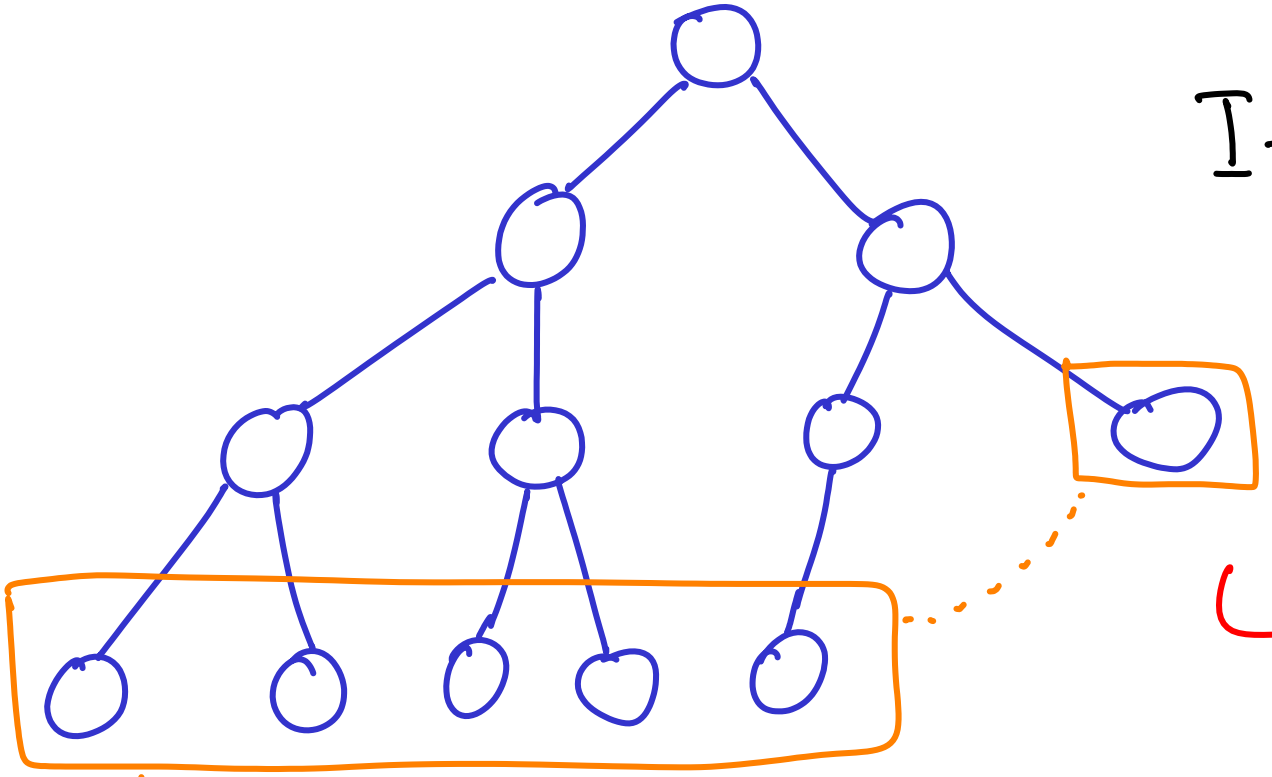
and in-place : iterate on array. Only swaps used.

CLRS 6.3 , p156

“Reverse” heap-build



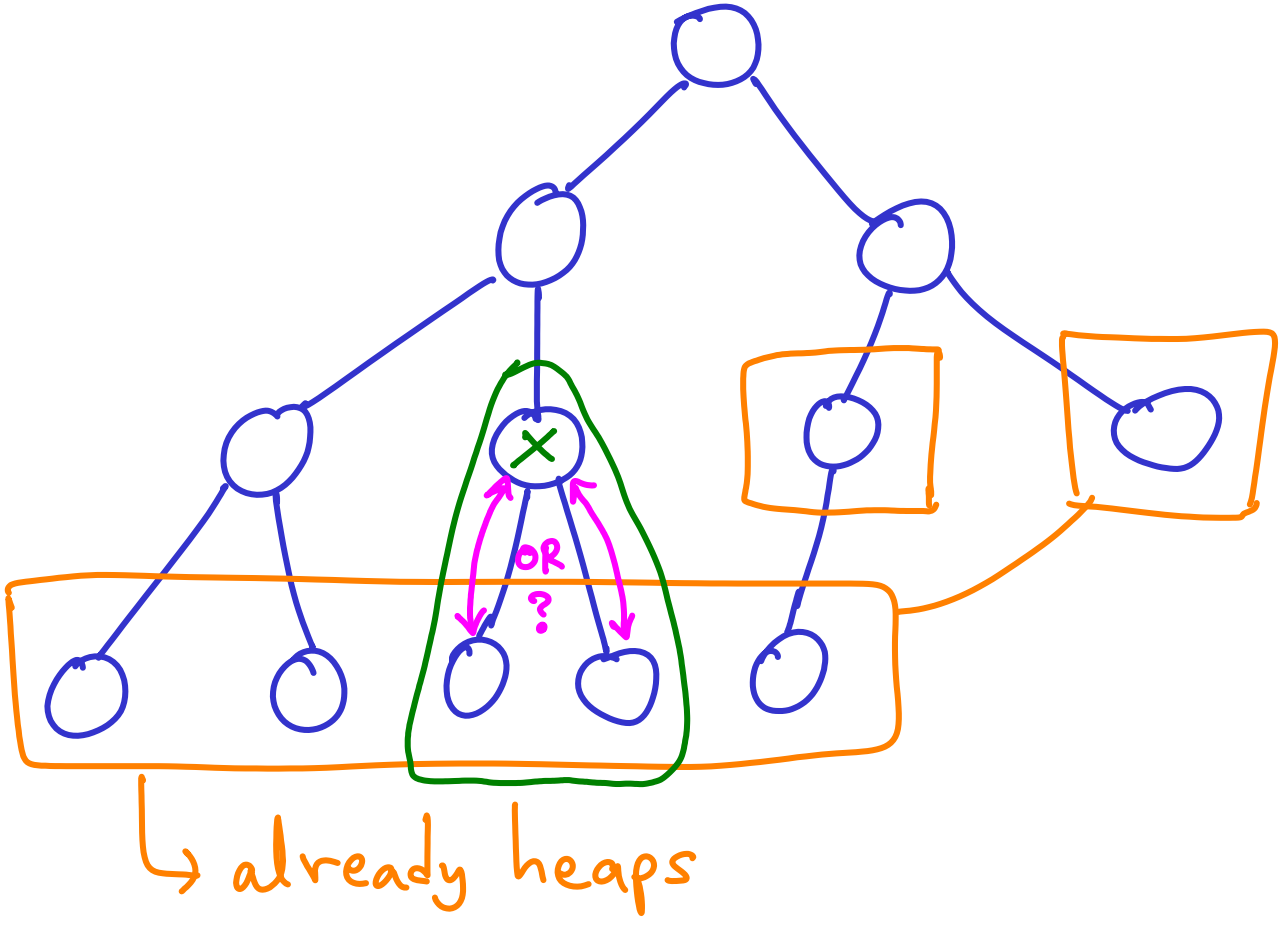
(also works in-place)



Iterate from end of array
↳ from right to left
on each level,
starting at bottom

heapify each node x
↳ i.e. subtree at x.

↳ already heaps

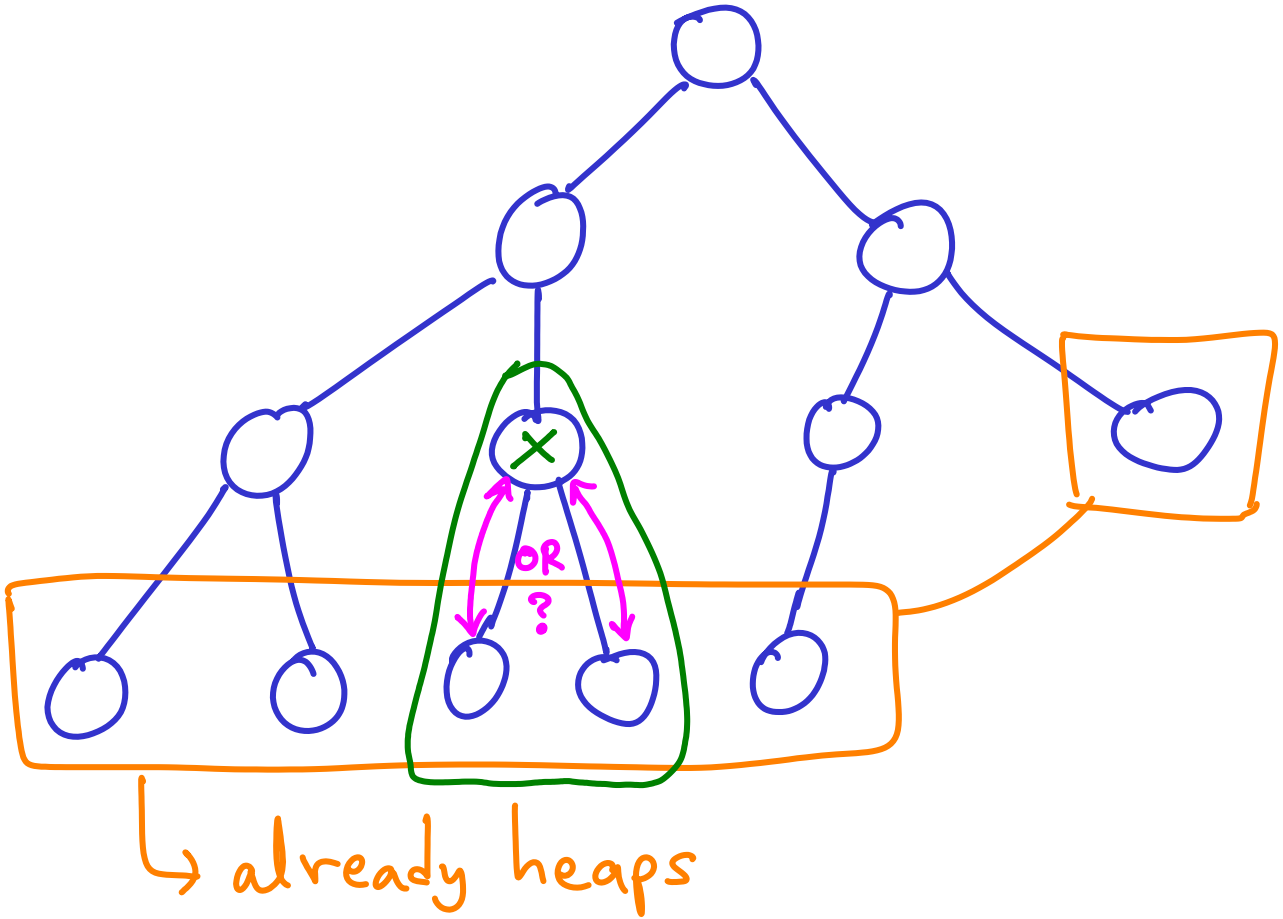


Heapify x:

‖ can assume each child is a heap.

Might have to swap x w/ one of its children & further down levels

Compare two children, determine max.
Then compare X with max, swap if needed



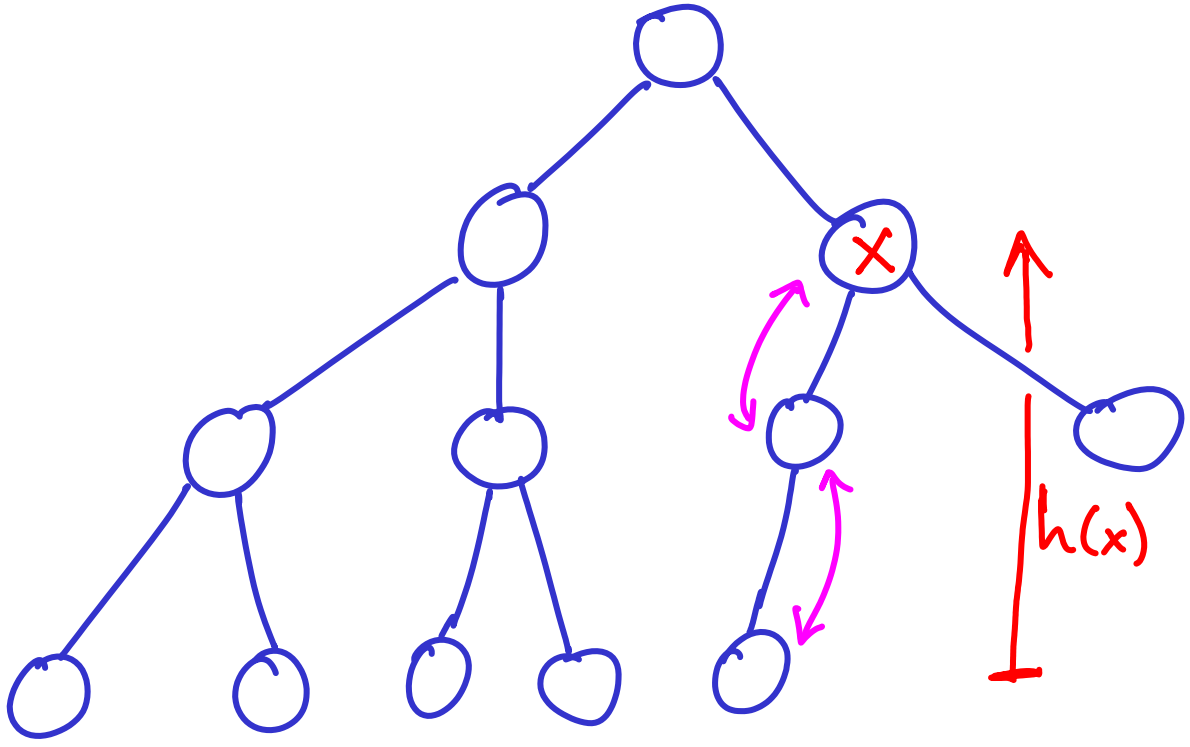
Heapify x:

‖ can assume each child is a heap.

Might have to swap x w/ one of its children & further down levels

Time for x = O(height(x))

↳ overall O(n log n)



Time = $O(h(x))$

Total time: $O\left(\sum_{\text{all } x} h(x)\right)$

$$\sum \leq \frac{n}{2}\cdot 1 + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \cdots + 2\cdot((\log n)-1) + 1\cdot \log n$$

$$= \sum_{h=1}^{\log n} \frac{n}{2^h}\cdot h = n\cdot\sum \frac{h}{2^h} \leq n\frac{1/2}{(1-1/2)^2} = O(n)$$

CLRS 1148

[use $\sum_0^\infty kx^k$]