MASTER METHOD

↳ a tool for solving recurrences of this form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

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$b > 1 \longrightarrow$ otherwise $T(n) = \infty$

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$a \geq 1$ → must recurse at least once.
$b > 1$ → otherwise $T(n) = \infty$
a & b are $O(1)$ → see next page

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required $\begin{cases} a \geq 1 \longrightarrow \text{must recurse at least once.} \\ b > 1 \longrightarrow \text{otherwise } T(n) = \infty \\ a \,\&\, b \text{ are } O(1) \longrightarrow \text{see next page} \end{cases}$

also $f(n) > 0$ for $n > n_0$

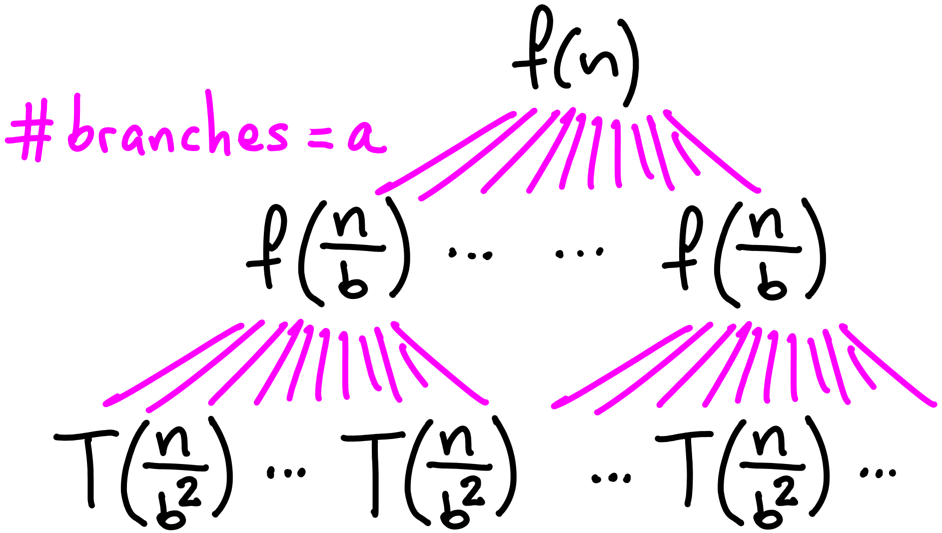
$$\boxed{T(n) = aT\left(\frac{n}{b}\right) + f(n)}$$

\# branches = a

$$f(n)$$

$$T\left(\frac{n}{b}\right) \quad \cdots \quad \cdots \quad T\left(\frac{n}{b}\right)$$

$$\boxed{T(n) = aT\left(\frac{n}{b}\right) + f(n)}$$



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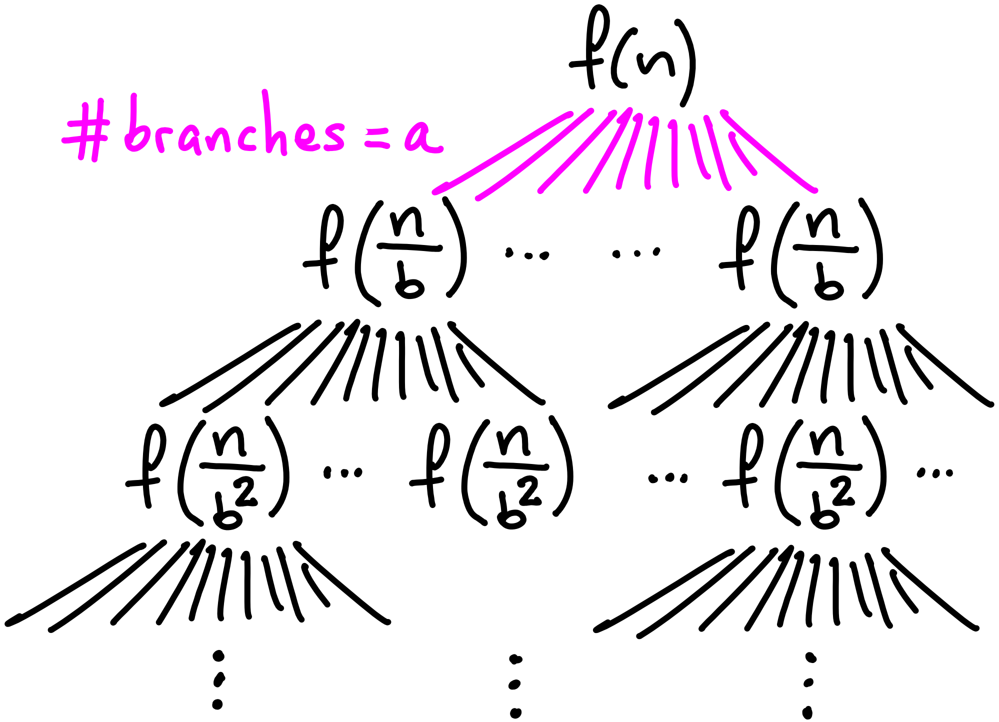

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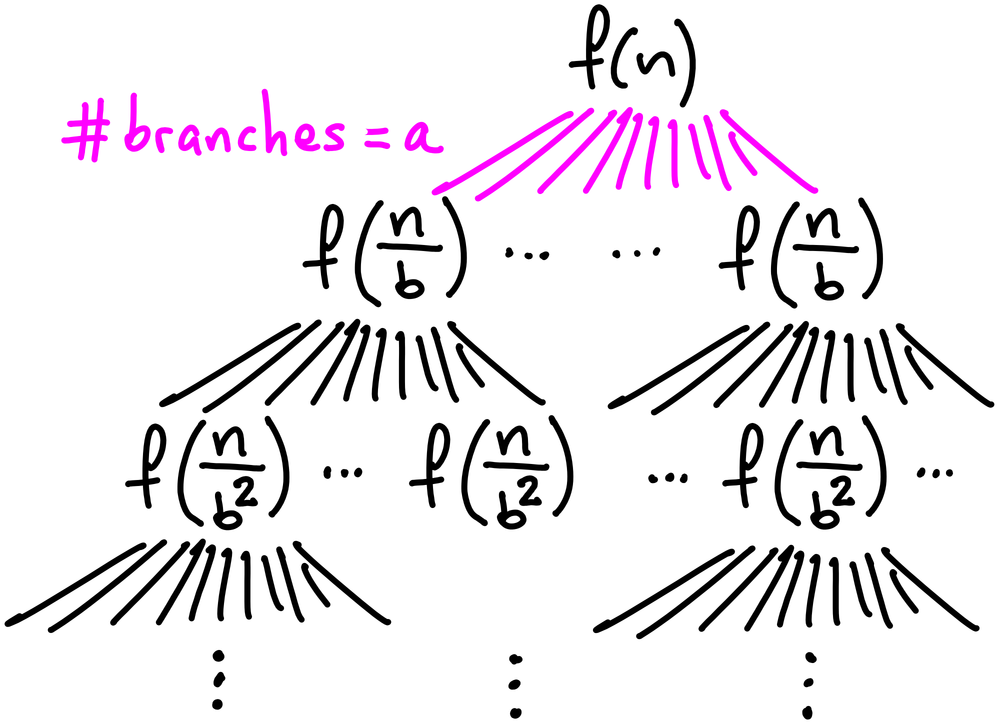


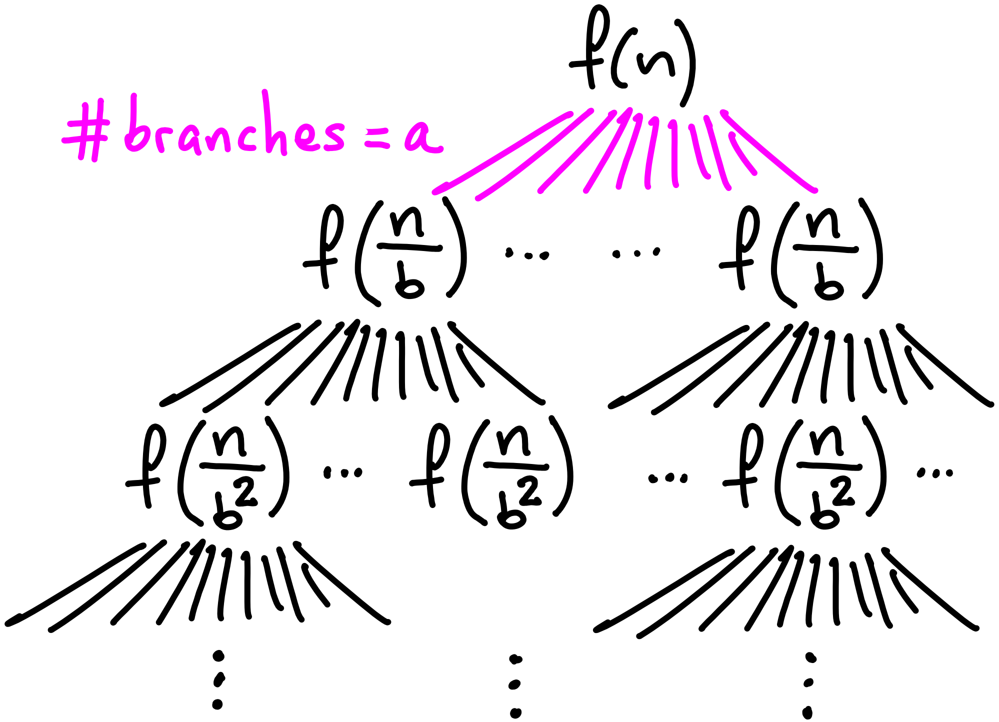
cost per level

$f(n)$

$a \cdot f\left(\frac{n}{b}\right)$

$a^2 \cdot f\left(\frac{n}{b^2}\right)$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



cost per level

$f(n)$

$a \cdot f\left(\frac{n}{b}\right)$

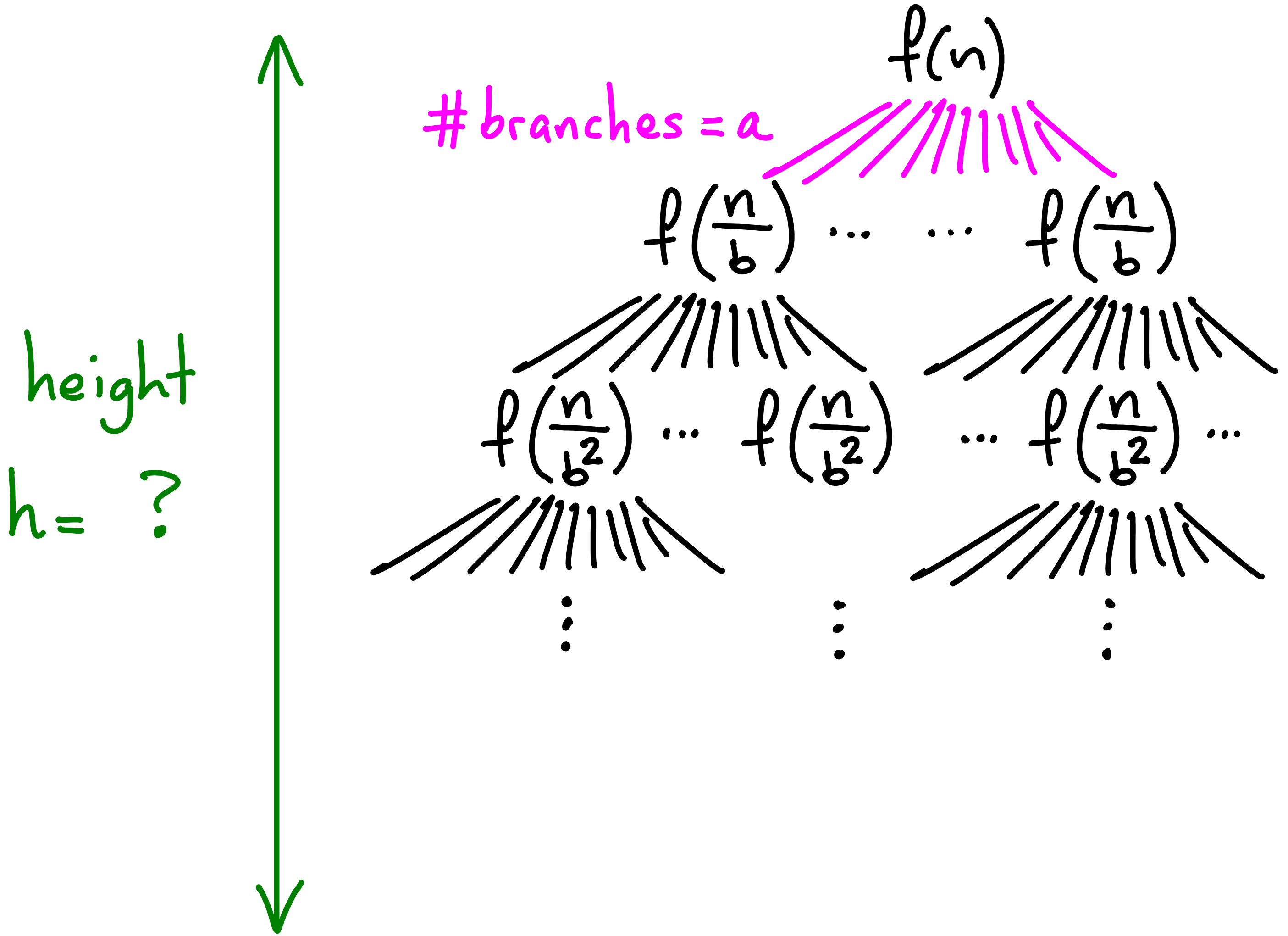
$a^2 \cdot f\left(\frac{n}{b^2}\right)$

$\vdots$

$a^i \cdot f\left(\frac{n}{b^i}\right)$

$\vdots$

$$\boxed{T(n) = aT\left(\frac{n}{b}\right) + f(n)}$$



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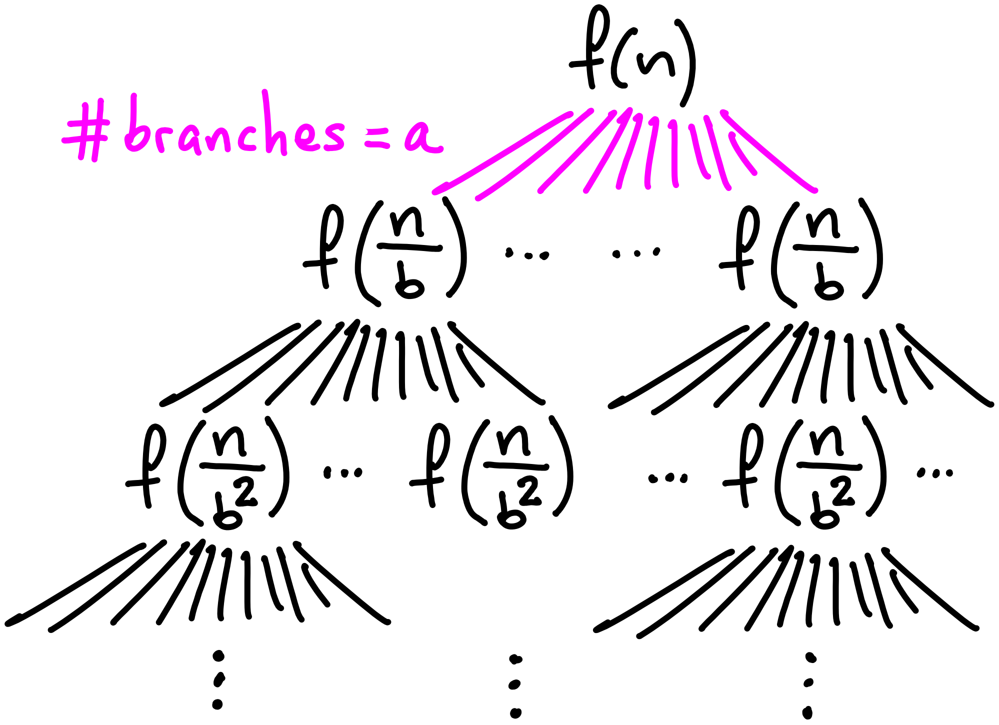
$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

height

$h = \log_b n$



leaves = ?

cost per level

$f(n)$

$a \cdot f\left(\frac{n}{b}\right)$

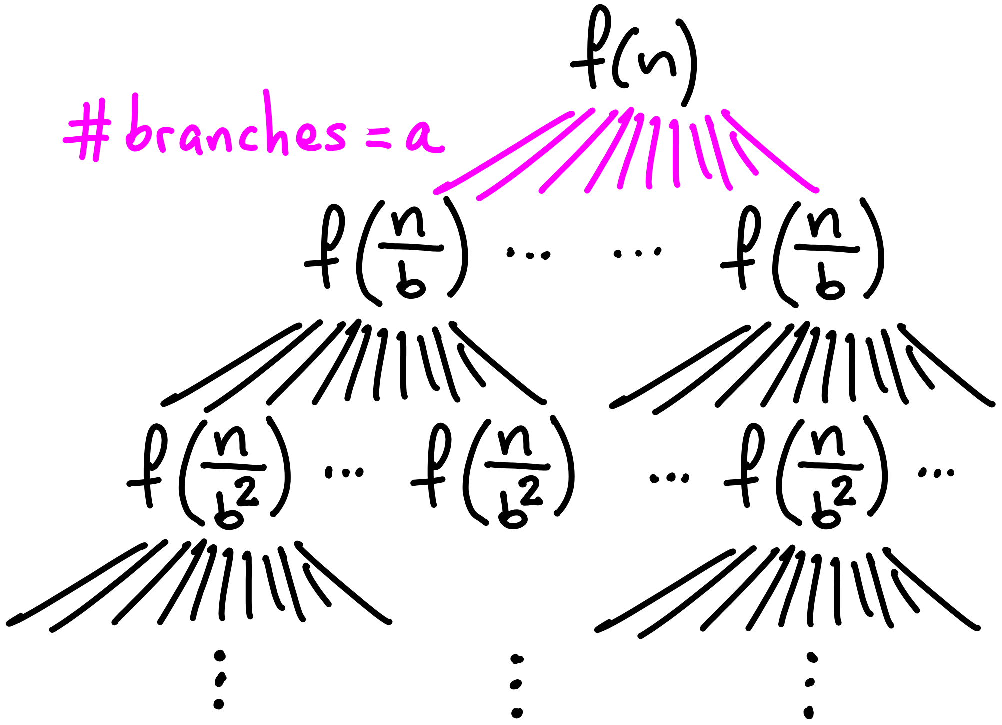
$a^2 \cdot f\left(\frac{n}{b^2}\right)$

⋮

$a^i \cdot f\left(\frac{n}{b^i}\right)$

⋮

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



height

$h = \log_b n$

leaves = $a^h = a^{\log_b n}$

cost per level

$f(n)$

$a \cdot f\left(\frac{n}{b}\right)$

$a^2 \cdot f\left(\frac{n}{b^2}\right)$

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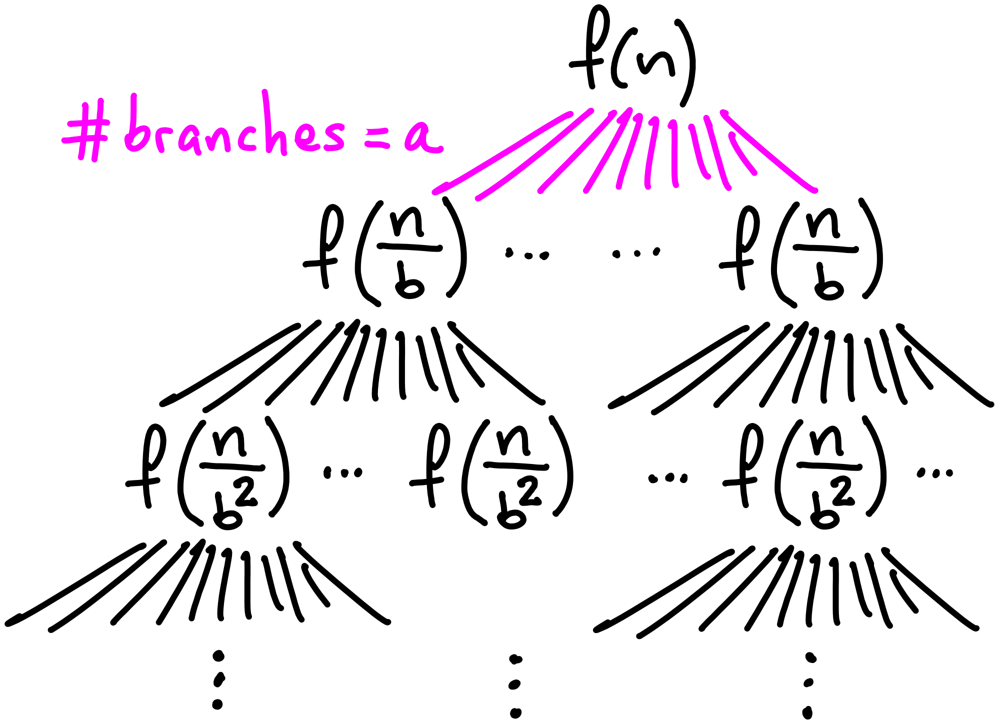

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$a^2 \cdot f\left(\frac{n}{b^2}\right)$

$\vdots$

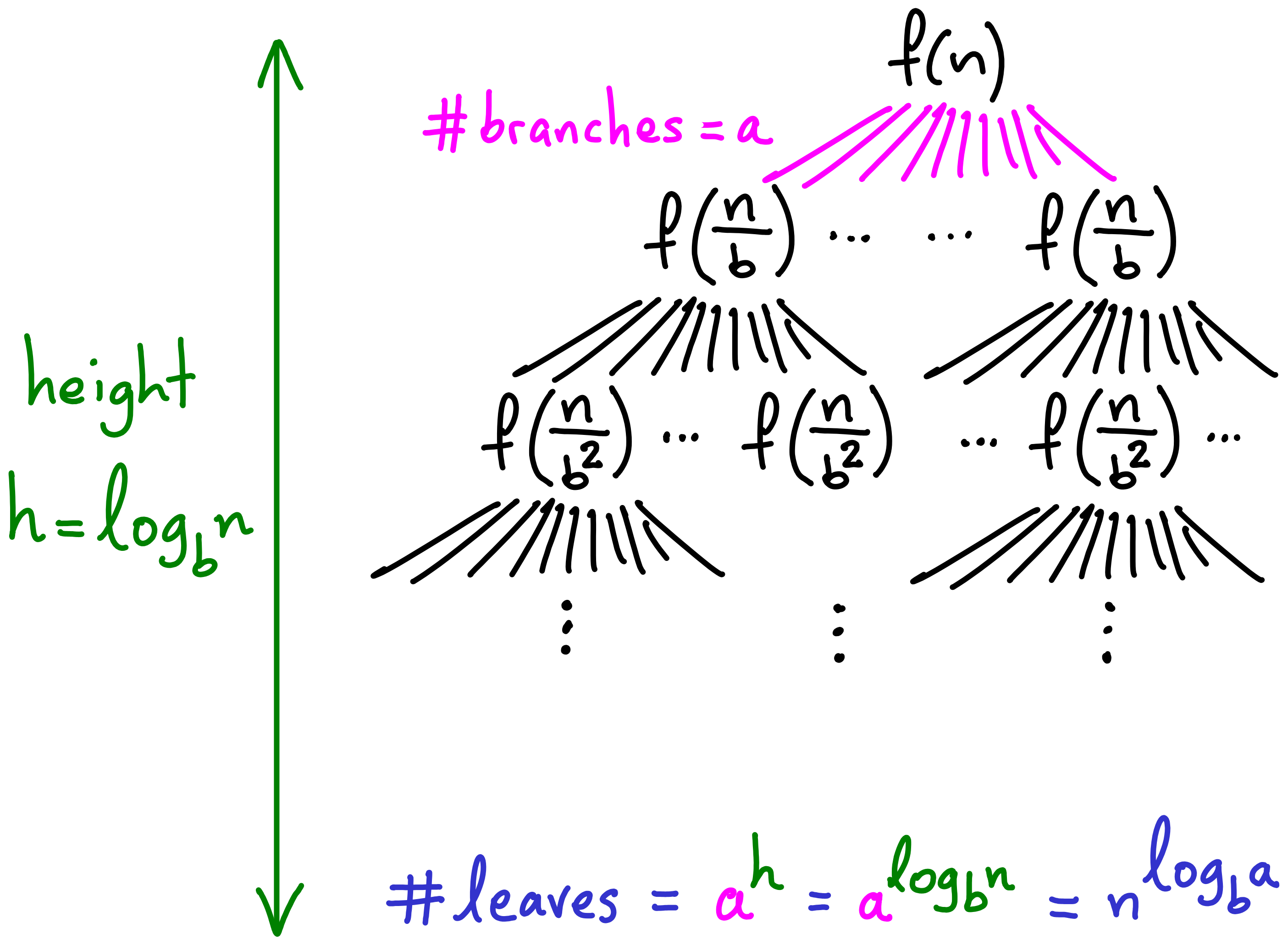
$a^i \cdot f\left(\frac{n}{b^i}\right)$

$\vdots$

by log rule

$$\#\text{leaves} = a^h = a^{\log_b n} = n^{\log_b a}$$

$$\boxed{T(n) = aT\left(\frac{n}{b}\right) + f(n)}$$



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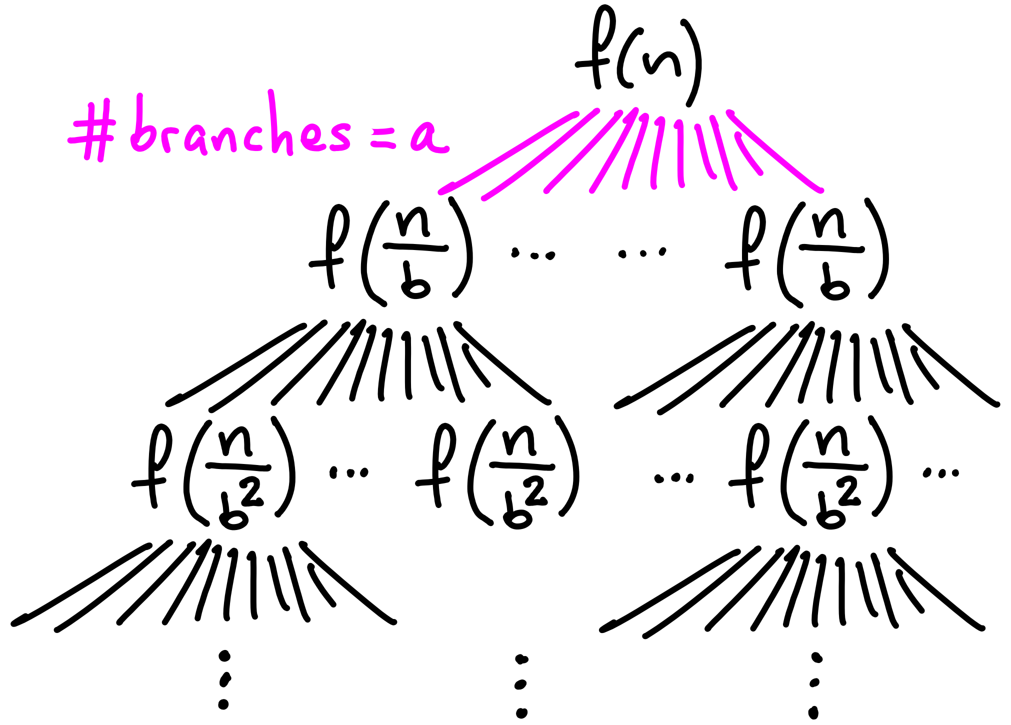
$\vdots$

$\Theta\left(n^{\log_b a}\right)$ } $\Theta(1)$ per leaf

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



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$h = \log_b n$

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$\vdots$

$a^i \cdot f\left(\frac{n}{b^i}\right)$

assuming a & b are $O(1)$

$\vdots$

$\Theta\left(n^{\log_b a}\right)$ $\Big\}$ $\Theta(1)$ per leaf

MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ compare $f(n)$ to $n^{\log_b a}$

root #leaves

MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ compare $f(n)$ to $n^{\log_b a}$

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---

1) #leaves dominate polynomially

---

2) all levels ~ same

---

3) root dominates polynomially

MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

compare $f(n)$ to $n^{\log_b a}$

root #leaves

1\)

#leaves dominate polynomially

solution: $T(n) = \Theta\left(n^{\log_b a}\right)$

2\)

all levels ~ same

solution: $T(n) = \Theta\left(f(n) \cdot \log n\right)$

3\)

root dominates polynomially

solution: $T(n) = \Theta\left(f(n)\right)$

MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ compare $f(n)$ to $n^{\log_b a}$

root #leaves

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1) #leaves dominate polynomially

solution: $T(n) = \Theta(n^{\log_b a})$

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3) root dominates polynomially

solution: $T(n) = \Theta(f(n))$

e.g.: #leaves $= n^4$, $f(n) = n^5$

MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

compare $f(n)$ to $n^{\log_b a}$

root #leaves

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3) $f(n) = \Omega(n^{\log_b a} \cdot n^{\varepsilon})$ $(\varepsilon > 0)$

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e.g.: #leaves $= n^4$, $f(n) = n^5$

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MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ compare $f(n)$ to $n^{\log_b a}$

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solution: $T(n) = \Theta\left(n^{\log_b a}\right)$

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e.g.: #leaves $= n^3$, $f(n) = 2n^3$

solution: $T(n) = \Theta(f(n) \cdot \log n)$

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3) $f(n) = \Omega\left(n^{\log_b a} \cdot n^{\varepsilon}\right)$ $(\varepsilon > 0)$

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MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

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1) $n^{\log_b a} = \Omega(f(n) \cdot n^{\varepsilon})$ $(\varepsilon > 0)$ #leaves dominate polynomially

e.g.: #leaves $= n^2$, $f(n) = 30n^{1.5} \cdot \log^2 n$

solution: $T(n) = \Theta(n^{\log_b a})$

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2) $f(n) = \Theta(n^{\log_b a})$ all levels ~ same

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Technicality for case 3

Also required: $af\left(\frac{n}{b}\right) \leq \delta \cdot f(n)$ $\qquad 0 < \delta < 1$

Good news: For commonly encountered functions this will hold.

You don't need to check this condition.

Examples

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

Divide & Conquer

$T(n) = 2T(\frac{n}{2}) + f(n)$

$n^{\log_b a} = n$

if $f(n) = \Theta(n)$

?

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = \Theta(n)$ → case 2 ?

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

Divide & Conquer

$T(n) = 2T(\frac{n}{2}) + f(n)$

$n^{\log_b a} = n$

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

if $f(n) = O(n^d)$ $(d<1)$?

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

if $f(n) = O(n^d)$ $(d<1)$ $\rightarrow$ case 1 ?

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

if $f(n) = O(n^d)$ $(d<1)$ $\rightarrow$ case 1 $\Theta(n)$

e.g., $O(1)$, $O(\log n)$, $O(\sqrt{n})$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

if $f(n) = O(n^d)$ $(d<1)$ $\rightarrow$ case 1 $\Theta(n)$

e.g., $O(1)$, $O(\log n)$, $O(\sqrt{n})$

if $f(n) = \Omega(n^d)$ $(d>1)$? ?

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n\log n)$

if $f(n) = O(n^d)$ $(d<1)$ $\rightarrow$ case 1 $\Theta(n)$

e.g., $O(1)$, $O(\log n)$, $O(\sqrt{n})$

if $f(n) = \Omega(n^d)$ $(d>1)$ $\rightarrow$ case 3 $\Theta(f(n))$

Divide & Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

$$n^{\log_b a} = n$$

$$T(n) = 4T\left(\frac{n}{4}\right) + f(n)$$

?

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

if $f(n) = O(n^d)$ $(d<1)$ $\rightarrow$ case 1 $\Theta(n)$

e.g., $O(1)$, $O(\log n)$, $O(\sqrt{n})$

if $f(n) = \Omega(n^d)$ $(d>1)$ $\rightarrow$ case 3 $\Theta(f(n))$

Divide & Conquer

$T(n) = 2T(\frac{n}{2}) + f(n)$

$n^{\log_b a} = n$

$T(n) = 4T(\frac{n}{4}) + f(n)$

SAME

if $f(n) = \Theta(n)$ $\rightarrow$ case 2 $\Theta(n \log n)$

if $f(n) = O(n^d)$ $(d<1)$ $\rightarrow$ case 1 $\Theta(n)$

e.g., $O(1)$, $O(\log n)$, $O(\sqrt{n})$

if $f(n) = \Omega(n^d)$ $(d>1)$ $\rightarrow$ case 3 $\Theta(f(n))$

$n^{\log_b a} = ?$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \quad = ?$$

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \qquad = \Theta(n^2)$$

leaves dominate polynomially

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \qquad = \Theta(n^2)$$

leaves dominate polynomially

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \qquad = ?$$

$n^{\log_b a} = n^{\log_2 4} = n^2$

$T(n) = 4T(\frac{n}{2}) + n$ $= \Theta(n^2)$

leaves dominate polynomially

$T(n) = 4T(\frac{n}{2}) + n^2$ $= \Theta(n^2 \log n)$

case 2

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \quad = \Theta(n^2)$$

leaves dominate polynomially

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \quad = \Theta(n^2 \log n)$$

case 2

$$T(n) = 4T\left(\frac{n}{2}\right) + n^3 \quad = ?$$

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \quad = \Theta(n^2)$$

leaves dominate polynomially

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \quad = \Theta(n^2 \log n)$$

case 2

$$T(n) = 4T\left(\frac{n}{2}\right) + n^3 \quad = \Theta(n^3)$$

root dominates polynomially

Next:

An extension of Case 2
that can sometimes come in handy

EXTENDED CASE 2

$f(n) = \Theta\left(n^{\log_b a} \cdot \log^k n\right)$ $k \geq 0$ ($k = 0$ is regular case 2)

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$f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$ $k \geq 0$ ($k = 0$ is regular case 2)

$\hookrightarrow T(n) = \Theta(f(n) \cdot \log n)$ same result as regular case 2

EXTENDED CASE 2

$f(n) = \Theta\left(n^{\log_b a} \cdot \log^k n\right)$ $\quad k \geq 0 \quad$ ($k=0$ is regular case 2)

$\hookrightarrow T(n) = \Theta\left(f(n) \cdot \log n\right)$ same result as regular case 2

Examples: $\quad T(n) = 2T\left(\frac{n}{2}\right) + n\log^5 n \quad = \ ?$

EXTENDED CASE 2

$f(n) = \Theta\left(n^{\log_b a} \cdot \log^k n\right)$ $\quad k \geq 0 \quad$ ($k = 0$ is regular case 2)

$\hookrightarrow T(n) = \Theta\left(f(n) \cdot \log n\right)$ $\quad$ same result as regular case 2

Examples:

$T(n) = 2T\left(\frac{n}{2}\right) + n\log^5 n \quad = \Theta\left(n\log^6 n\right)$

$T(n) = 4T\left(\frac{n}{2}\right) + n^2\log n \quad = \ ?$

EXTENDED CASE 2

$f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$ $\quad$ $k \geq 0$ $\quad$ ($k=0$ is regular case 2)

$\rightarrow T(n) = \Theta(f(n) \cdot \log n)$ $\quad$ same result as regular case 2

Examples:

$T(n) = 2T(\frac{n}{2}) + n\log^5 n \quad = \Theta(n\log^6 n)$

$T(n) = 4T(\frac{n}{2}) + n^2 \log n \quad = \Theta(n^2 \log^2 n)$

$T(n) = T(\frac{n}{6}) + \log^2 n \quad = ?$

EXTENDED CASE 2

$f(n) = \Theta\left(n^{\log_b a} \cdot \log^k n\right)$ $\quad k \geq 0$ $\quad$ ($k=0$ is regular case 2)

$\hookrightarrow T(n) = \Theta(f(n) \cdot \log n)$ $\quad$ same result as regular case 2

Examples:

$$T(n) = 2T\left(\frac{n}{2}\right) + n\log^5 n \quad = \Theta(n\log^6 n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2\log n \quad = \Theta(n^2\log^2 n)$$

$$T(n) = T\left(\frac{n}{6}\right) + \log^2 n \quad = \Theta(\log^3 n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n} \quad = ?$$

EXTENDED CASE 2

$f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$ $\quad k \geq 0 \quad$ ($k = 0$ is regular case 2)

$\hookrightarrow T(n) = \Theta(f(n) \cdot \log n)$ $\quad$ same result as regular case 2

Examples:

$T(n) = 2T(\frac{n}{2}) + n\log^5 n \quad = \Theta(n\log^6 n)$

$T(n) = 4T(\frac{n}{2}) + n^2\log n \quad = \Theta(n^2\log^2 n)$

$T(n) = T(\frac{n}{6}) + \log^2 n \quad = \Theta(\log^3 n)$

$T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n} \quad$ N/A $\quad$ *based on what we've seen

Next: Recap of cases

and one last extension (FYI)

MASTER METHOD $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ compare $f(n)$ to $n^{\log_b a}$

---

1) $f(n) = O\left(n^{(\log_b a)-\varepsilon}\right)$ $\#\text{leaves} = \Omega\left(f(n)\cdot n^{\varepsilon}\right)$

$\Leftrightarrow f(n) = O\left(\#\text{leaves}/n^{\varepsilon}\right)$

leaf level dominates polynomially

$T(n) = \Theta\left(n^{\log_b a}\right)$

---

2) $f(n) = \Theta\left(n^{\log_b a}\cdot \log^k n\right) = \Theta\left(\#\text{leaves}\cdot \log^k n\right)$

all levels ~ same

$T(n) = \Theta\left(f(n)\cdot \log n\right)$

---

3) $f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) = \Omega\left(\#\text{leaves}\cdot n^{\varepsilon}\right)$

root dominates polynomially

AND $af\left(\frac{n}{b}\right) \leq \delta \cdot f(n)$

work reduced by constant fraction in each level

$T(n) = \Theta\left(f(n)\right)$

$\varepsilon > 0 \quad k \geq 0 \quad 0 < \delta < 1$

FYI - EXTRA-EXTENDED CASE 2 $f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$

(Doesn't come up in any algorithms that we will see)

Standard extended case 2

$k \geq 0$ $T(n) = \Theta(n^{\log_b a} \cdot \log^{k+1} n)$ $T(n) = \Theta(f(n) \cdot \log n)$

$k = -1$ $T(n) = \Theta(n^{\log_b a} \cdot \log\log n)$ $T(n) = \Theta(f(n) \cdot \log n \cdot \log\log n)$

e.g., $T(n) = 8T(\frac{n}{2}) + \frac{n^3}{\log n}$ $= n^3 \log\log n$

$k \leq -2$ $T(n) = \Theta(n^{\log_b a})$ almost like an extended case 1:

Leaf level dominates by a "large" poly-log factor.