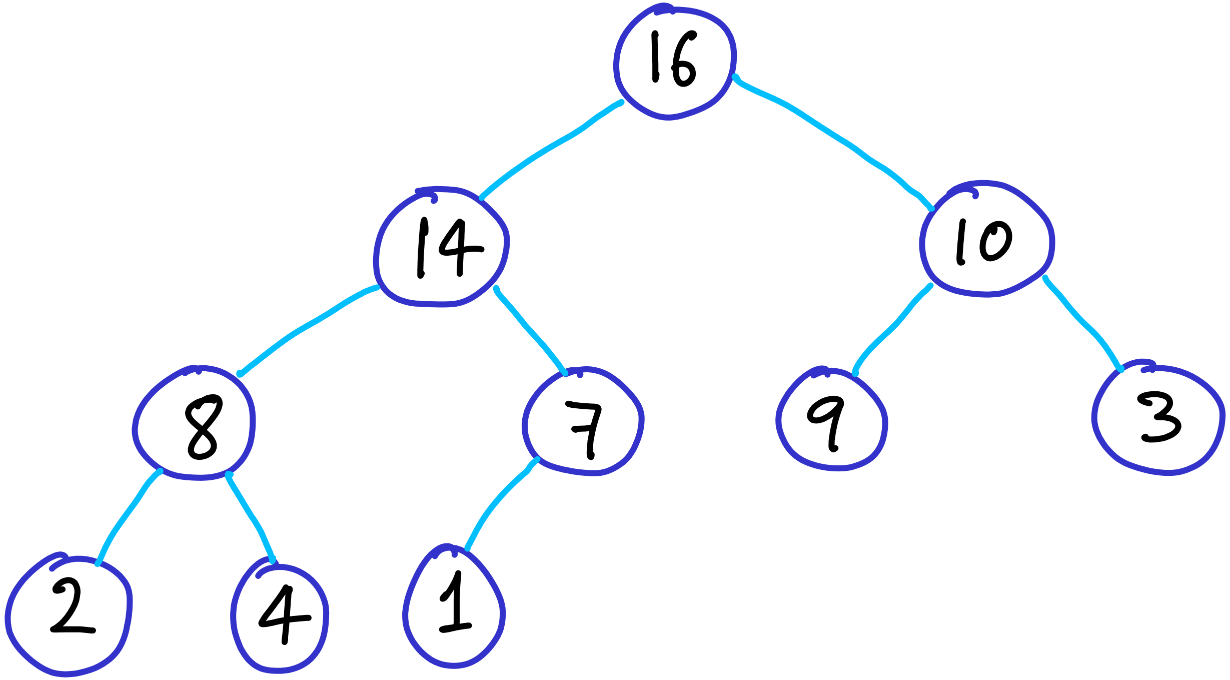
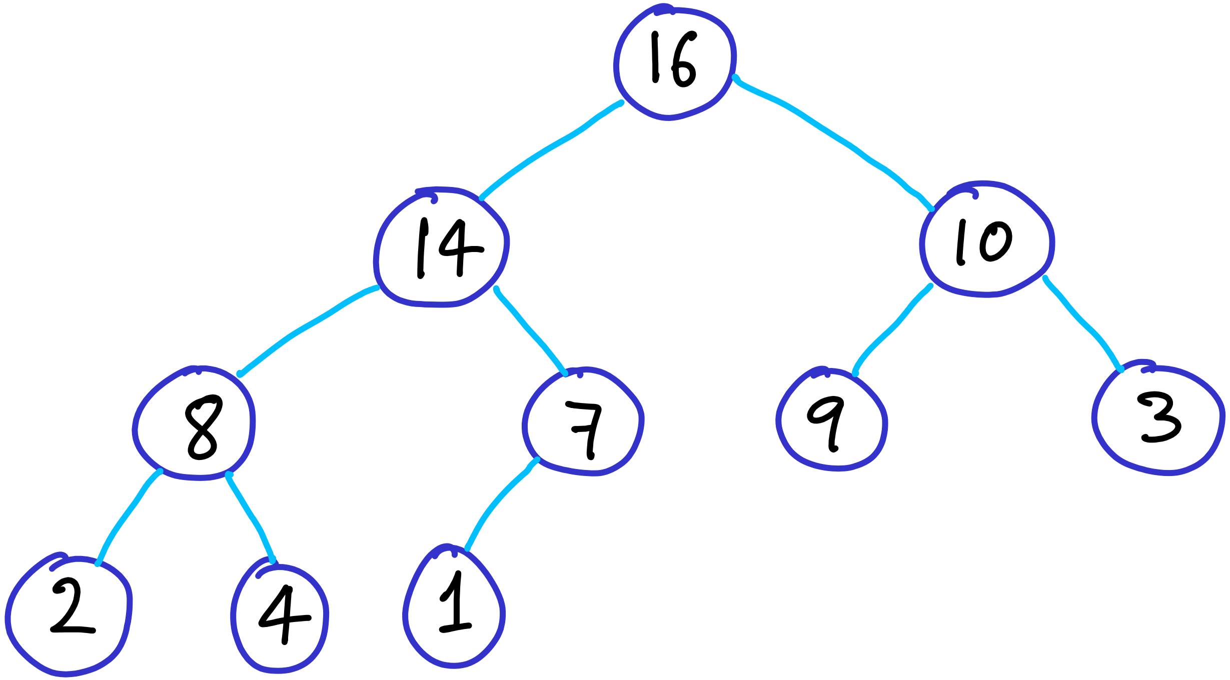
HEAPS and HEAP-SORT

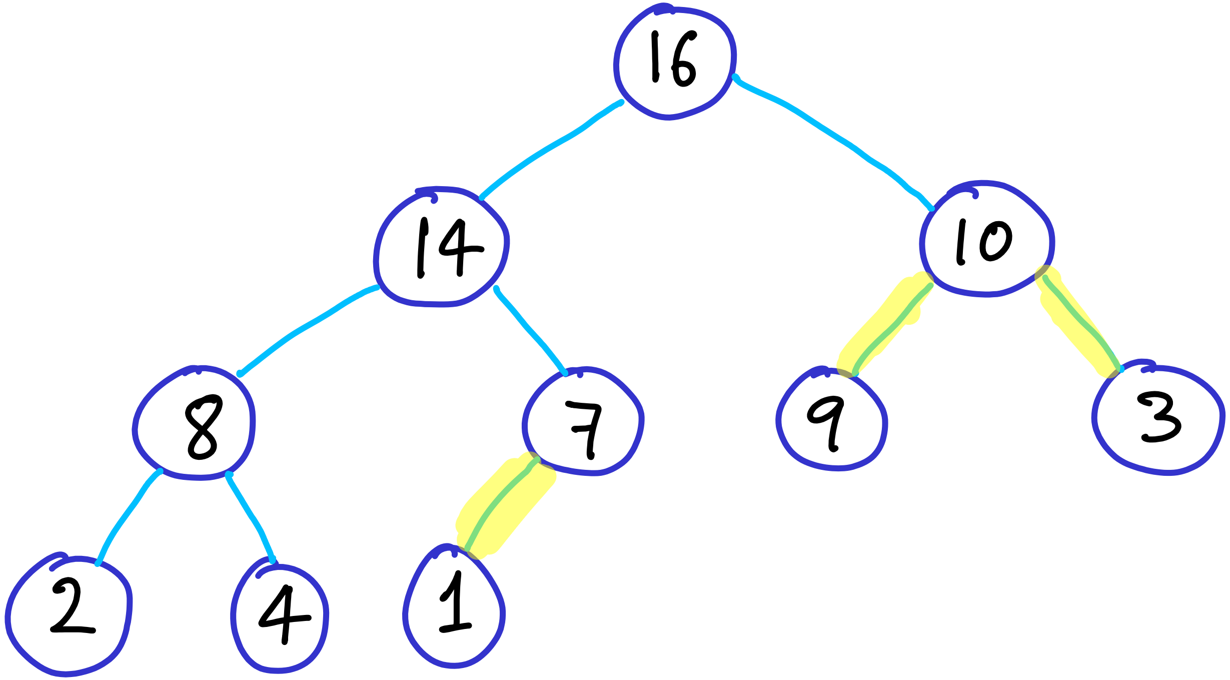
↳ specifically binary MAX-heaps


16
14
10
8
7
9
3
2
4
1



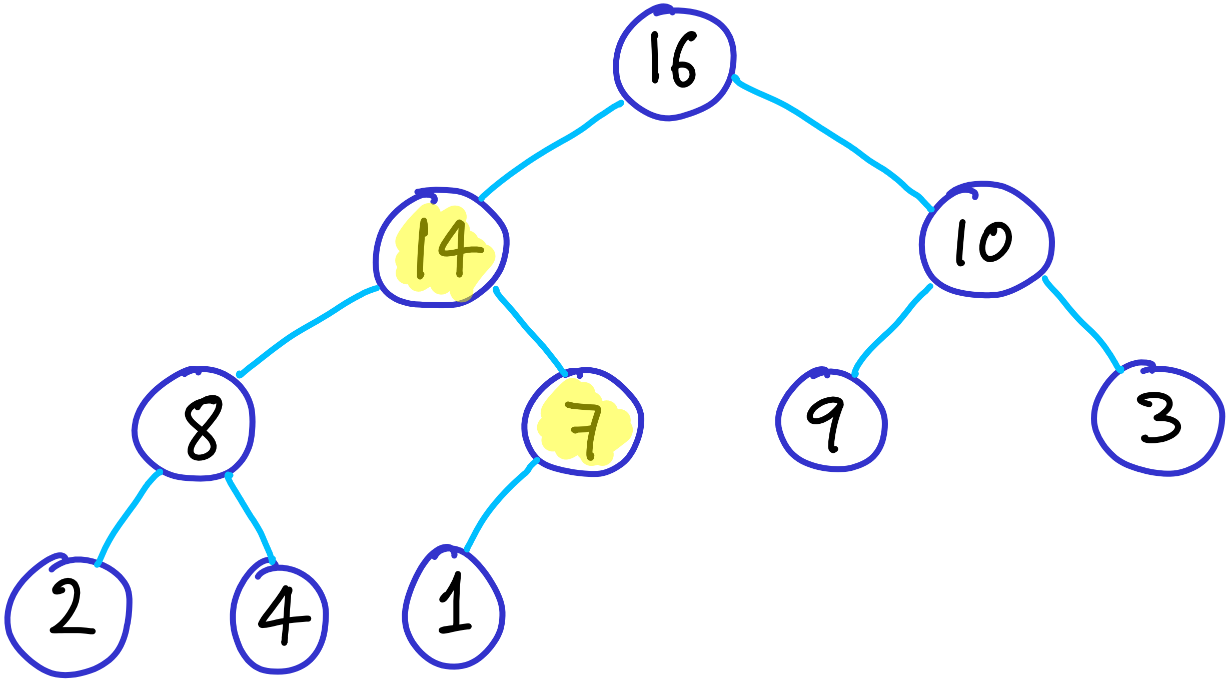
Rules:

- binary
- max
- complete



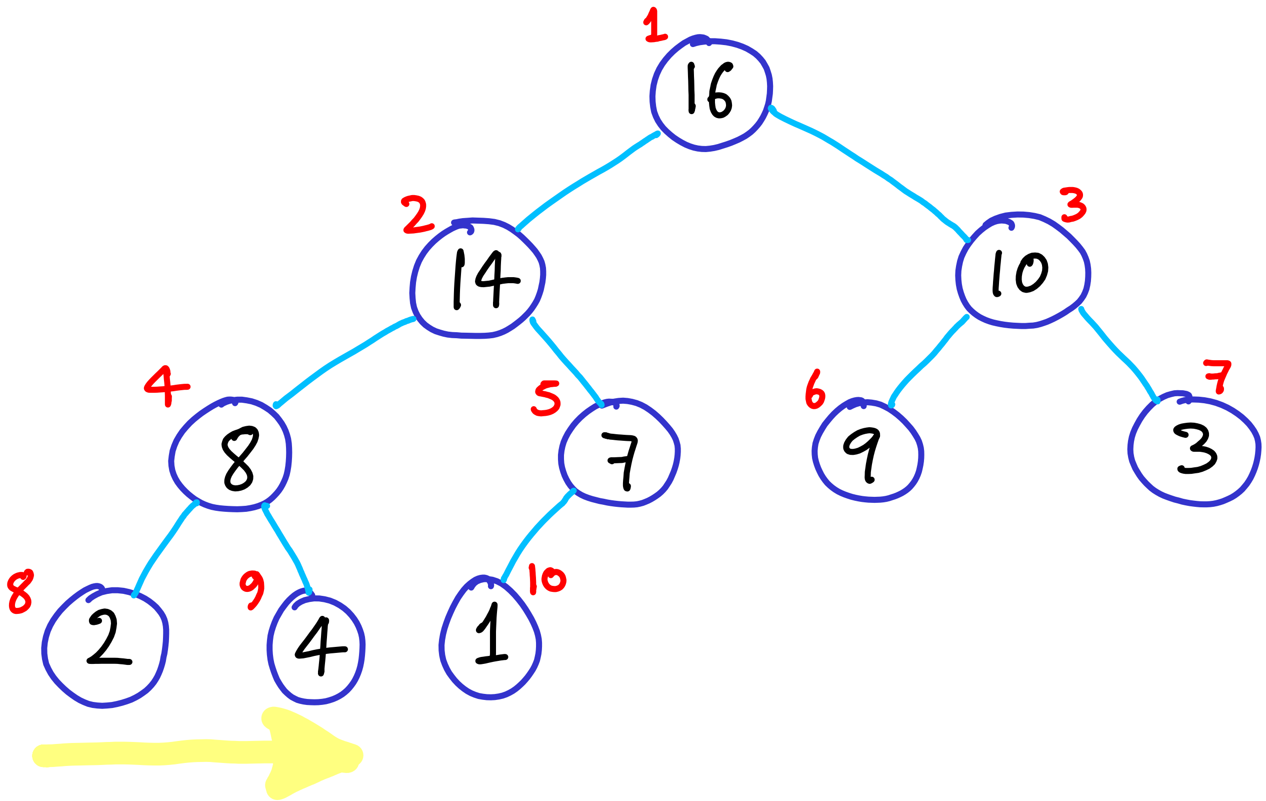
Rules:

- binary: internal nodes have 1 or 2 children
- max
- complete



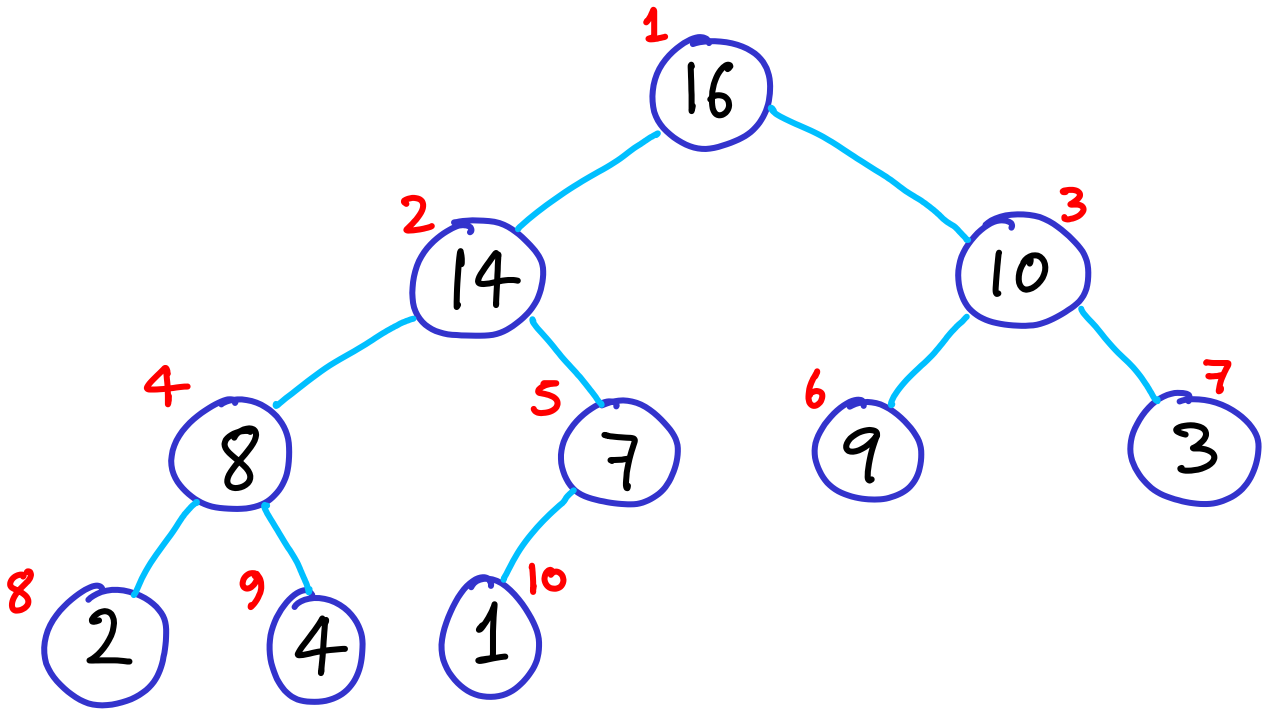
Rules:

- binary: internal nodes have 1 or 2 children
- max: parent ≥ child
- complete



Rules:

- **binary:** internal nodes have 1 or 2 children
- **max:** parent ≥ child
- **complete:** all levels filled (lowest can be partial, left to right)



Rules:

- **binary:** internal nodes have 1 or 2 children
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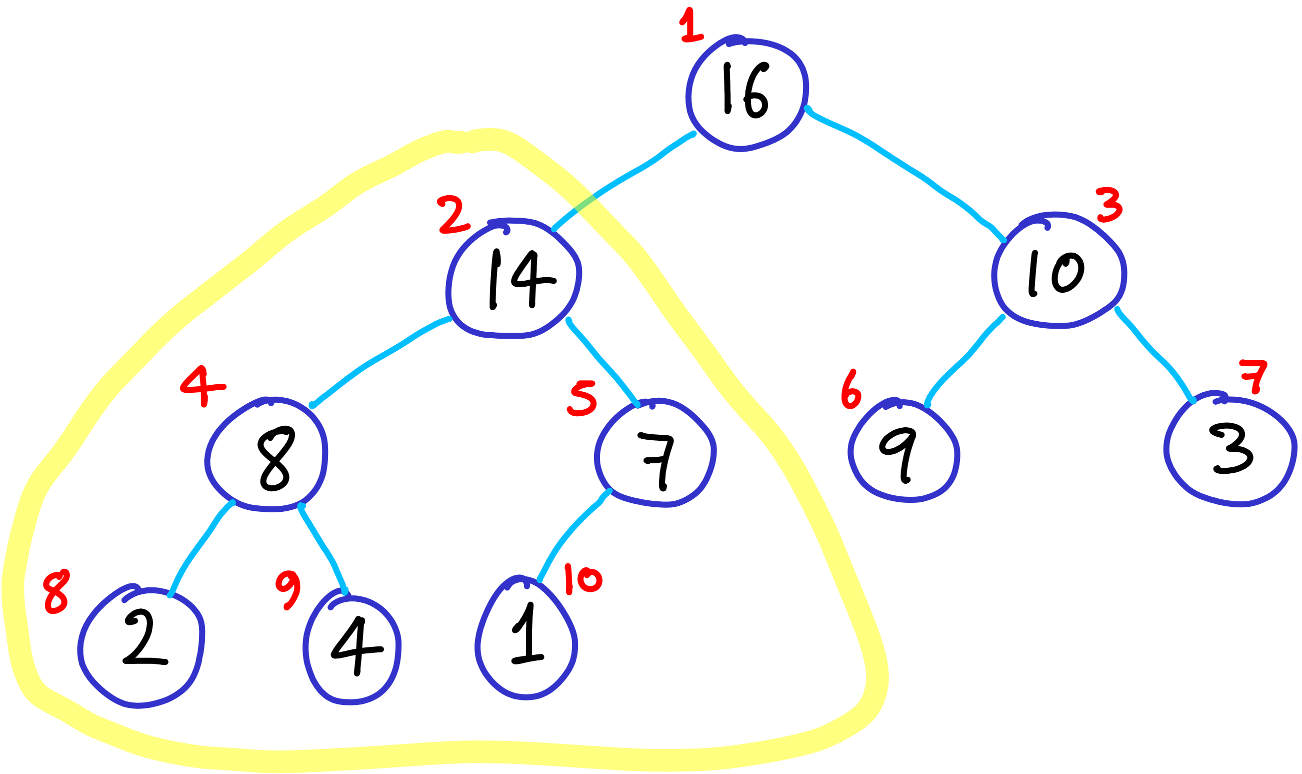
some applications don't need this but we will enforce it

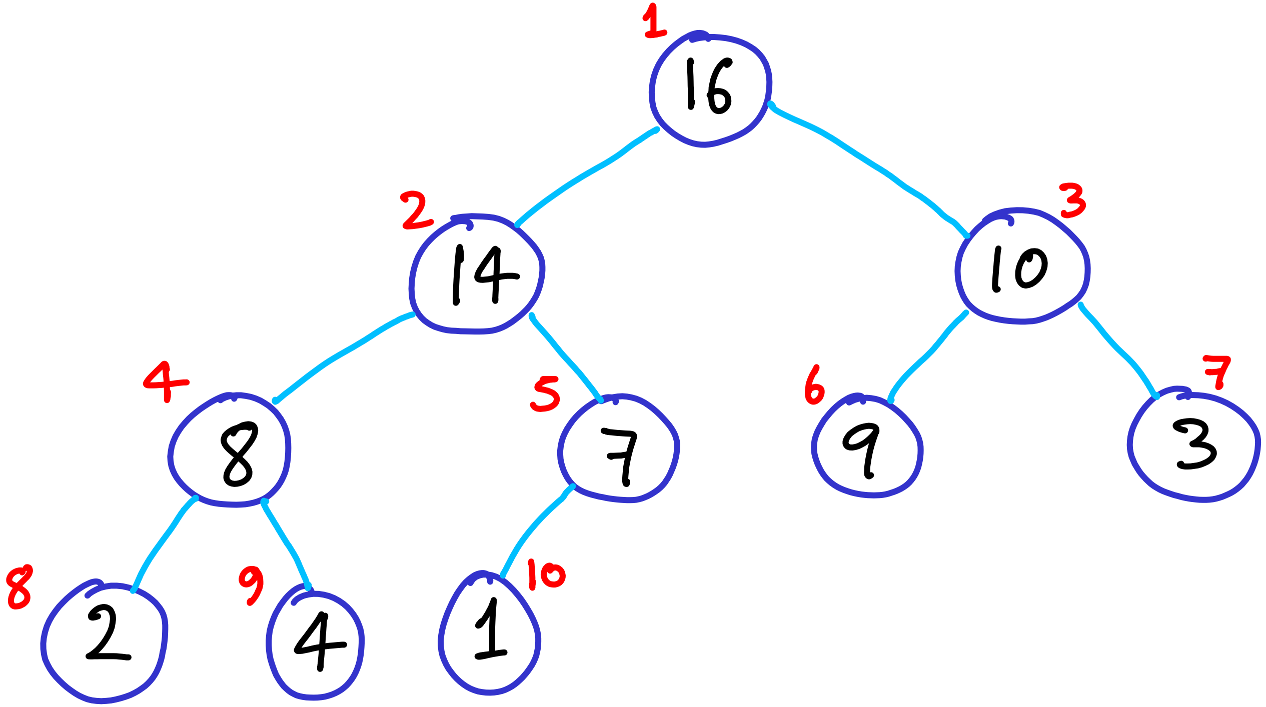


[Notice every subtree is also a heap]

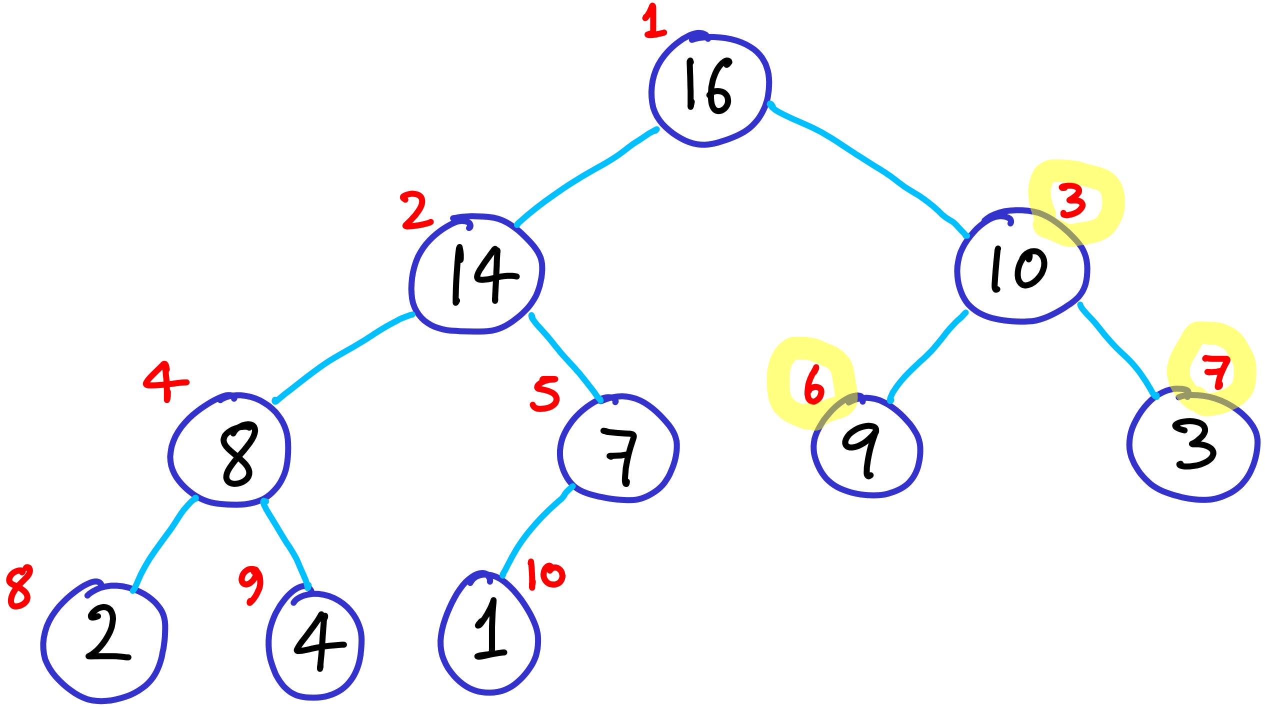
Rules:

- **binary:** internal nodes have 1 or 2 children
- **max:** parent ≥ child
- **complete:** all levels filled (lowest can be partial, left to right)

some applications don't need this but we will enforce it



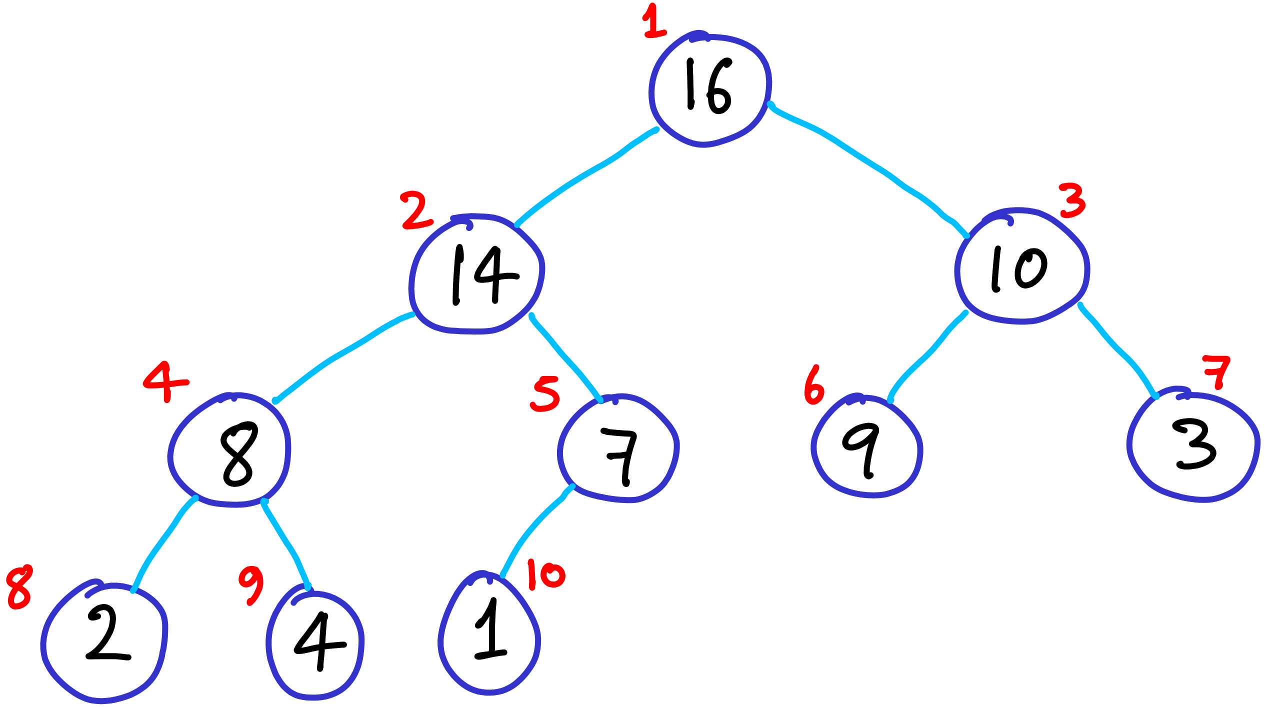
How can we identify the indices of the children of a given node?



How can we identify the indices of the children of a given node?

left-child(i) = 2i

right-child(i) = 2i+1

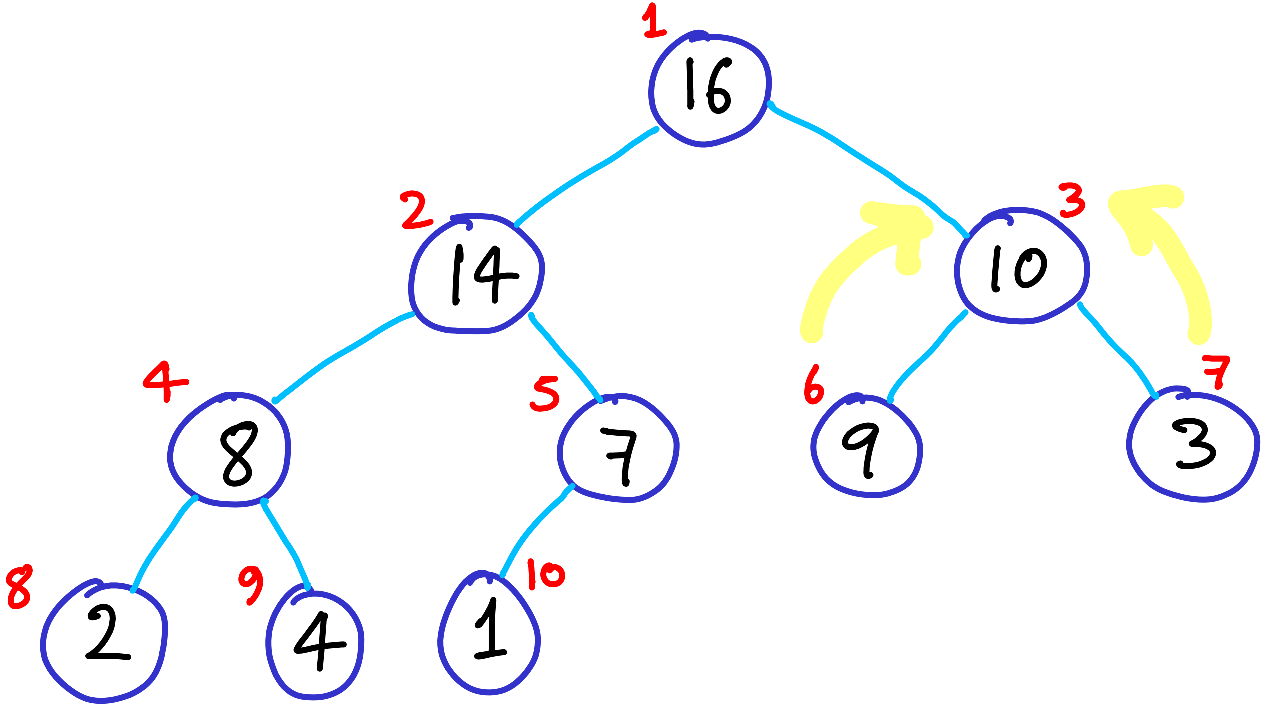


How can we identify the indices of the children of a given node?

left-child(i) = $2i$

right-child(i) = $2i+1$

parent(i) = ?

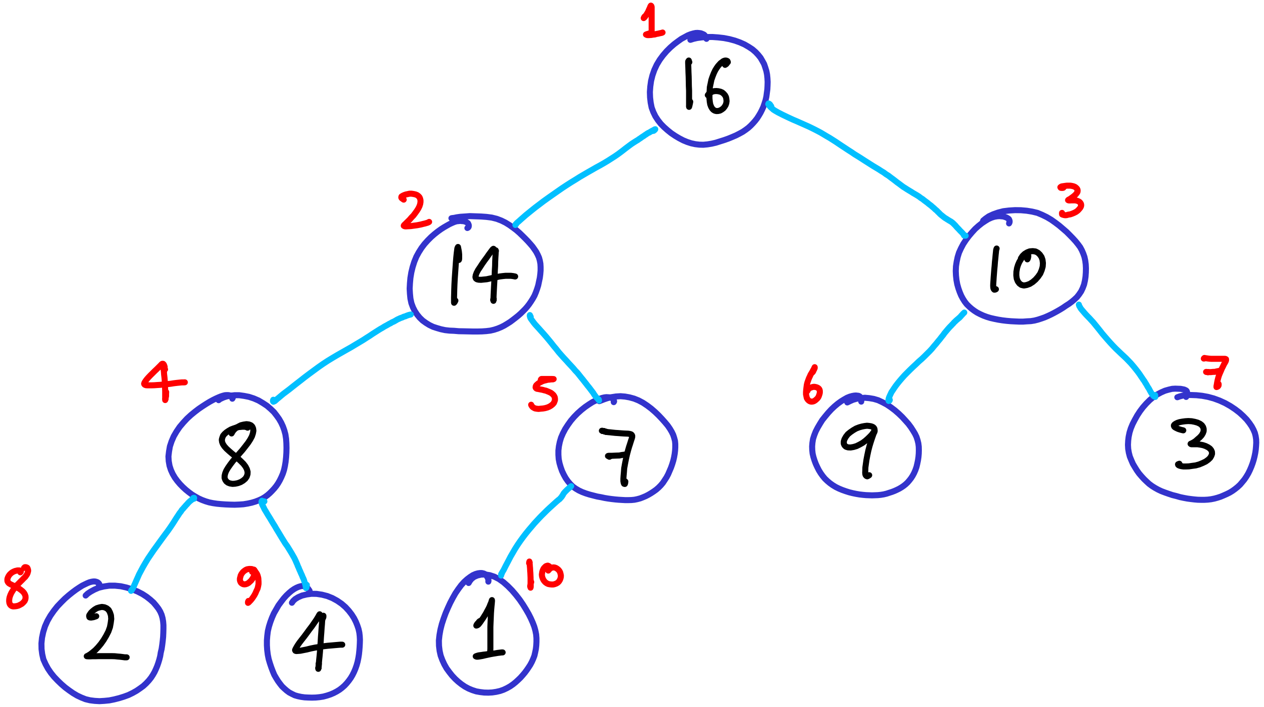


How can we identify the indices of the children of a given node?

left-child(i) = $2i$

right-child(i) = $2i+1$

parent(i) = $\lfloor i/2 \rfloor$



How can we identify the indices of the children of a given node?

Use array to store heap

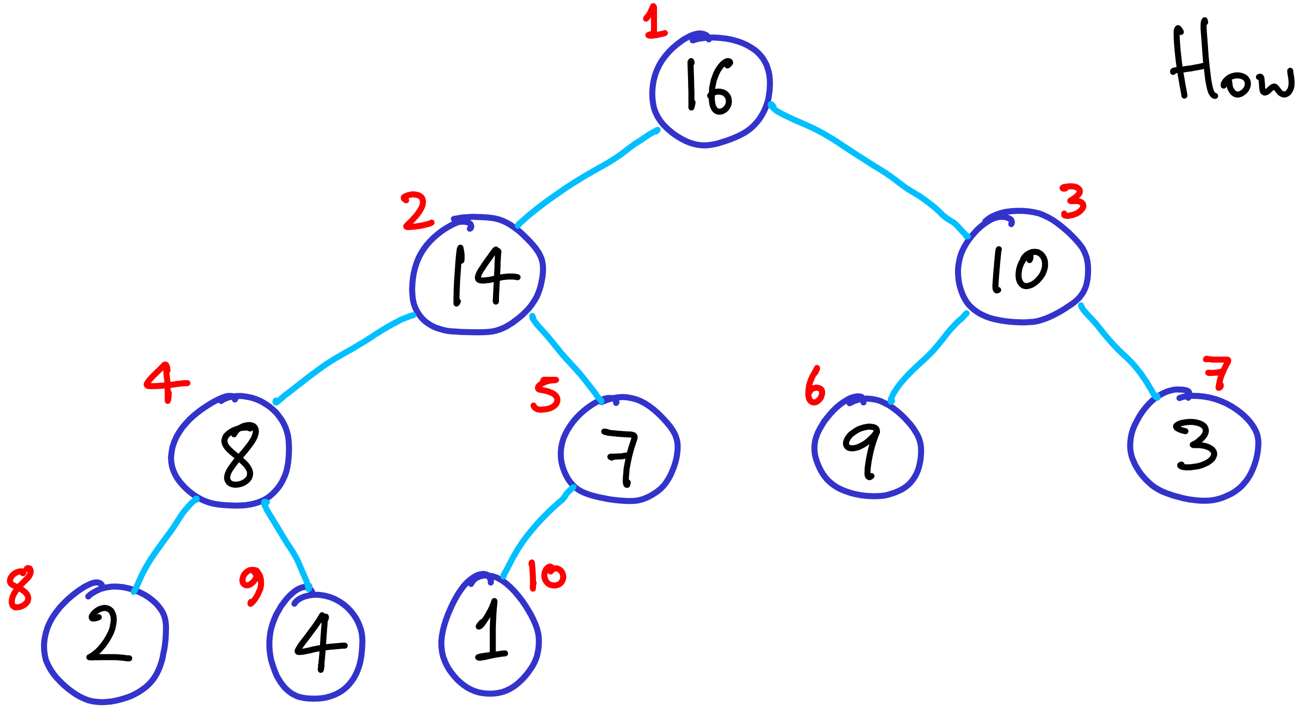
(avoid wasting space with pointers)

1	2	3	4	5	6	7	8	9	10	
16	14	10	8	7	9	3	2	4	1	.....

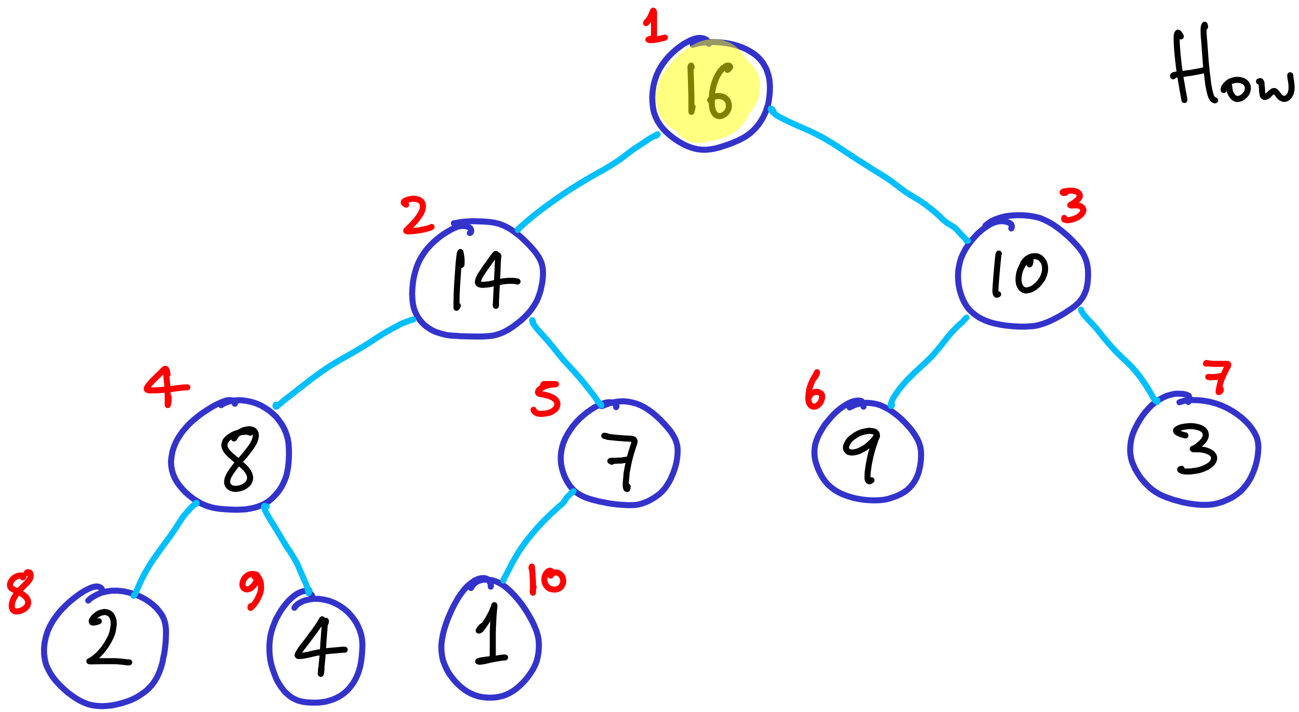
left-child(i) = $2i$

right-child(i) = $2i+1$

parent(i) = $\lfloor i/2 \rfloor$

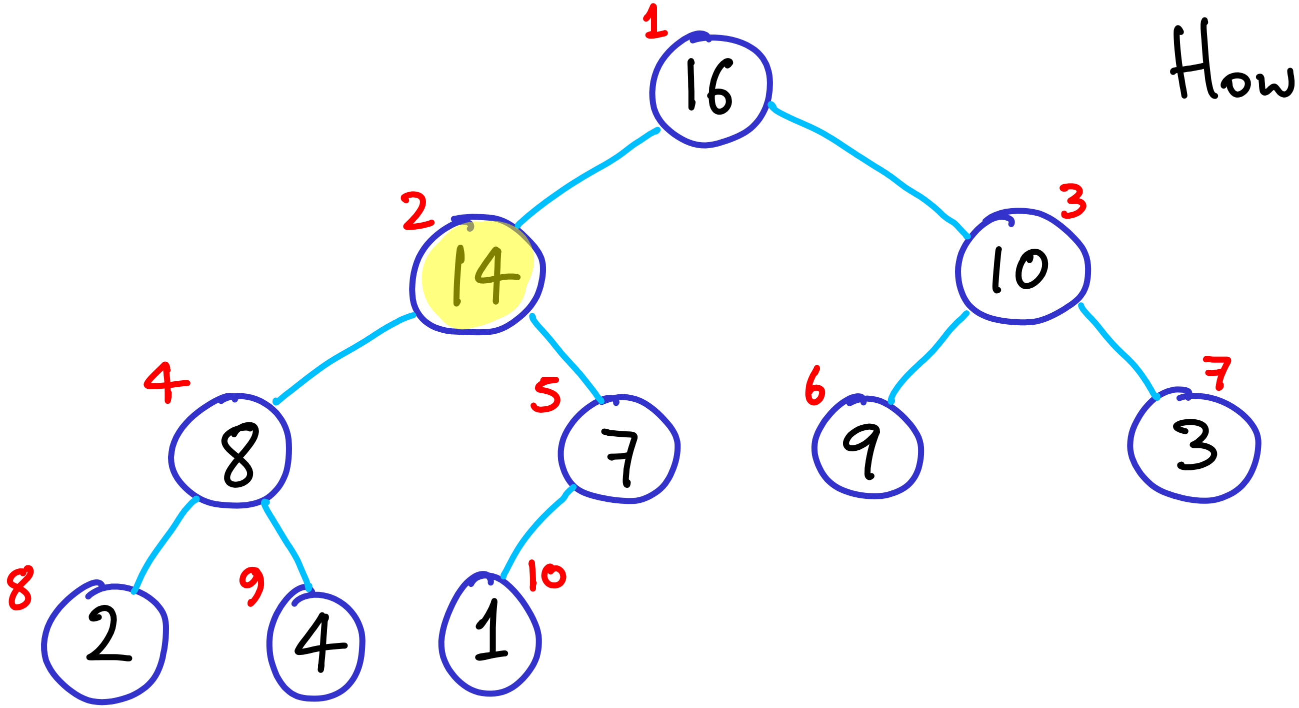


How does this relate to sorting?



How does this relate to sorting?

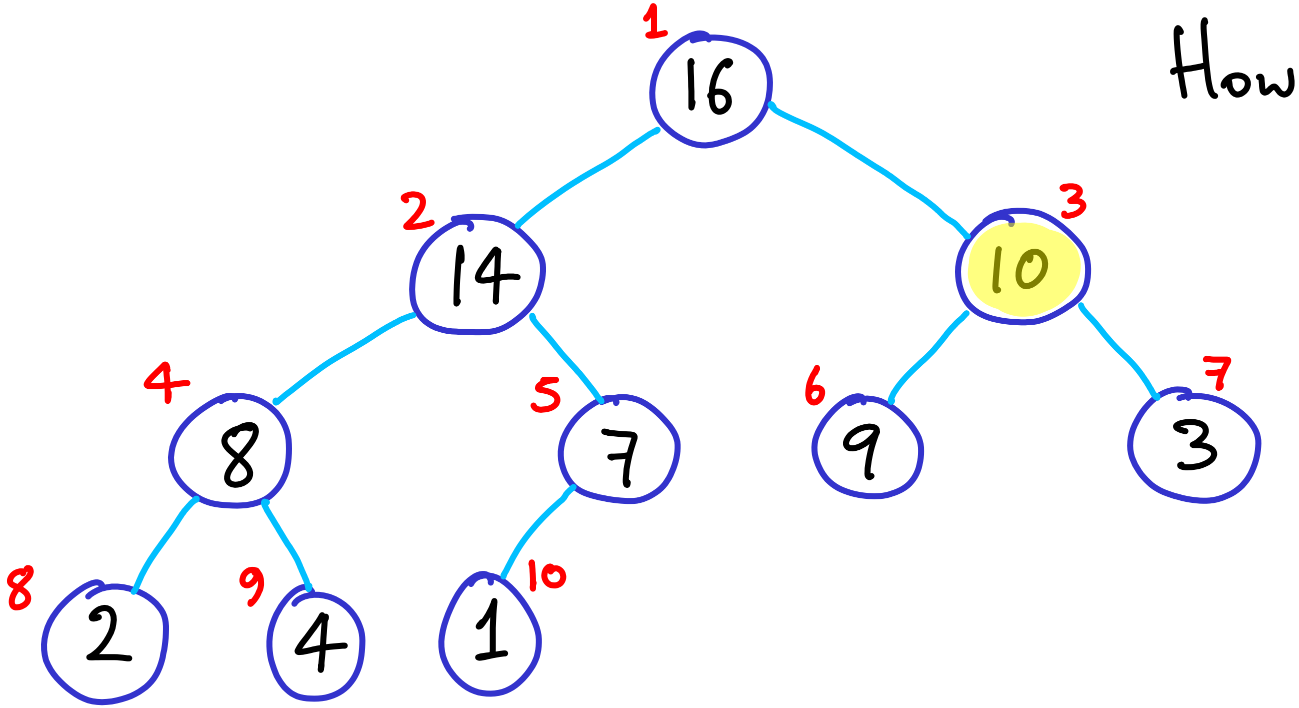
Largest element is on top.



How does this relate to sorting?

Largest element is on top.

2nd largest is in level 2.



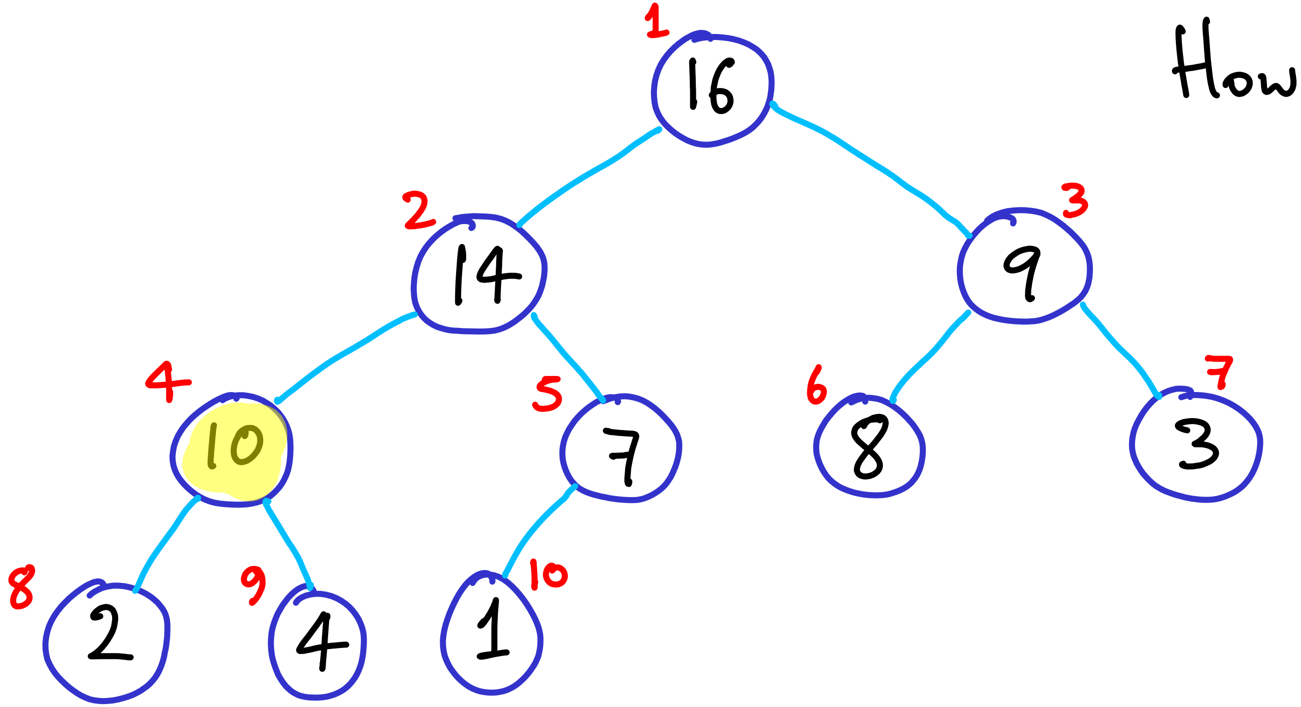
How does this relate to sorting?

Largest element is on top.

2nd largest is in level 2.

3rd largest is

↳ in level 2



How does this relate to sorting?

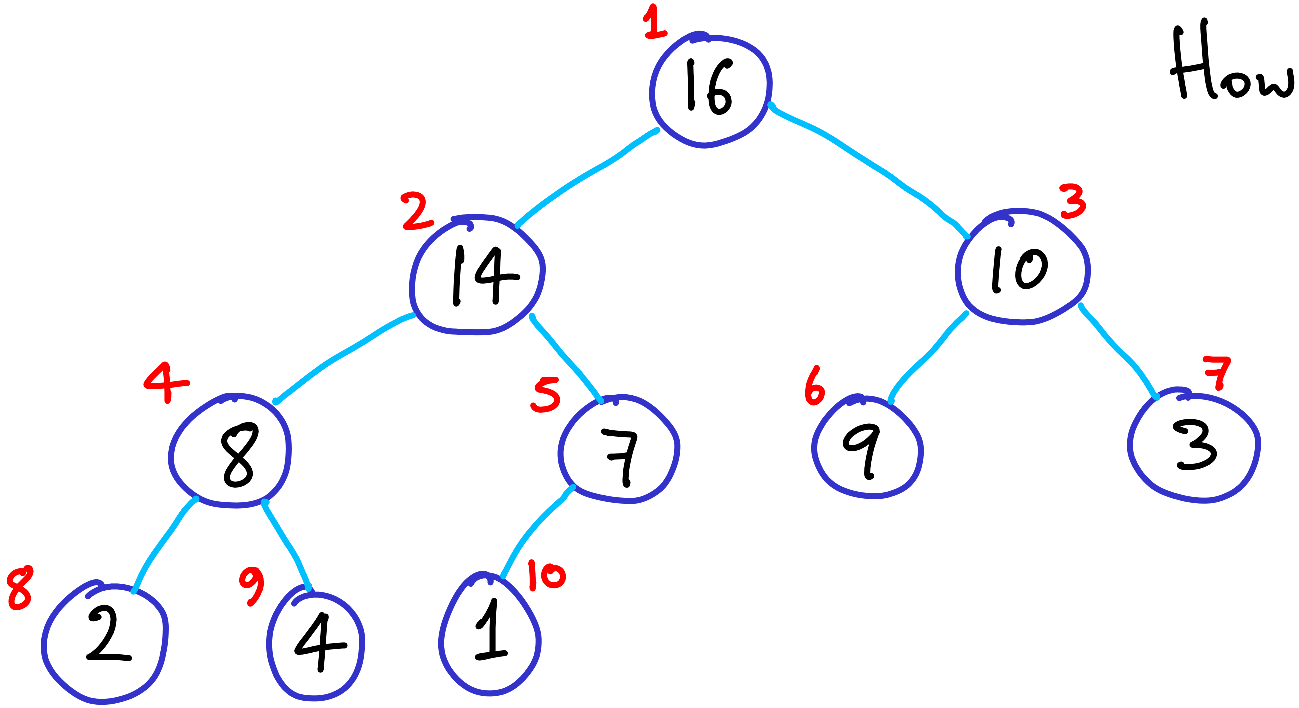
Largest element is on top.

2nd largest is in level 2.

3rd largest is
↳ in level 2
OR
↳ in level 3
& child of 2nd

⋮

getting messy



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Largest element is on top.

2nd largest is in level 2.

3rd largest is

↳ in level 2

OR

↳ in level 3

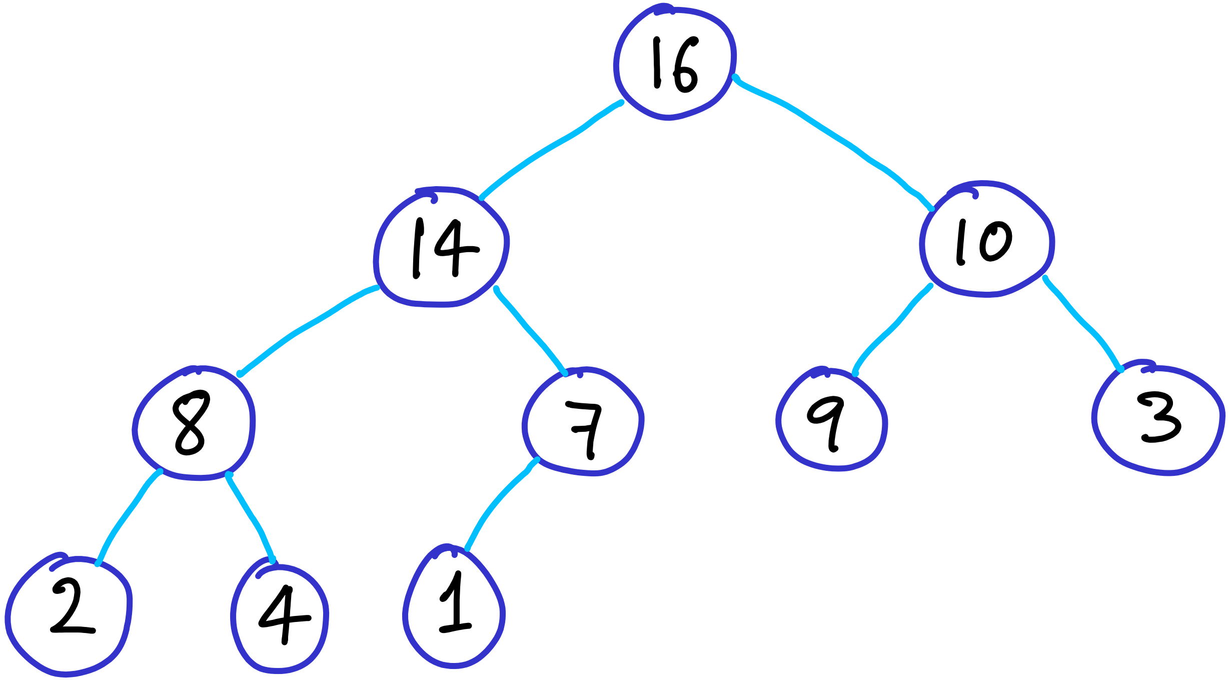
& child of 2nd

⋮

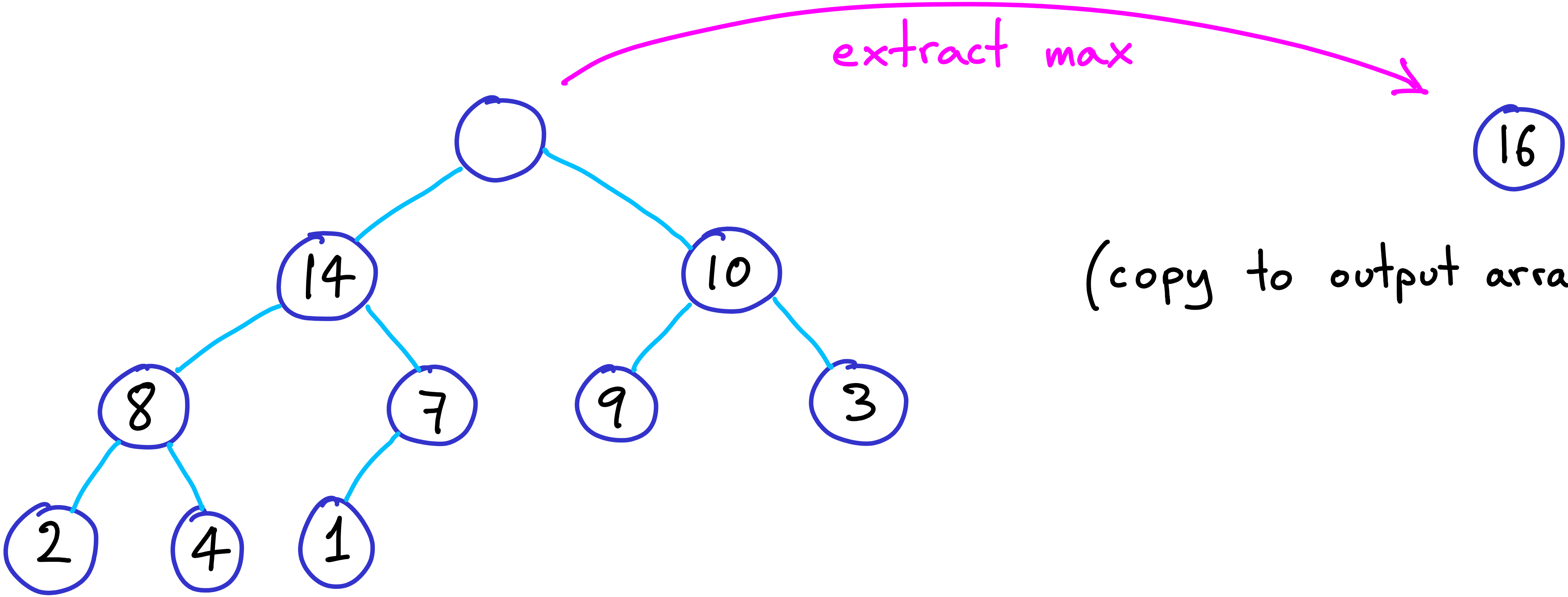
getting messy

Heaps are not "sorted"

How to sort data in a heap



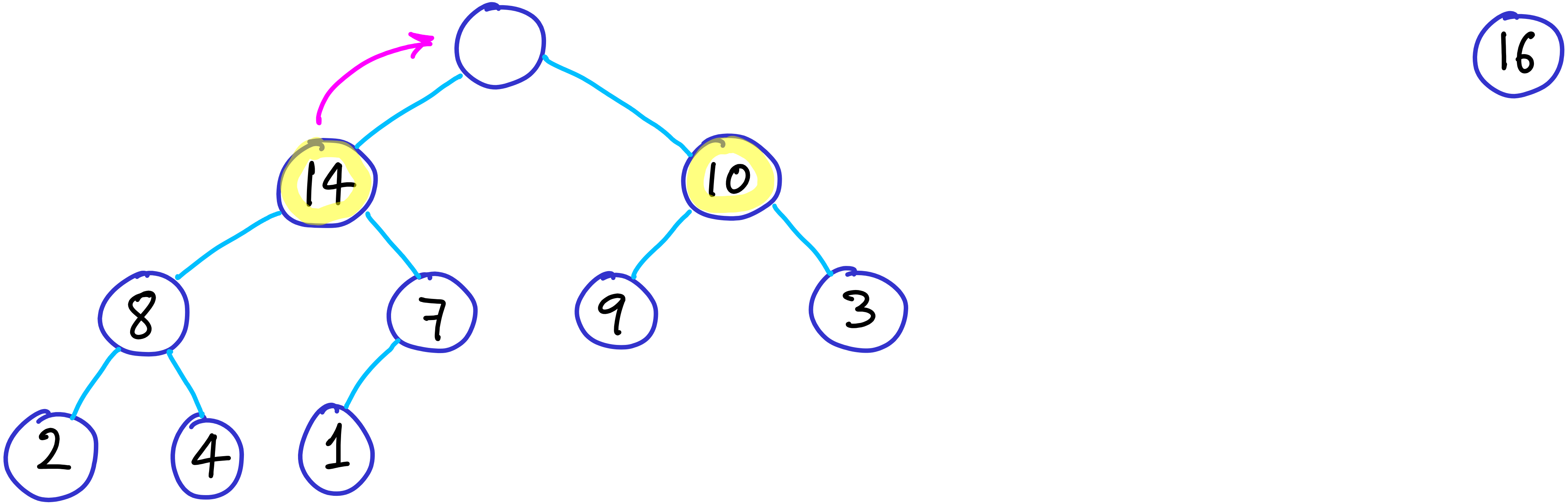
How to sort data in a heap



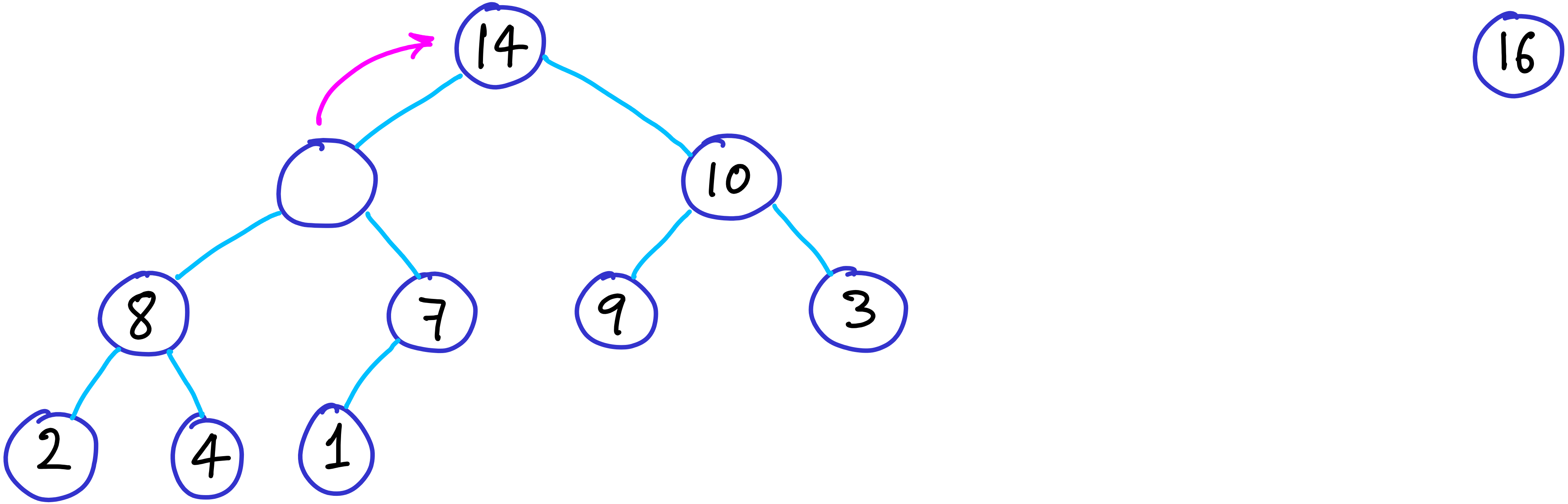
(copy to output array)

How to sort data in a heap

Update max : larger of 2 children

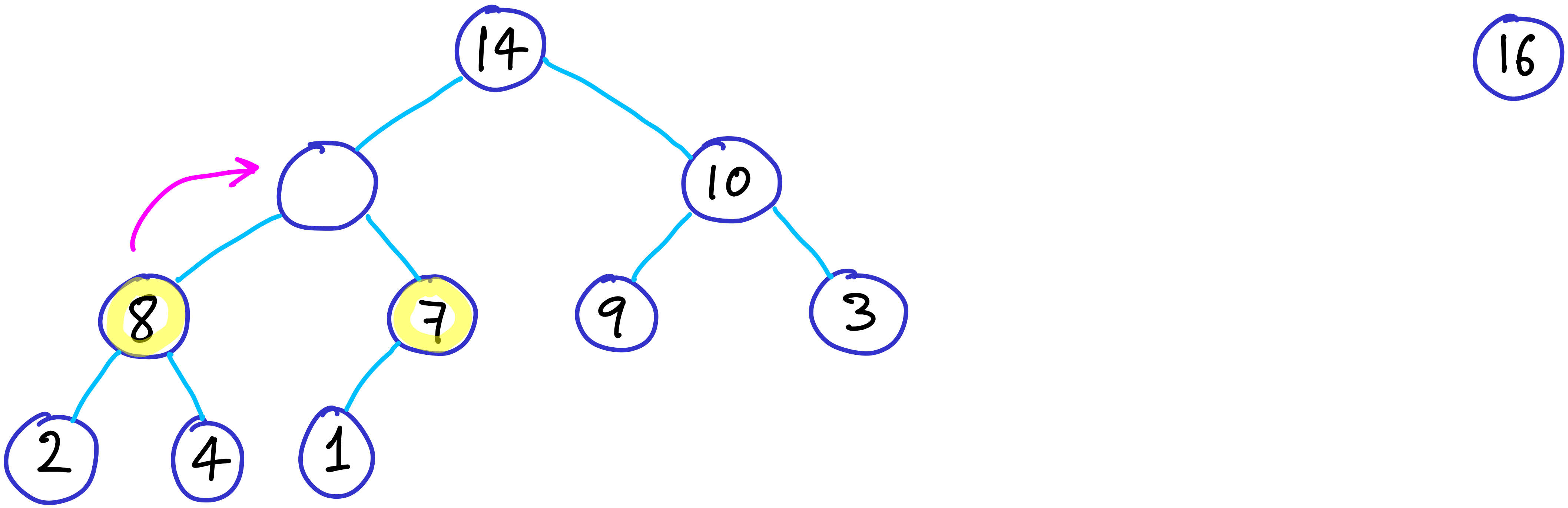


How to sort data in a heap

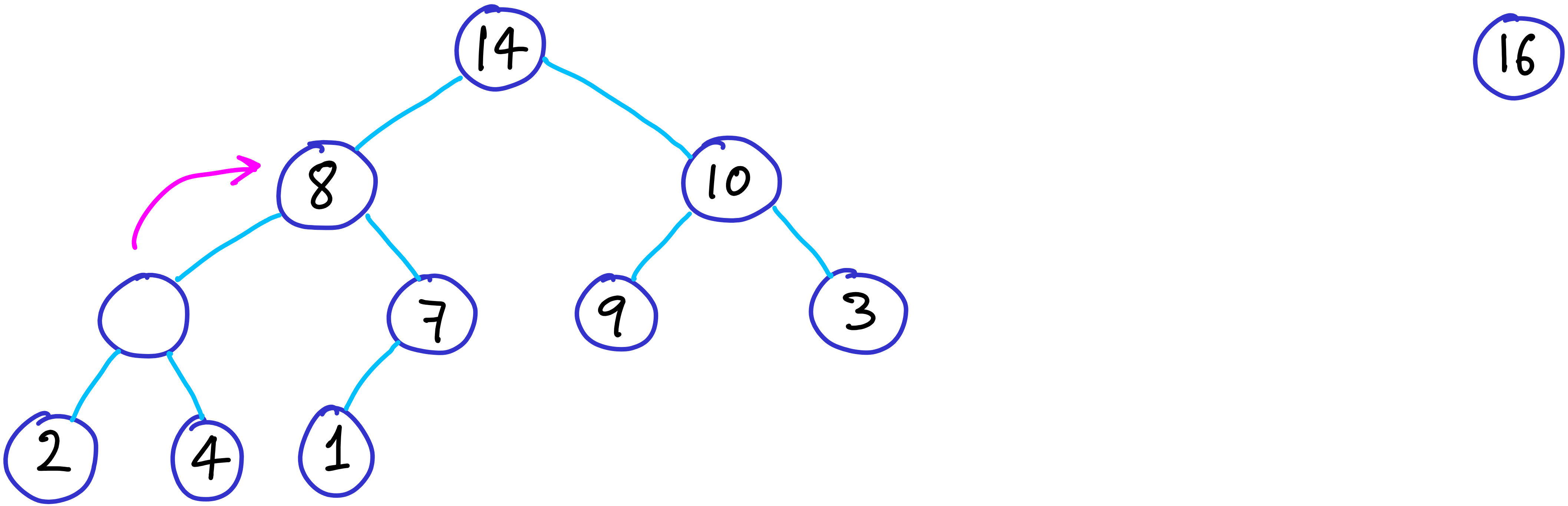


How to sort data in a heap

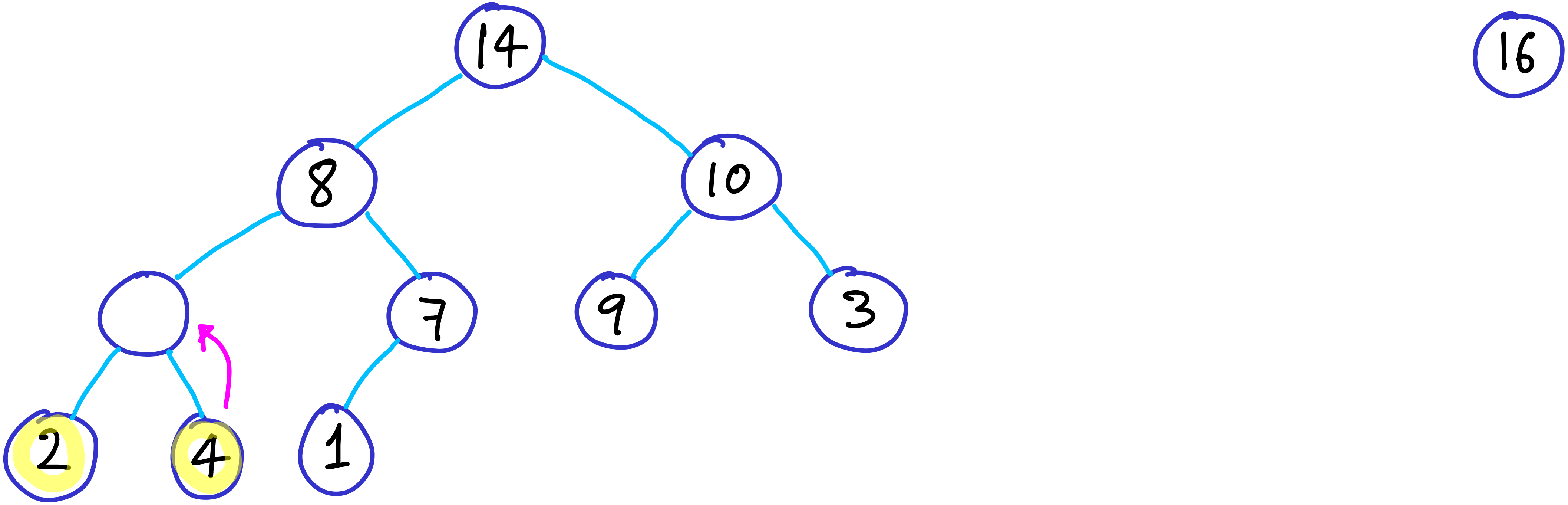
Update max recursively



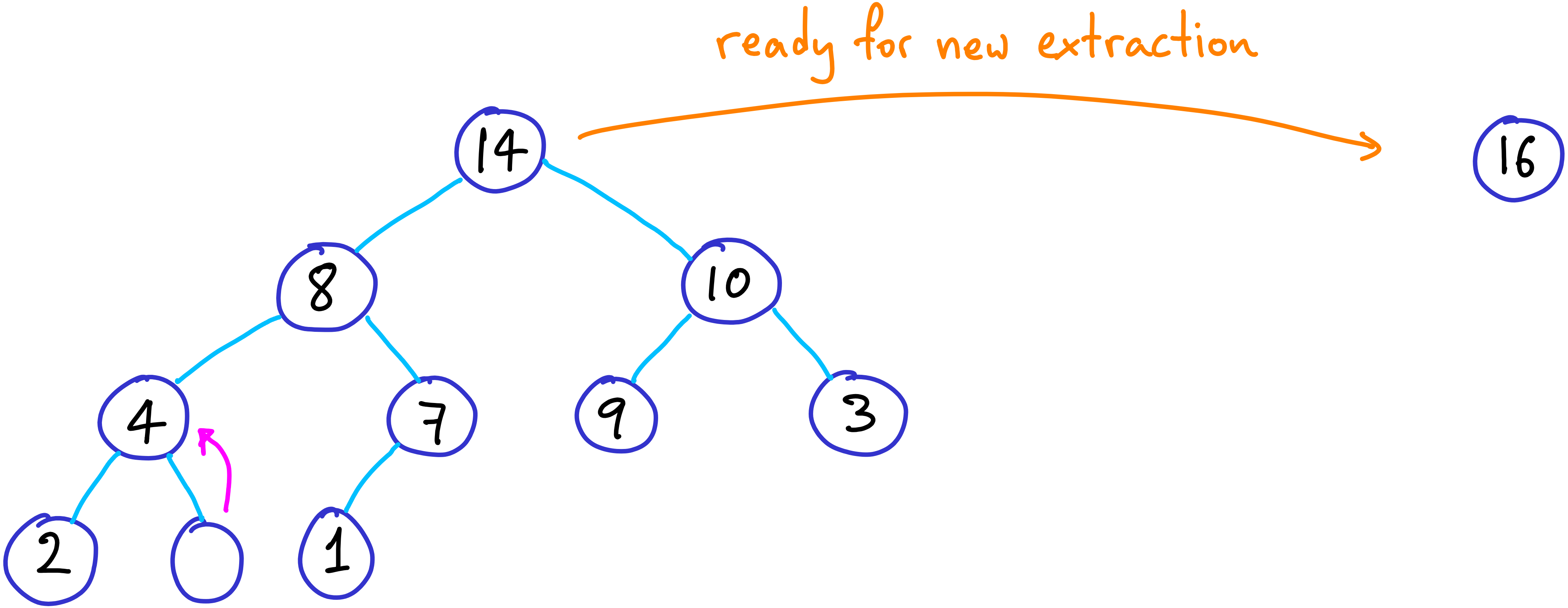
How to sort data in a heap



How to sort data in a heap

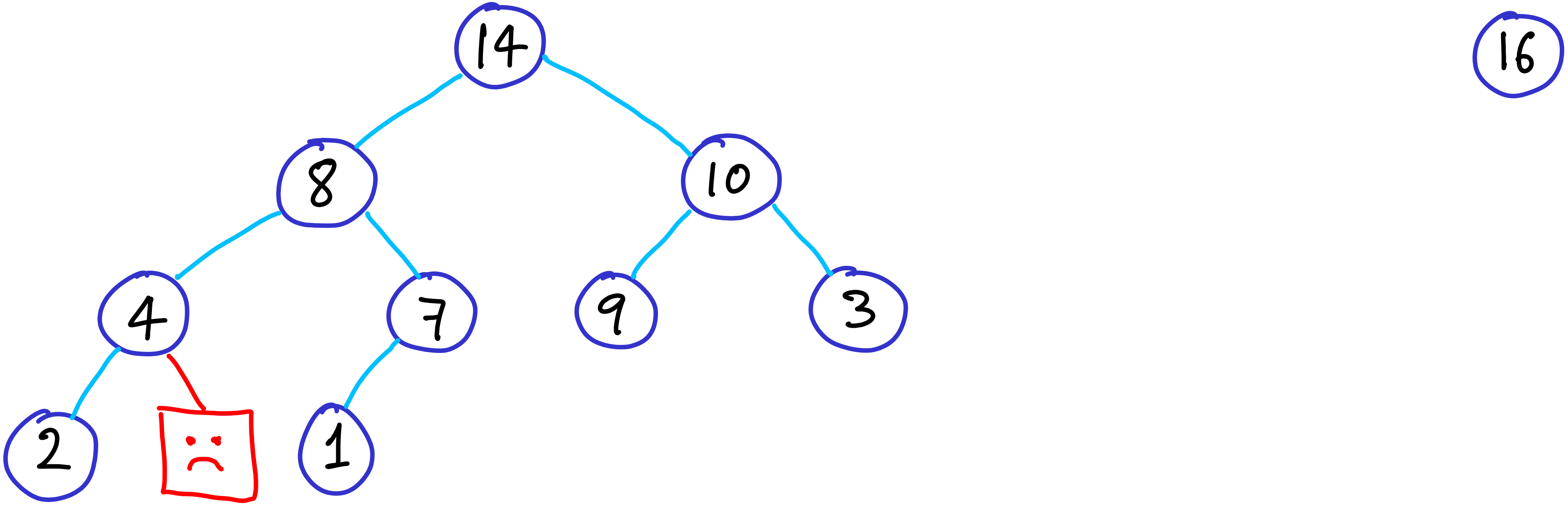


How to sort data in a heap



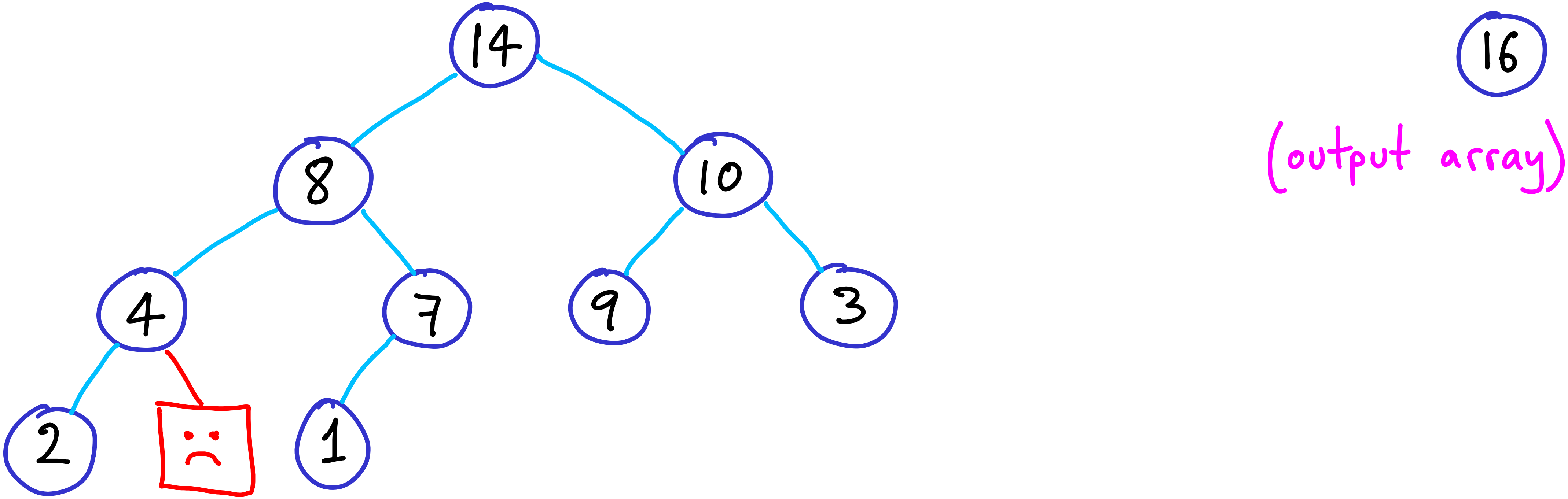
How to sort data in a heap

↳ if we don't care about keeping the heap complete



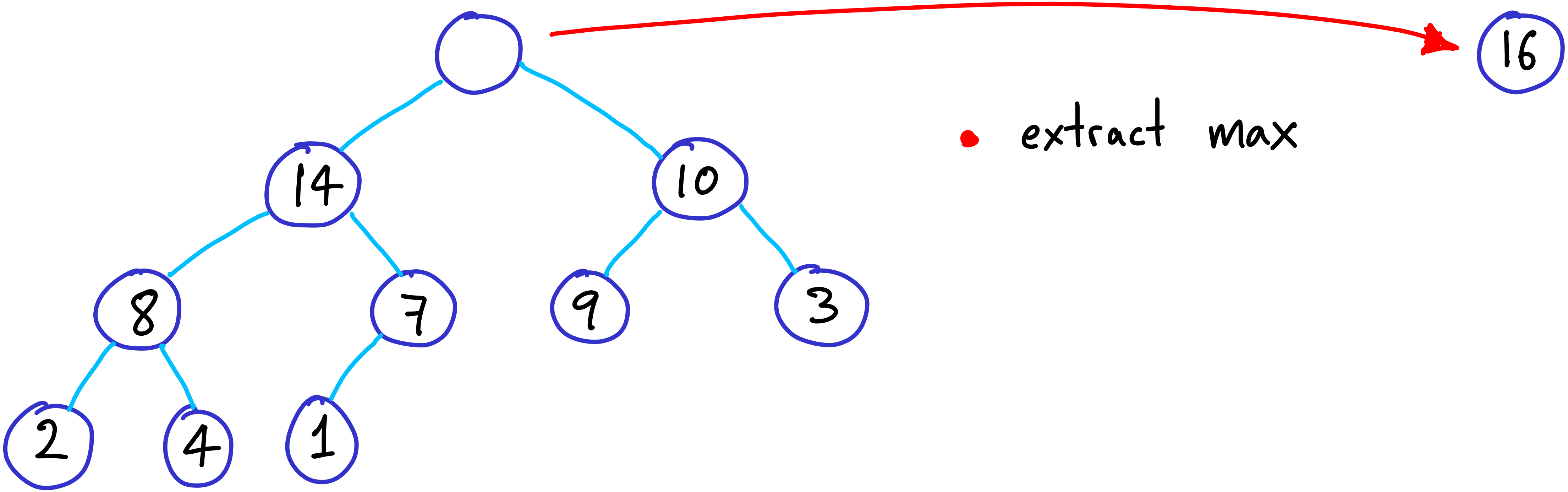
How to sort data in a heap

↳ if we don't care about
- keeping the heap complete
- using extra space



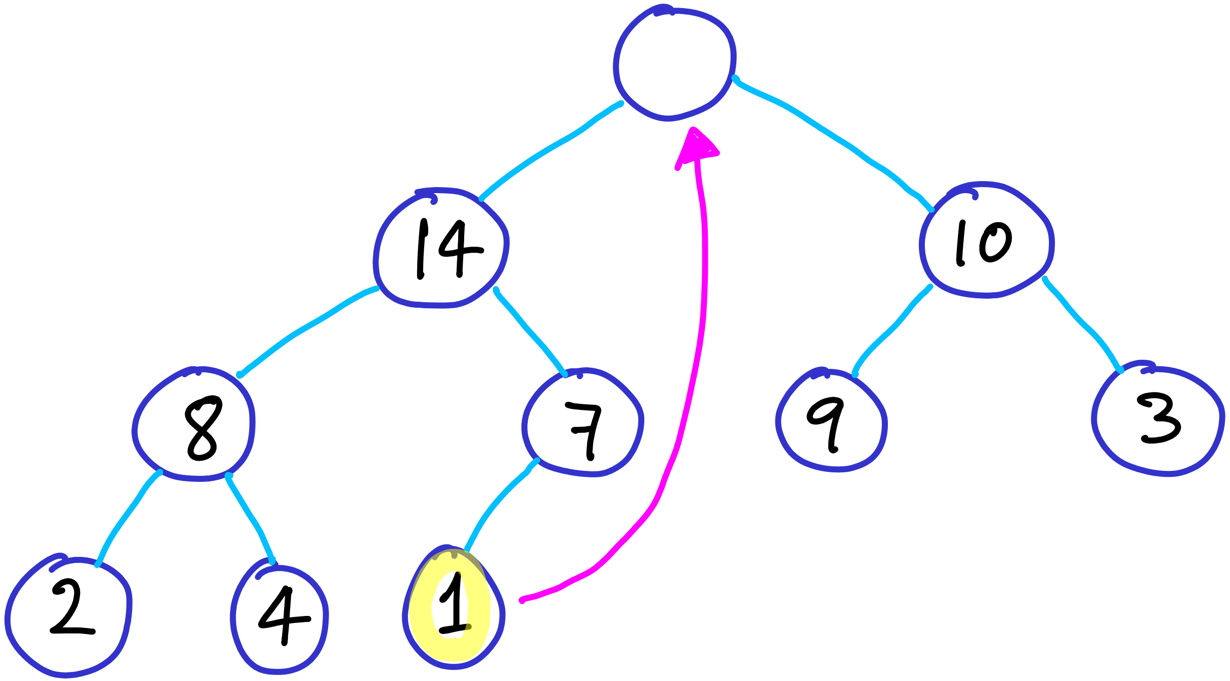
How to sort data in a complete heap ... using extra space

How to sort data in a complete heap ... using extra space



- extract max

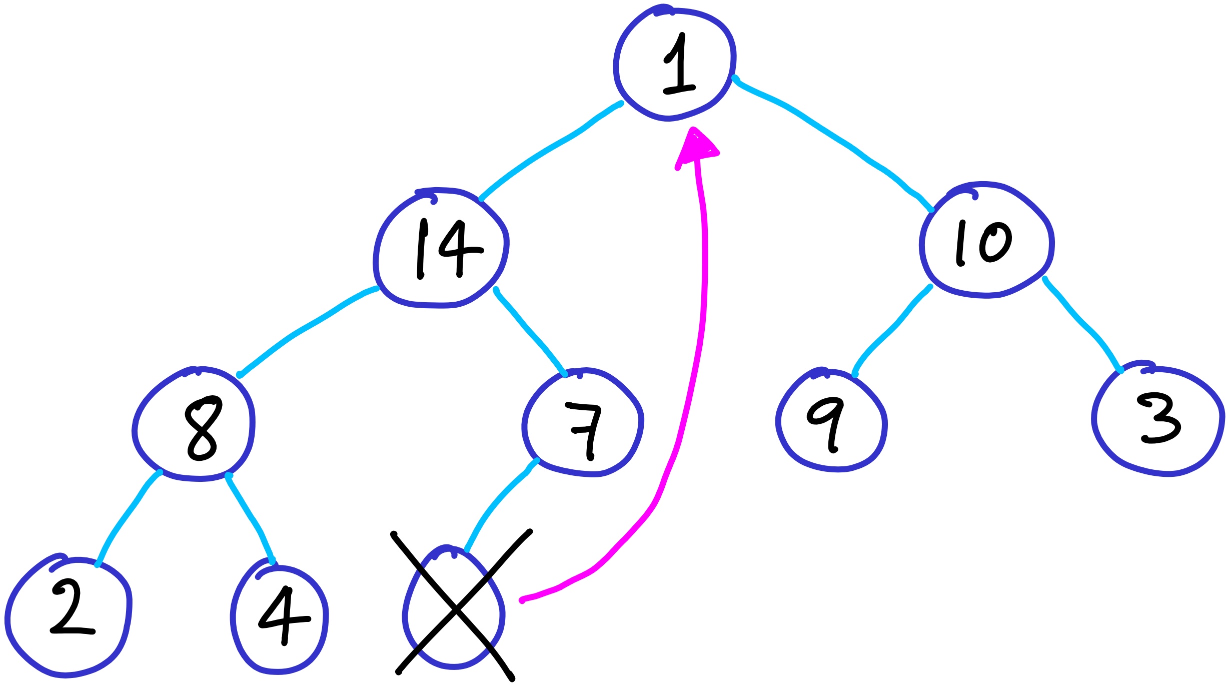
How to sort data in a complete heap ... using extra space

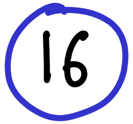


- extract max
- replace root

  with rightmost leaf
  from lowest level

How to sort data in a complete heap ... using extra space





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- replace root
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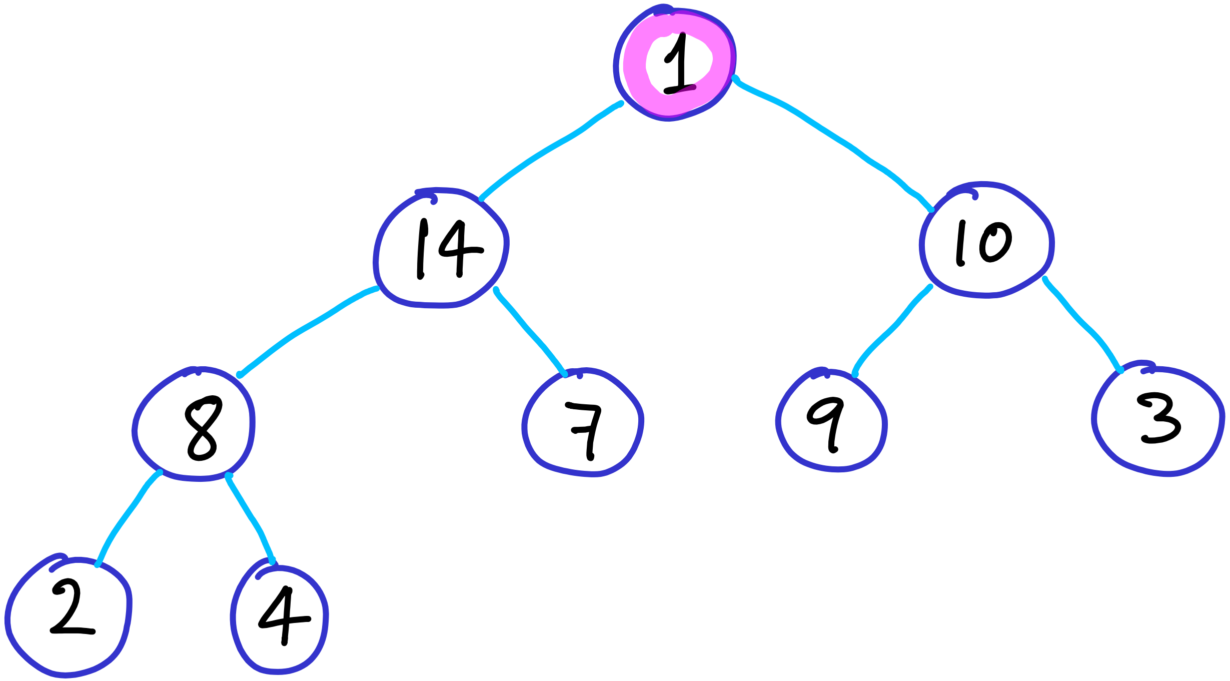

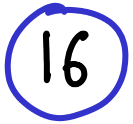

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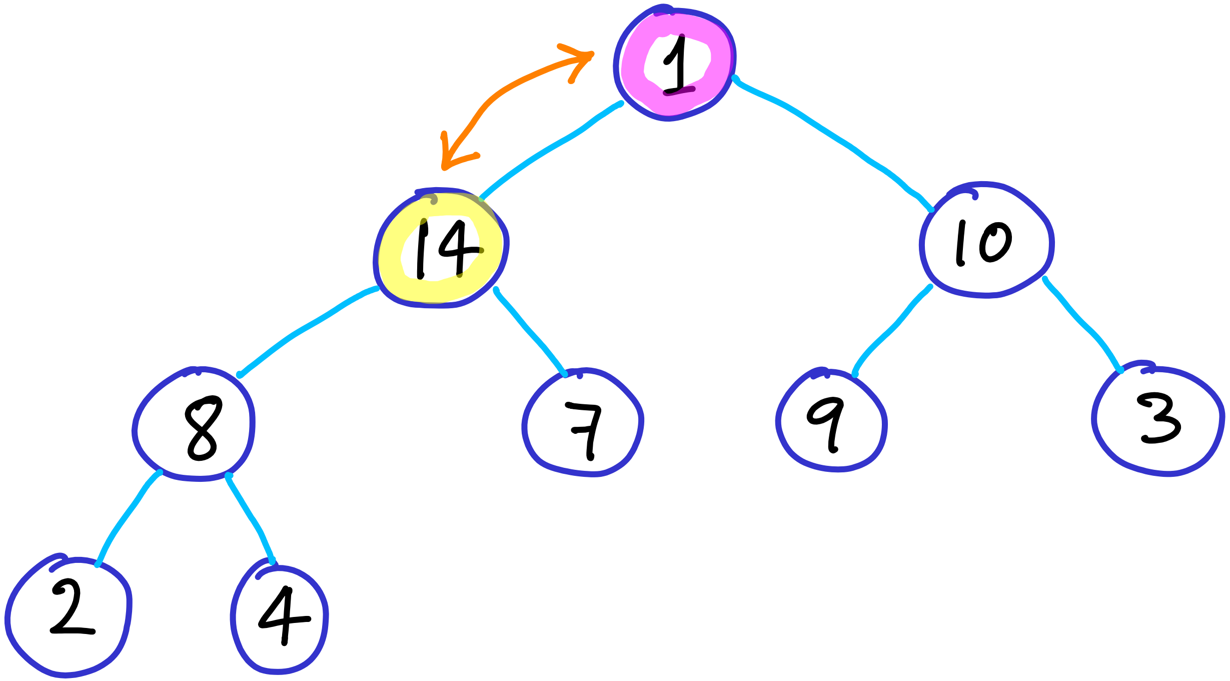
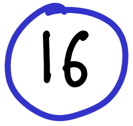
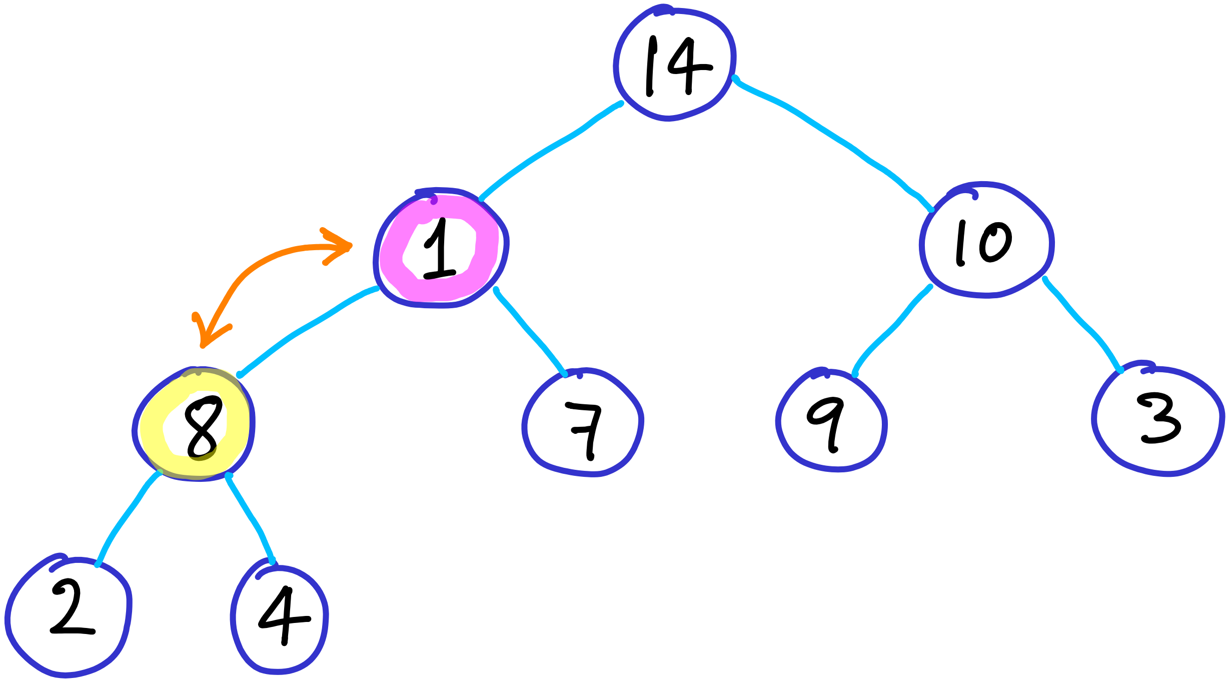
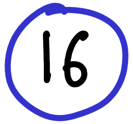




- extract max
- replace root
  with rightmost leaf
  from lowest level
- recursively swap
  with largest child
  while heap not restored

How to sort data in a complete heap ... using extra space





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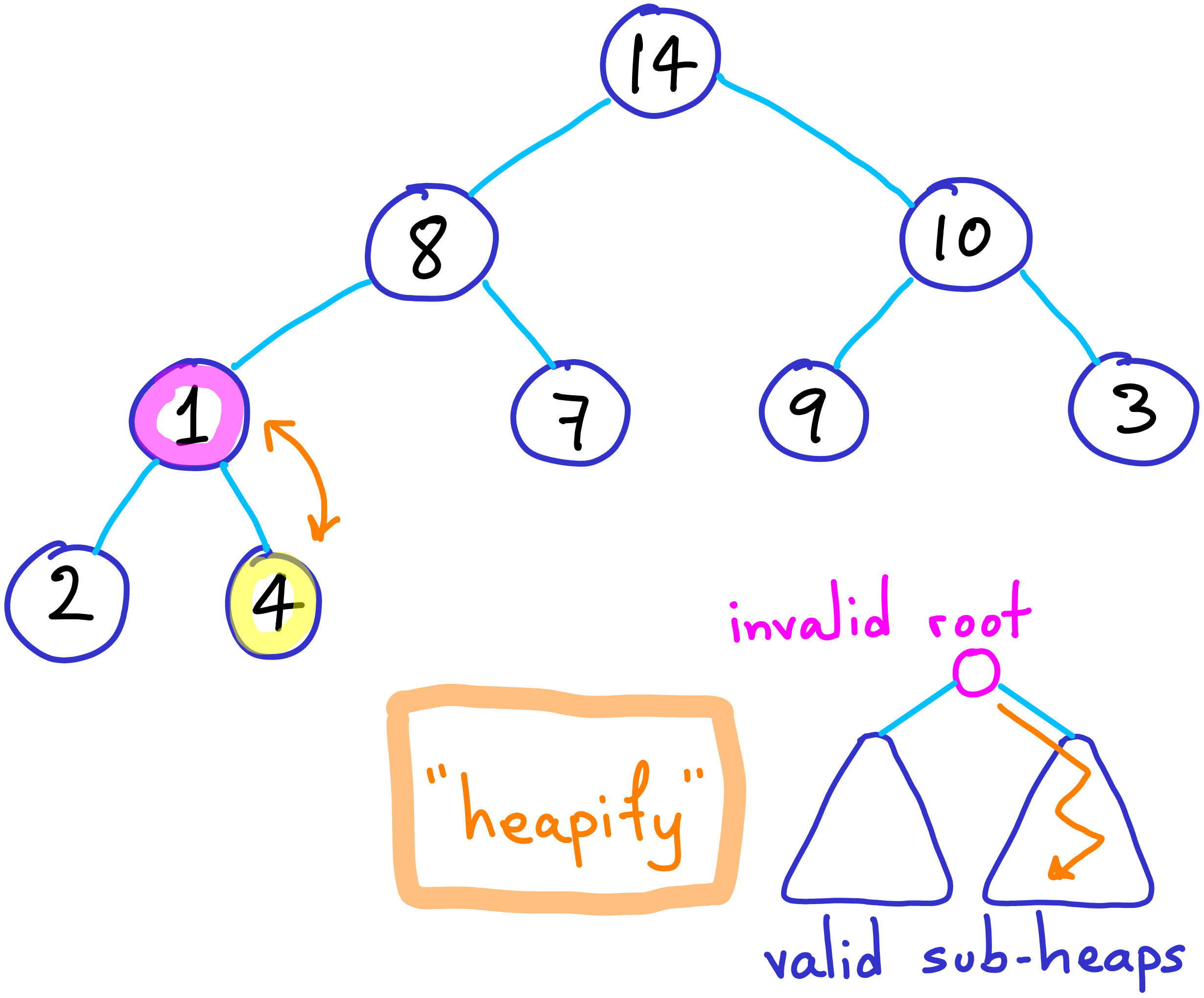
How to sort data in a complete heap ... using extra space



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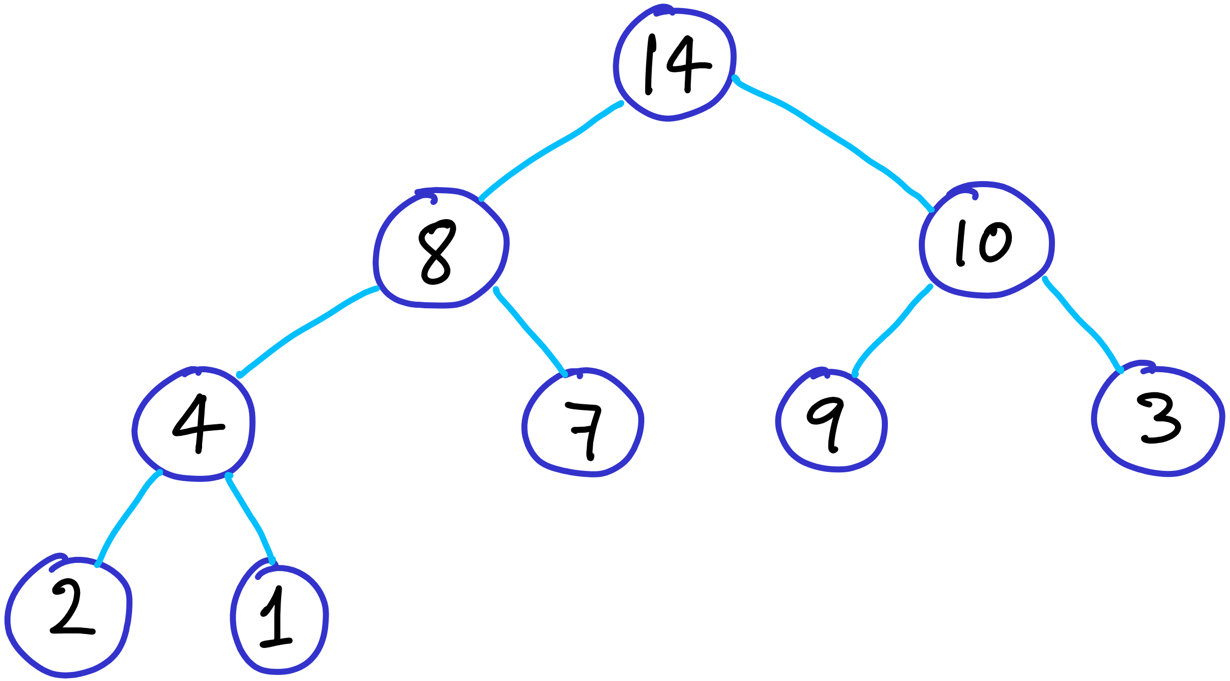
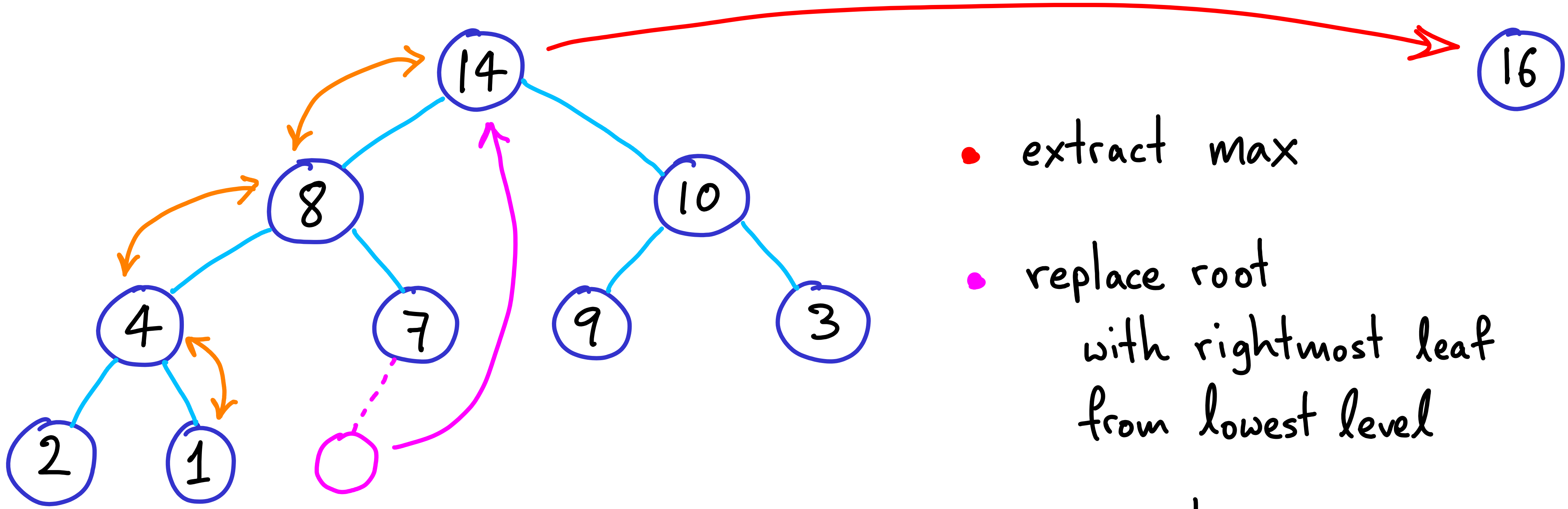


16

- extract max
- replace root with rightmost leaf from lowest level
- recursively swap with largest child while heap not restored

time ?

How to sort data in a complete heap ... using extra space

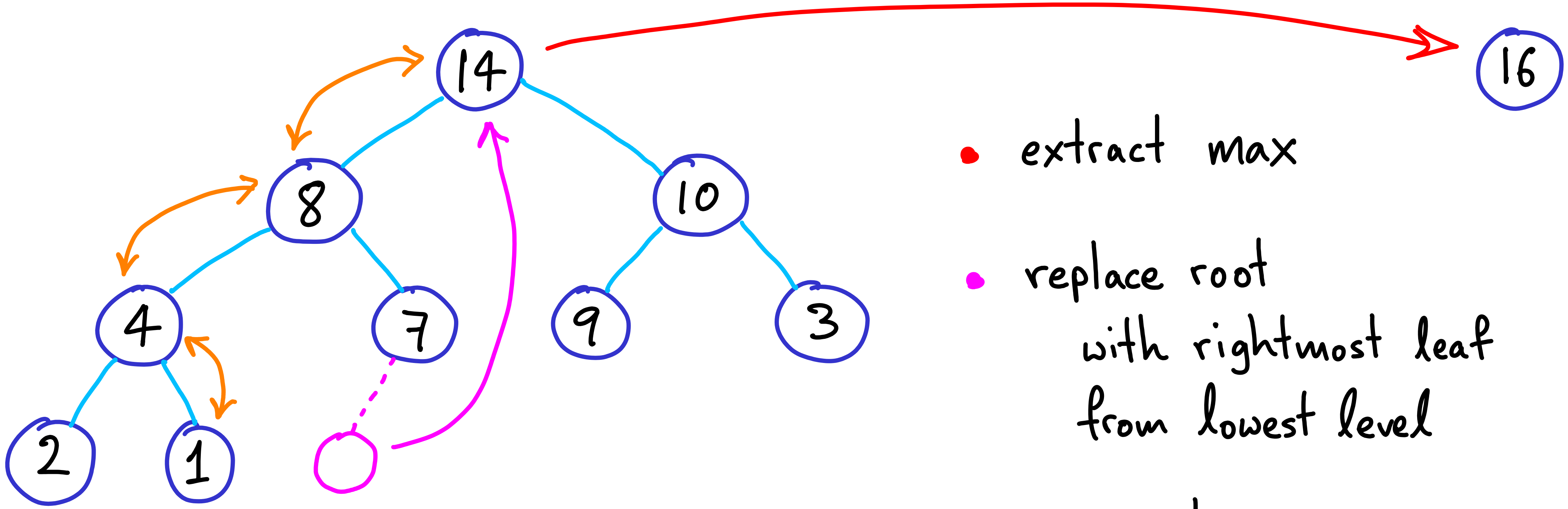


- extract max
- replace root with rightmost leaf from lowest level
- recursively swap with largest child while heap not restored

time ?

O(logn) per extraction

How to sort data in a complete heap ... using extra space



- extract max
- replace root with rightmost leaf from lowest level
- recursively swap with largest child while heap not restored

time = O(nlogn)

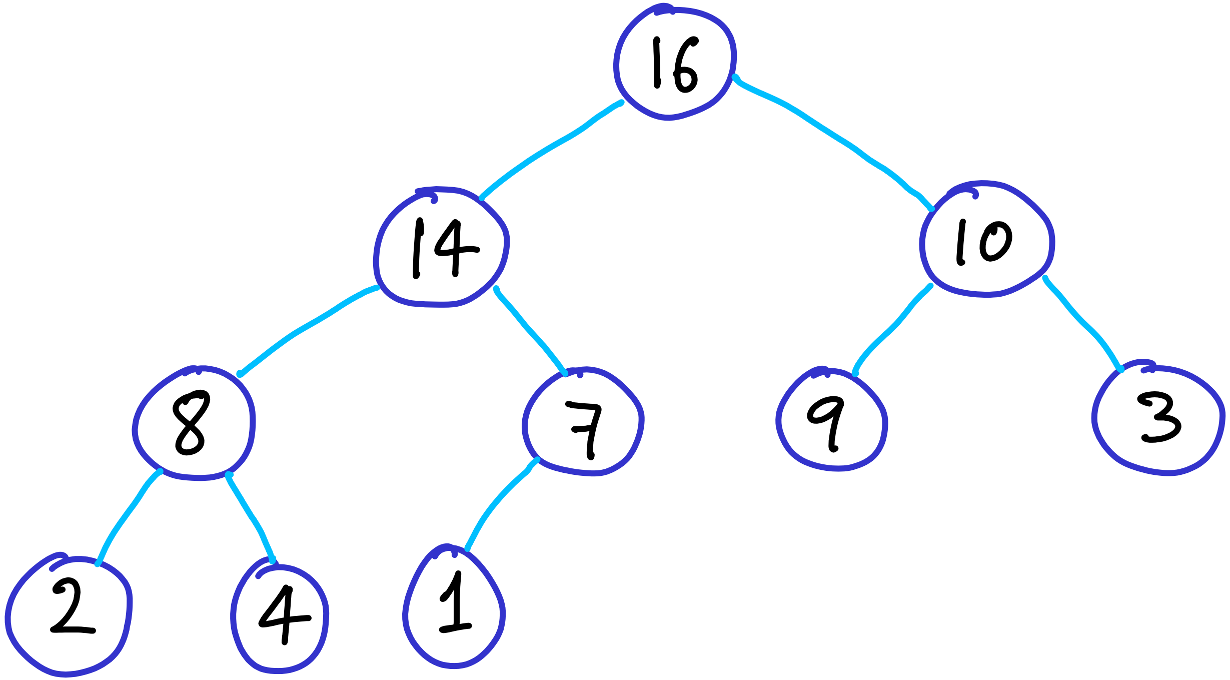
O(logn) per extraction

How to sort data in a complete heap in place

→ (without an output array)

How to sort data in a complete heap in place

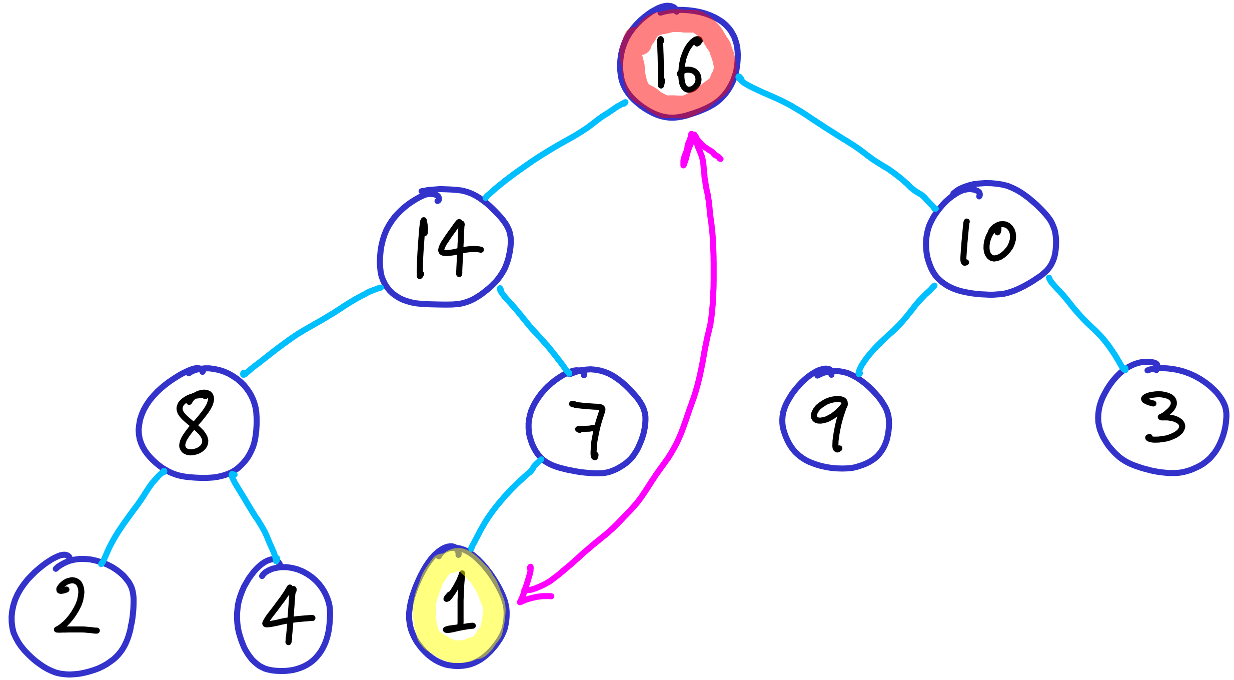
(without an output array)



1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

How to sort data in a complete heap in place

(without an output array)

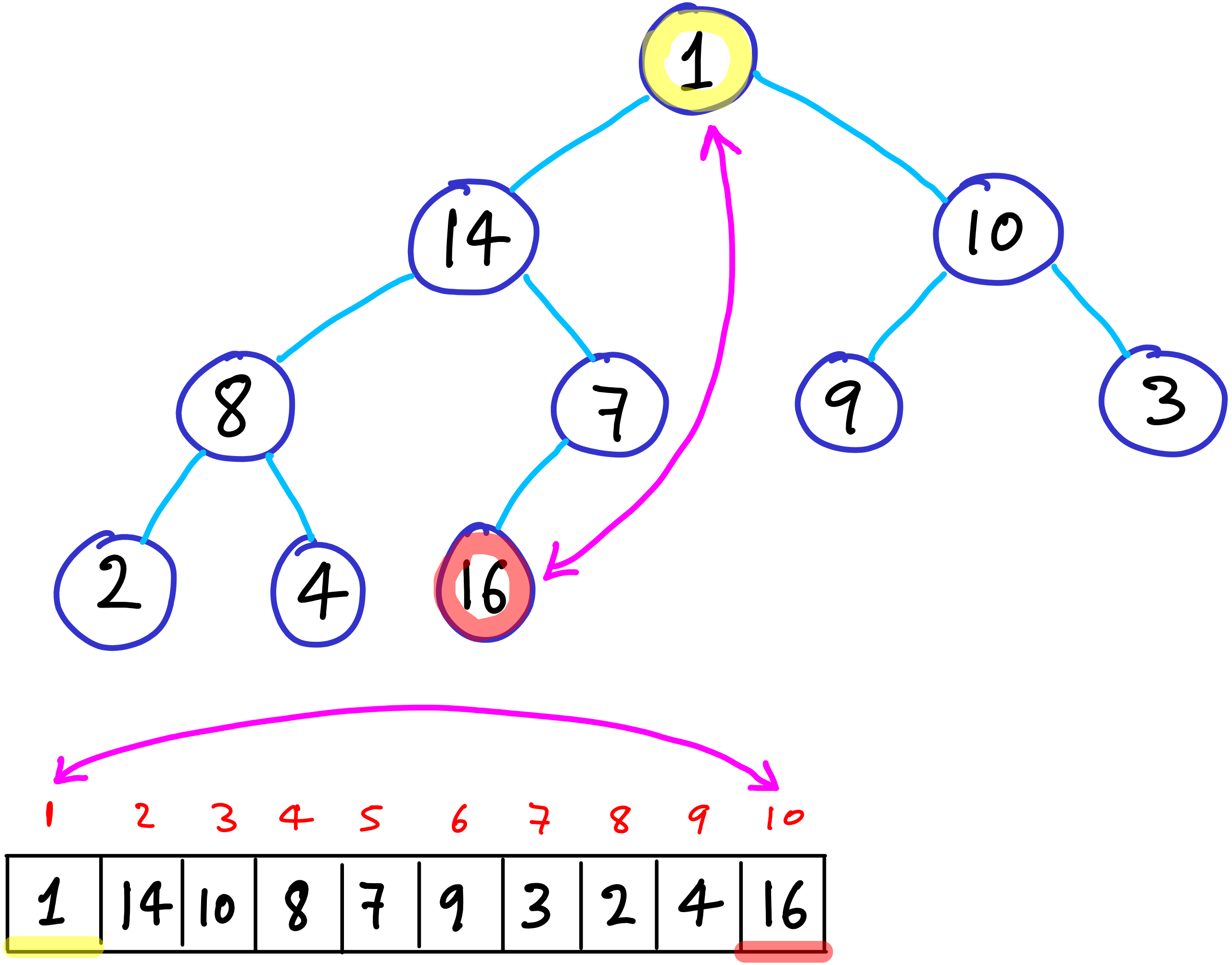


Same as before but we swap max with replacement

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

How to sort data in a complete heap in place

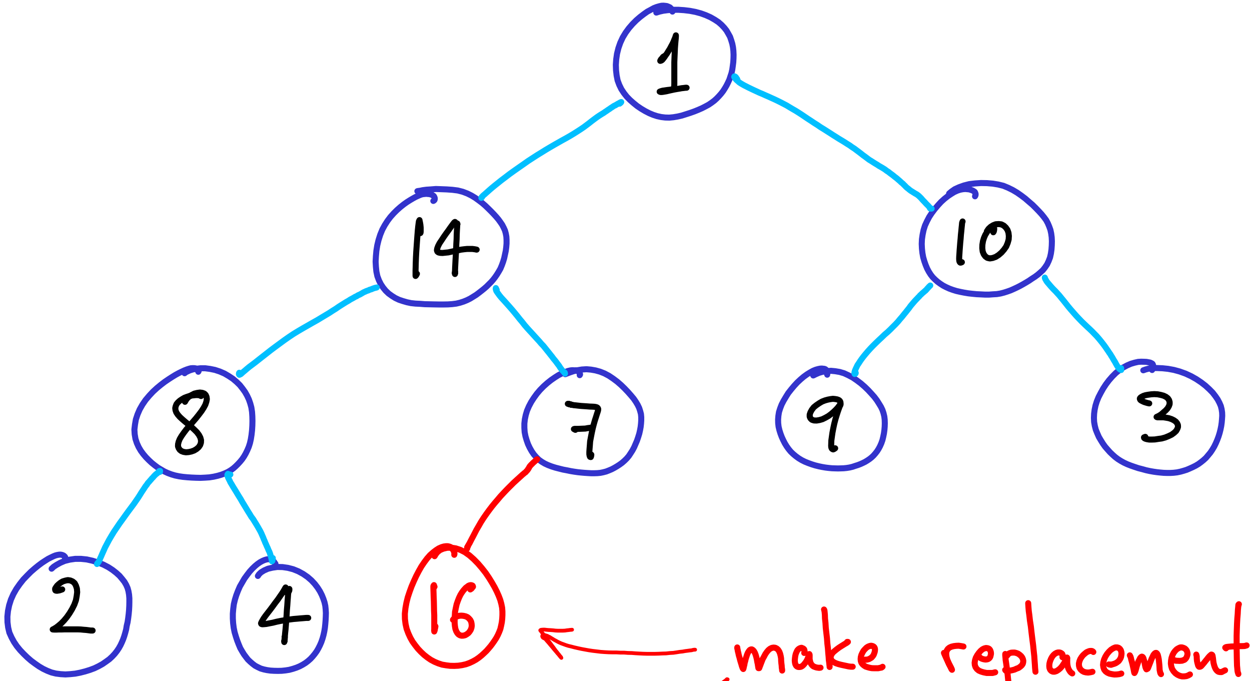
(without an output array)



1	2	3	4	5	6	7	8	9	10
1	14	10	8	7	9	3	2	4	16

Same as before but we swap max with replacement

How to sort data in a complete heap in place (without an output array)



Same as before but we swap max with replacement

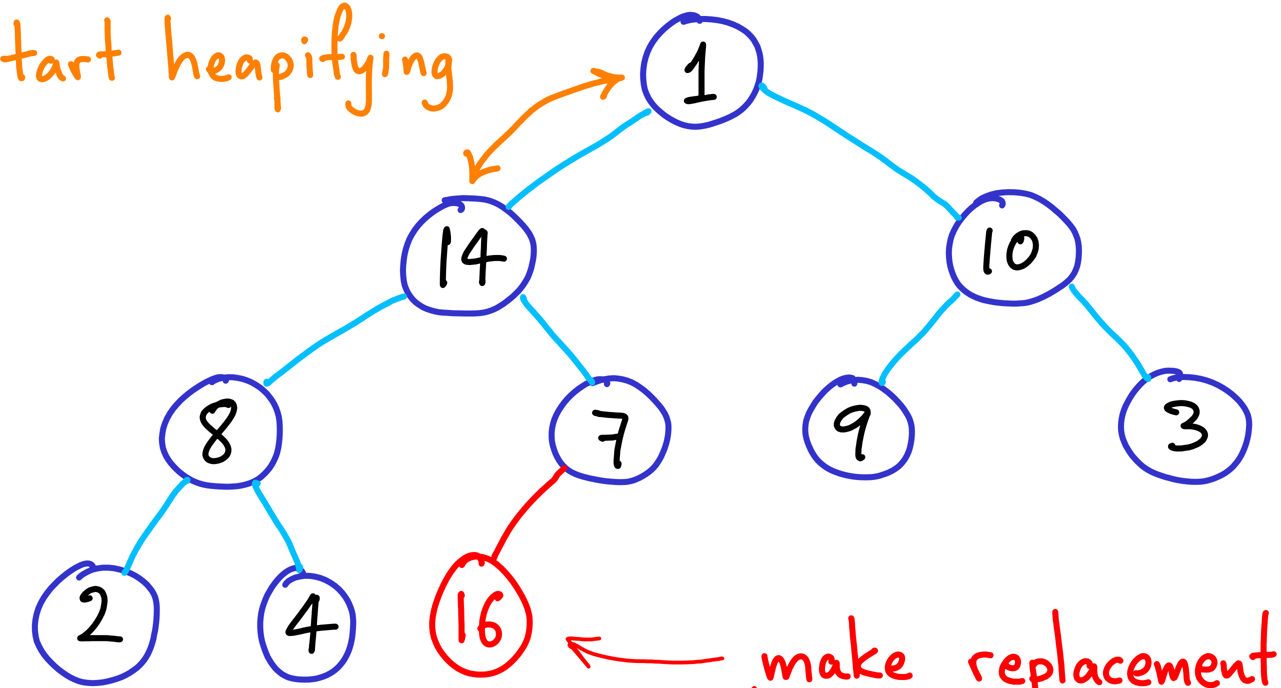
make replacement position inactive

as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
1	14	10	8	7	9	3	2	4	16

How to sort data in a complete heap in place (without an output array)

start heapifying



Same as before but we swap max with replacement

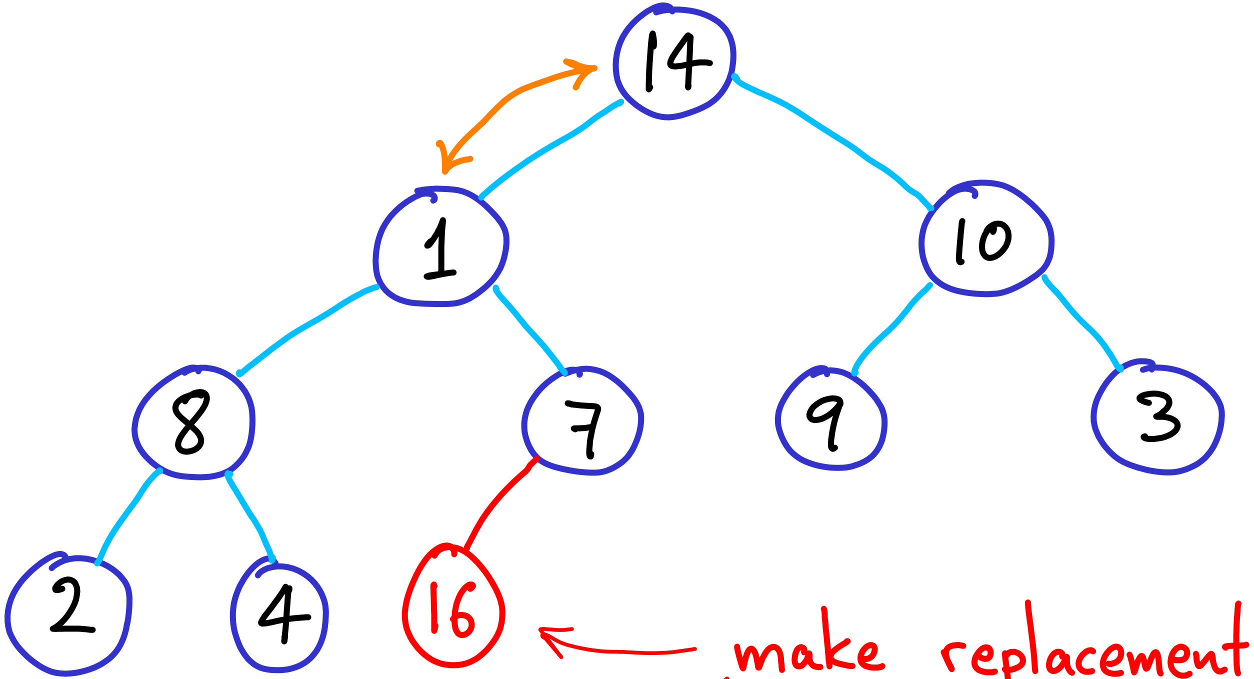
make replacement position inactive

as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
1	14	10	8	7	9	3	2	4	16

How to sort data in a complete heap in place

(without an output array)



Same as before but we swap max with replacement

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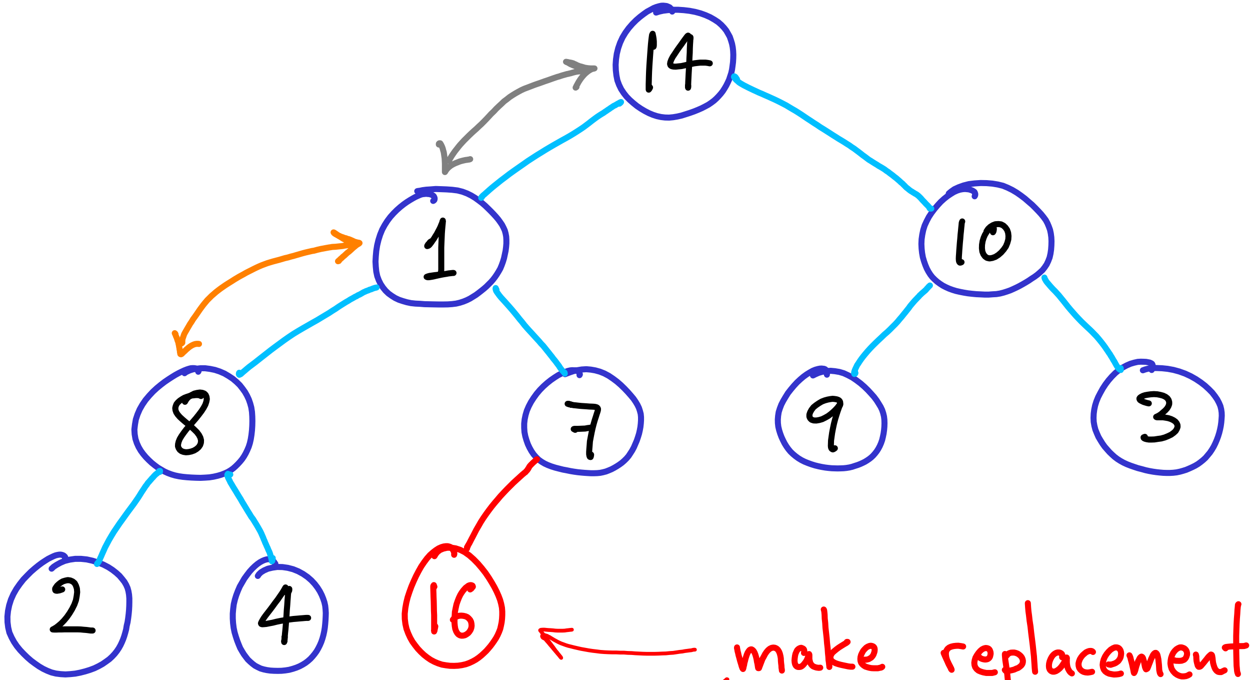
as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
14	1	10	8	7	9	3	2	4	16

How to sort data in a complete heap in place

(without an output array)



Same as before but we swap max with replacement

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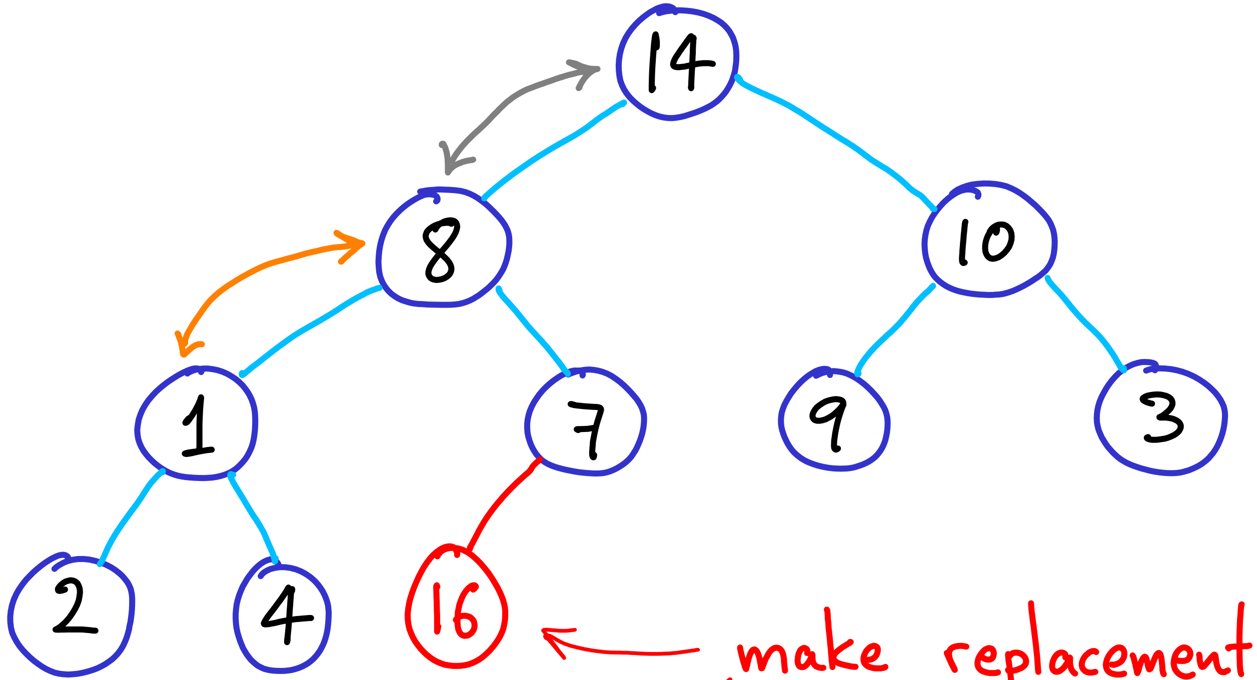
as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
14	1	10	8	7	9	3	2	4	16

How to sort data in a complete heap in place

(without an output array)



Same as before but we swap max with replacement

make replacement position inactive

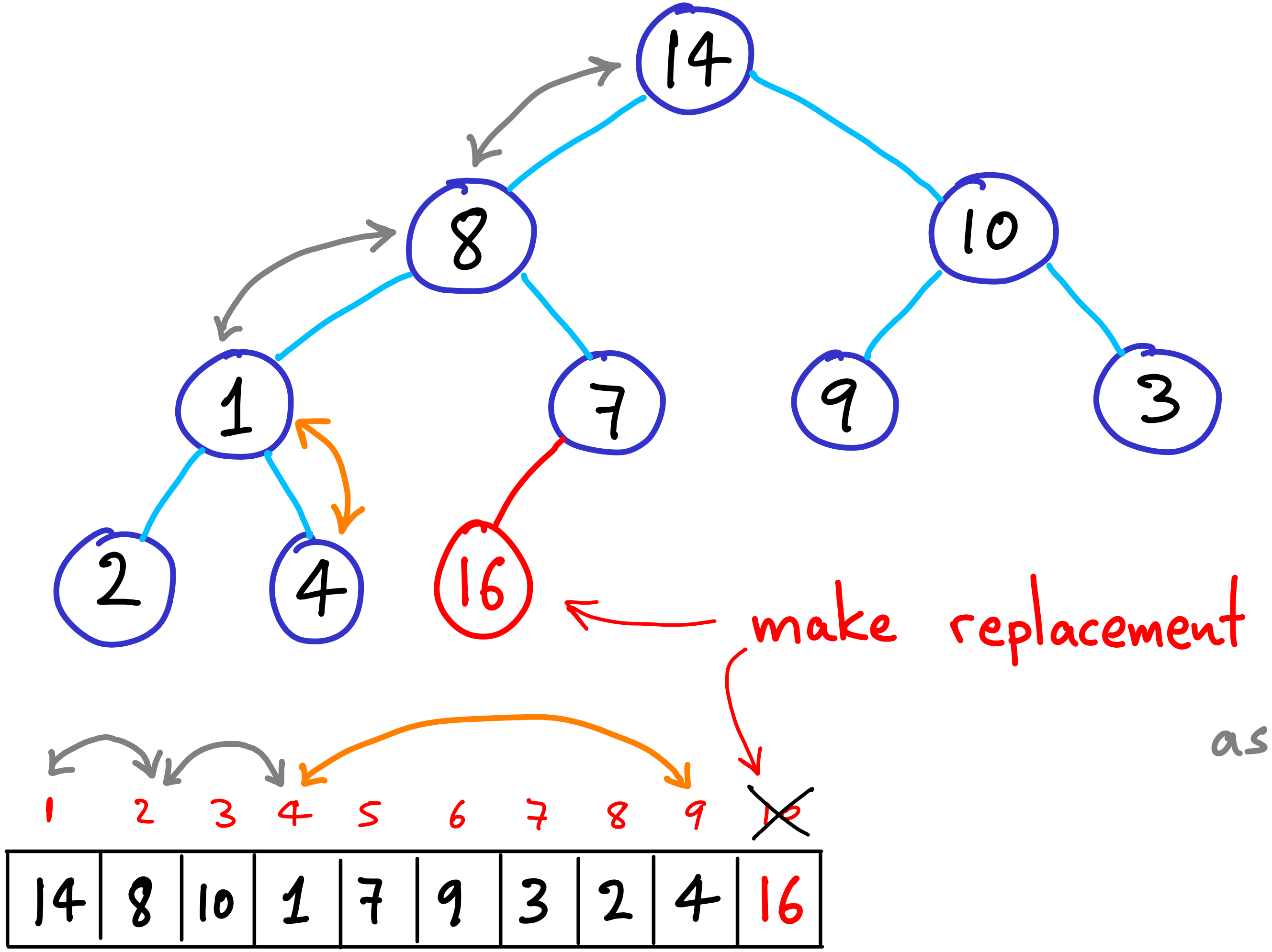
as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
14	8	10	1	7	9	3	2	4	16

How to sort data in a complete heap in place

(without an output array)

Same as before but we swap max with replacement



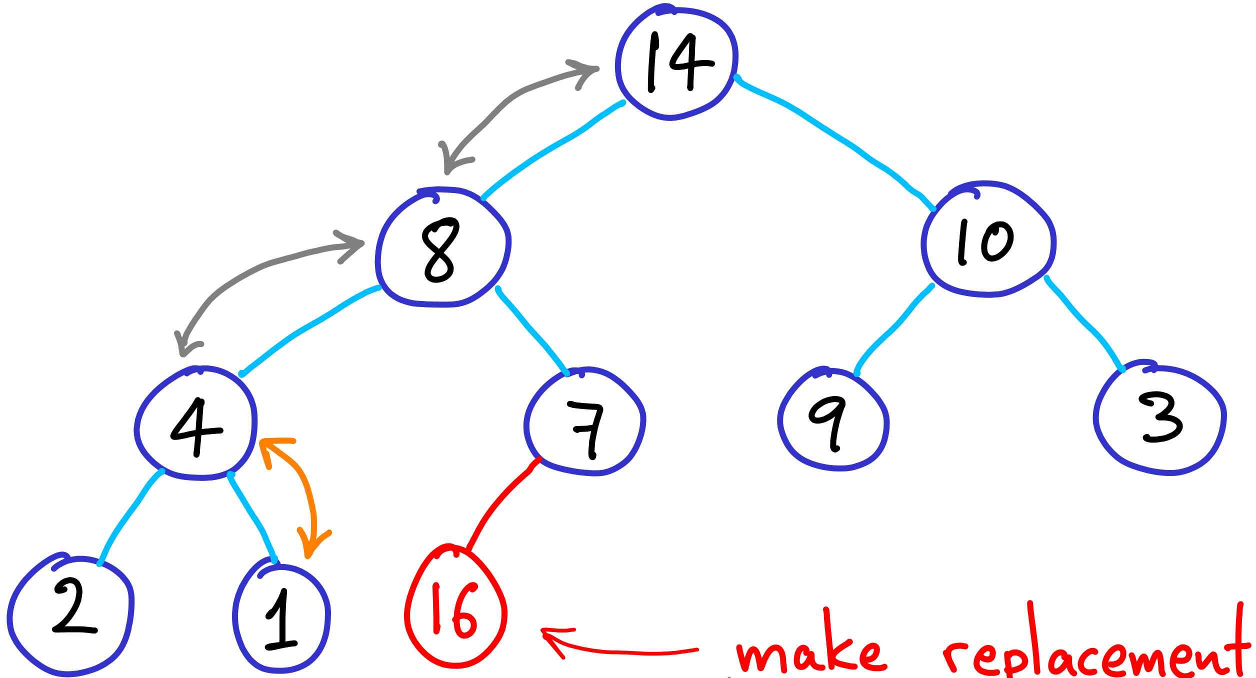
make replacement position inactive

as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
14	8	10	1	7	9	3	2	4	16

How to sort data in a complete heap in place

(without an output array)



Same as before but we swap max with replacement

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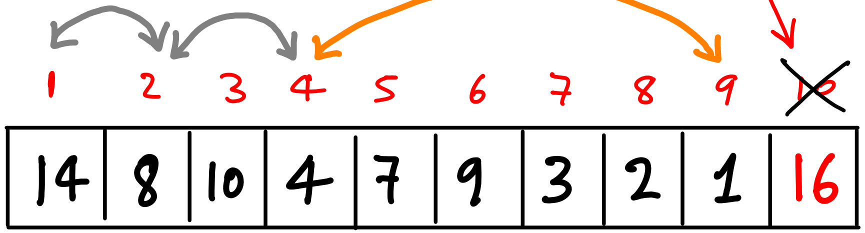
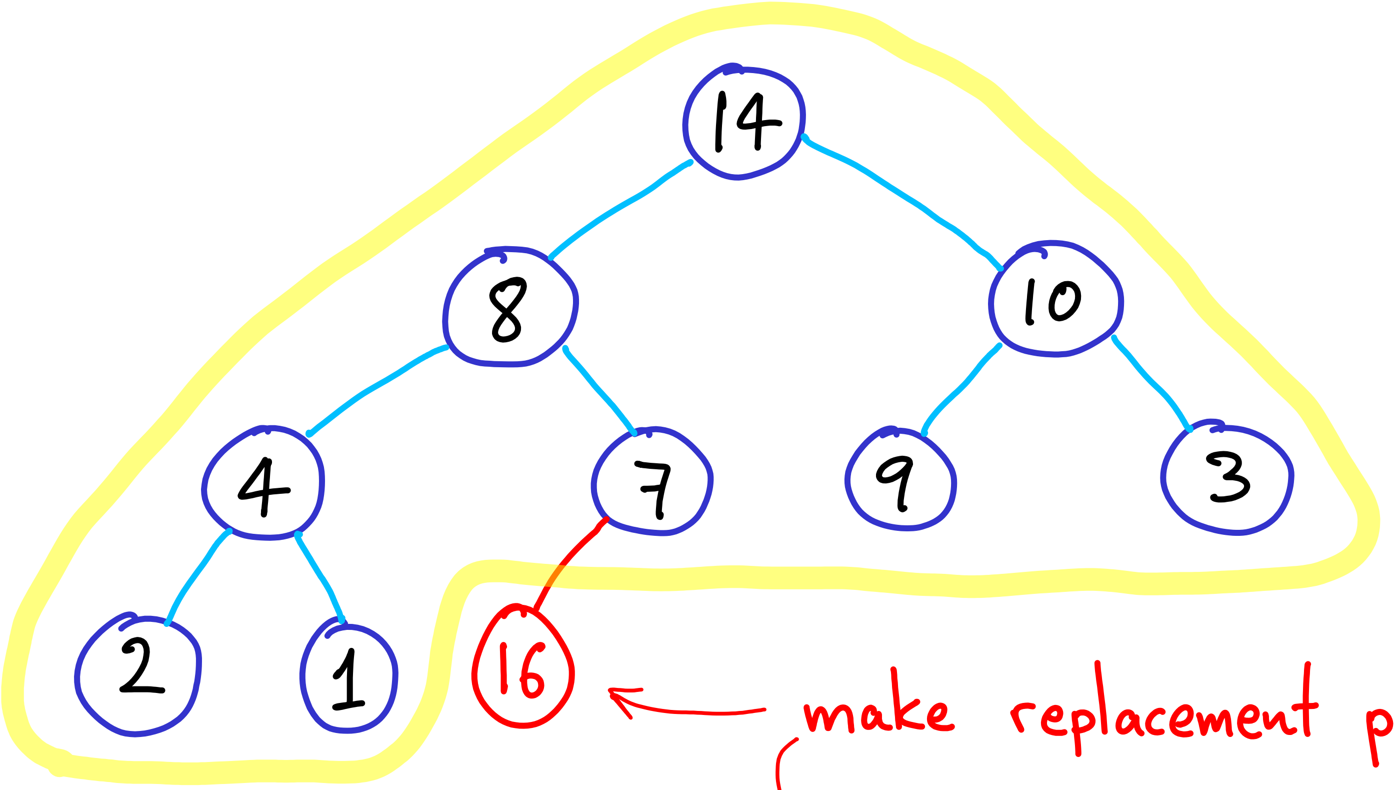
as though extracted



1	2	3	4	5	6	7	8	9	~~10~~
14	8	10	4	7	9	3	2	1	16

How to sort data in a complete heap in place

(without an output array)



Same as before but we swap max with replacement

make replacement position inactive

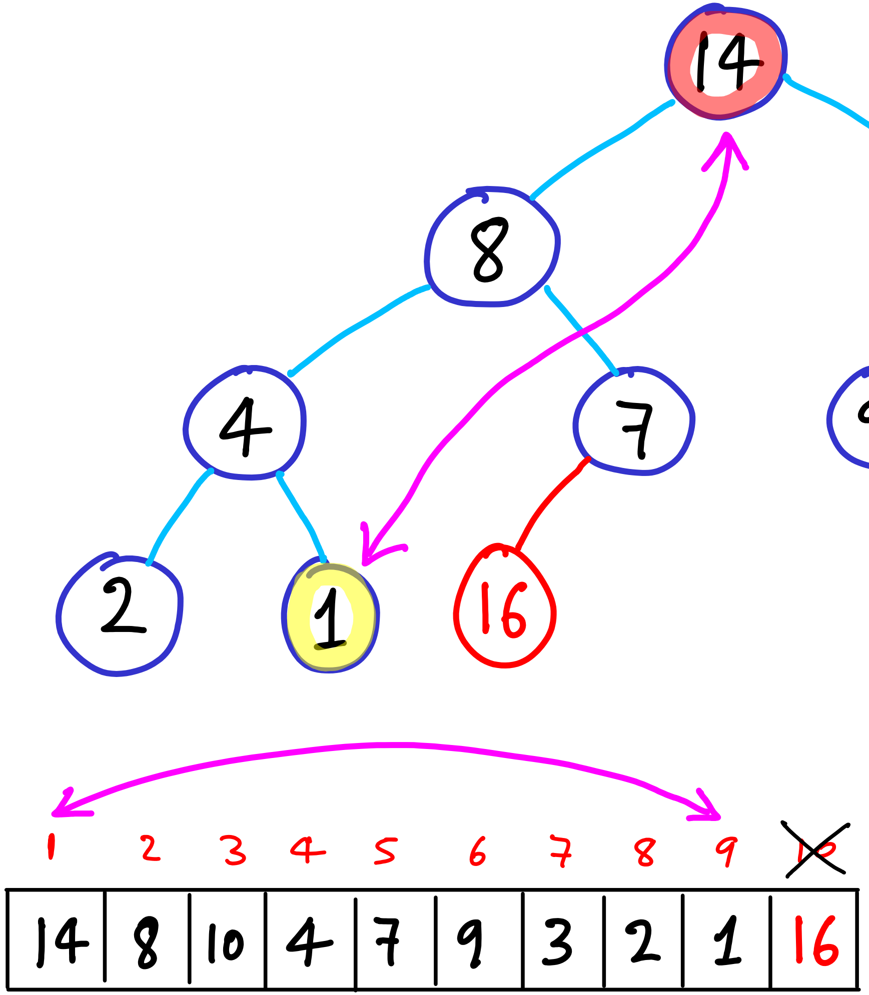
as though extracted

1	2	3	4	5	6	7	8	9	~~10~~
14	8	10	4	7	9	3	2	1	16

valid heap

How to sort data in a complete heap in place

(without an output array)



1	2	3	4	5	6	7	8	9	~~10~~
14	8	10	4	7	9	3	2	1	16

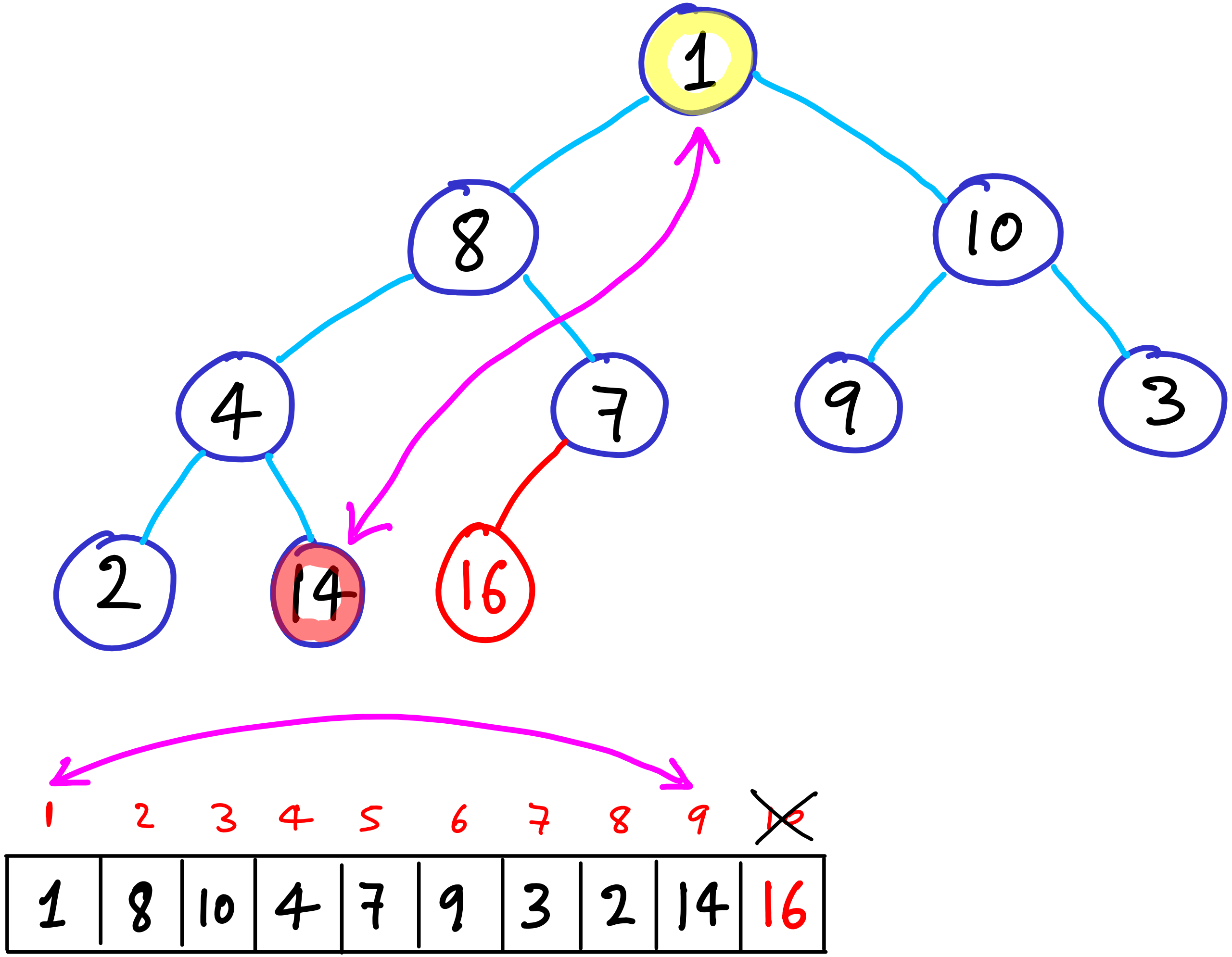
Same as before

but we swap

max with replacement

How to sort data in a complete heap in place

(without an output array)



1	2	3	4	5	6	7	8	9	~~10~~
1	8	10	4	7	9	3	2	14	16

Same as before

but we swap

max with replacement

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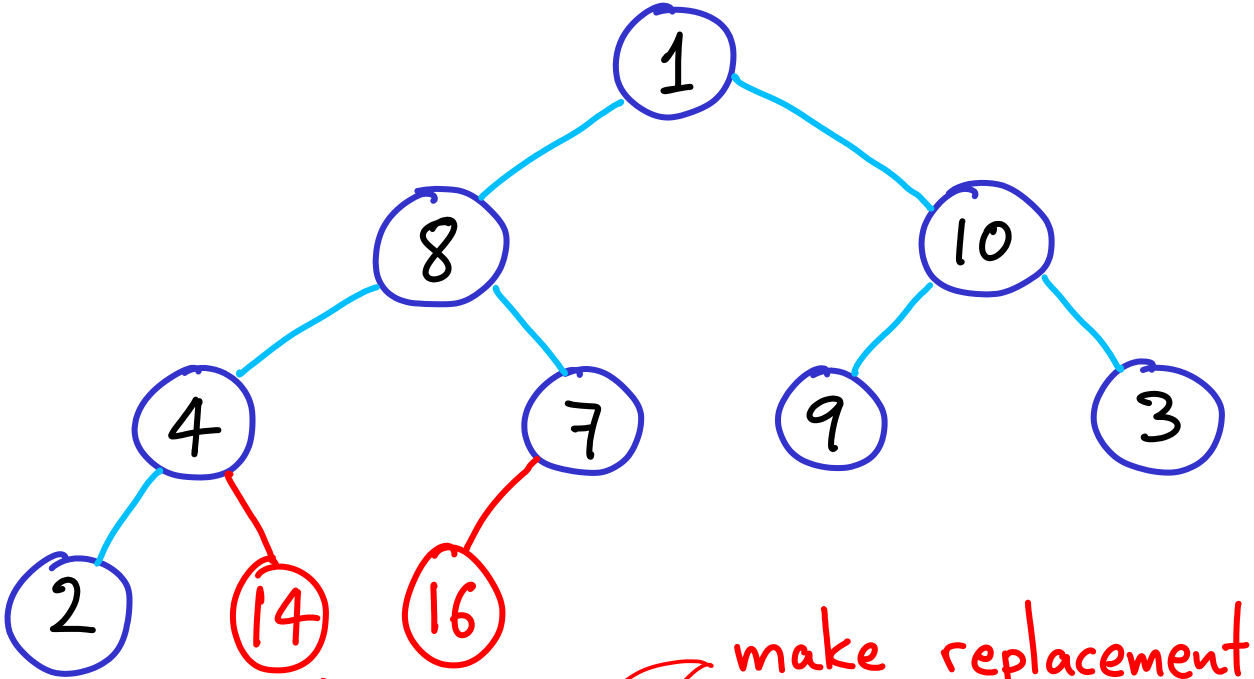
(without an output array)



Same as before
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make replacement position inactive

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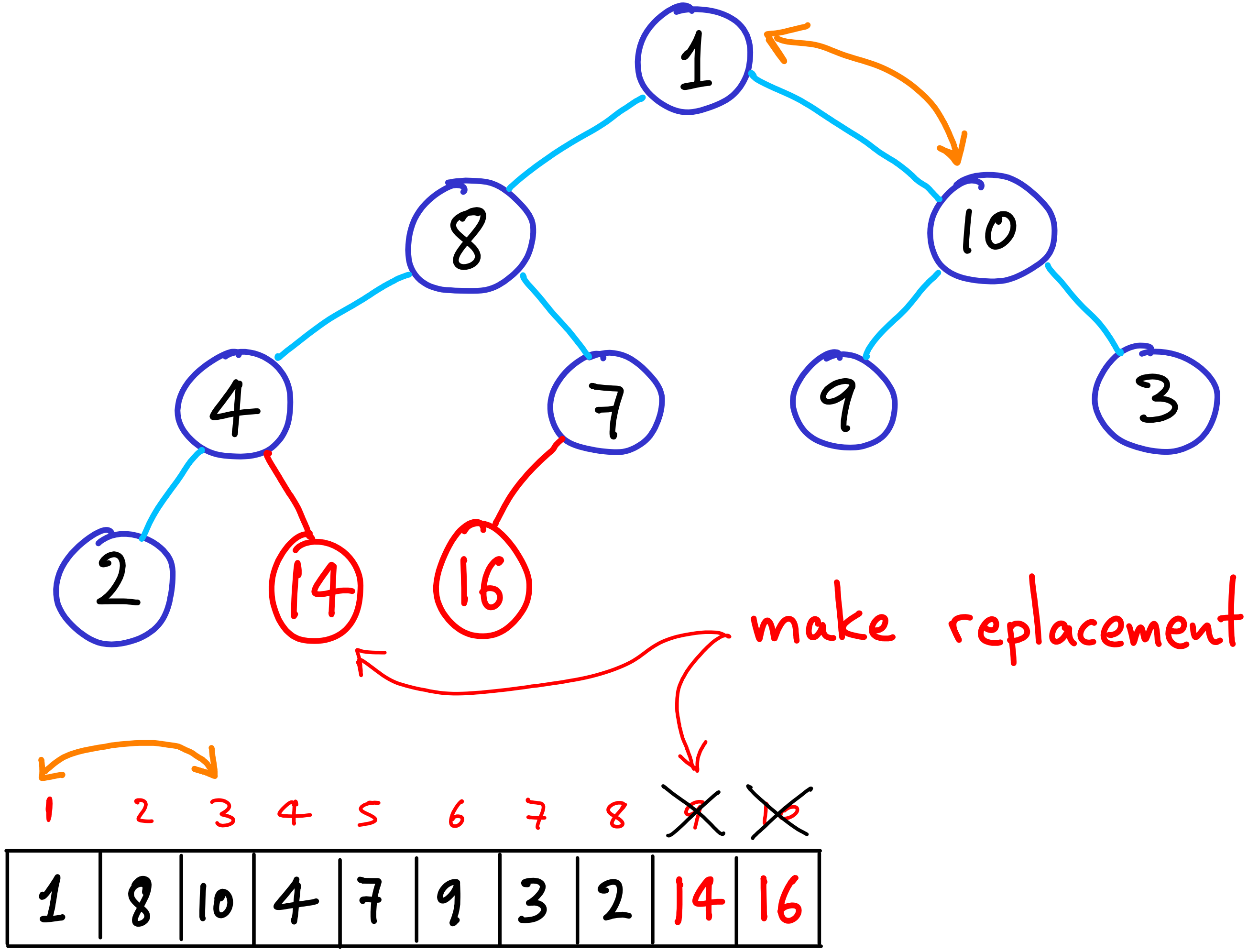

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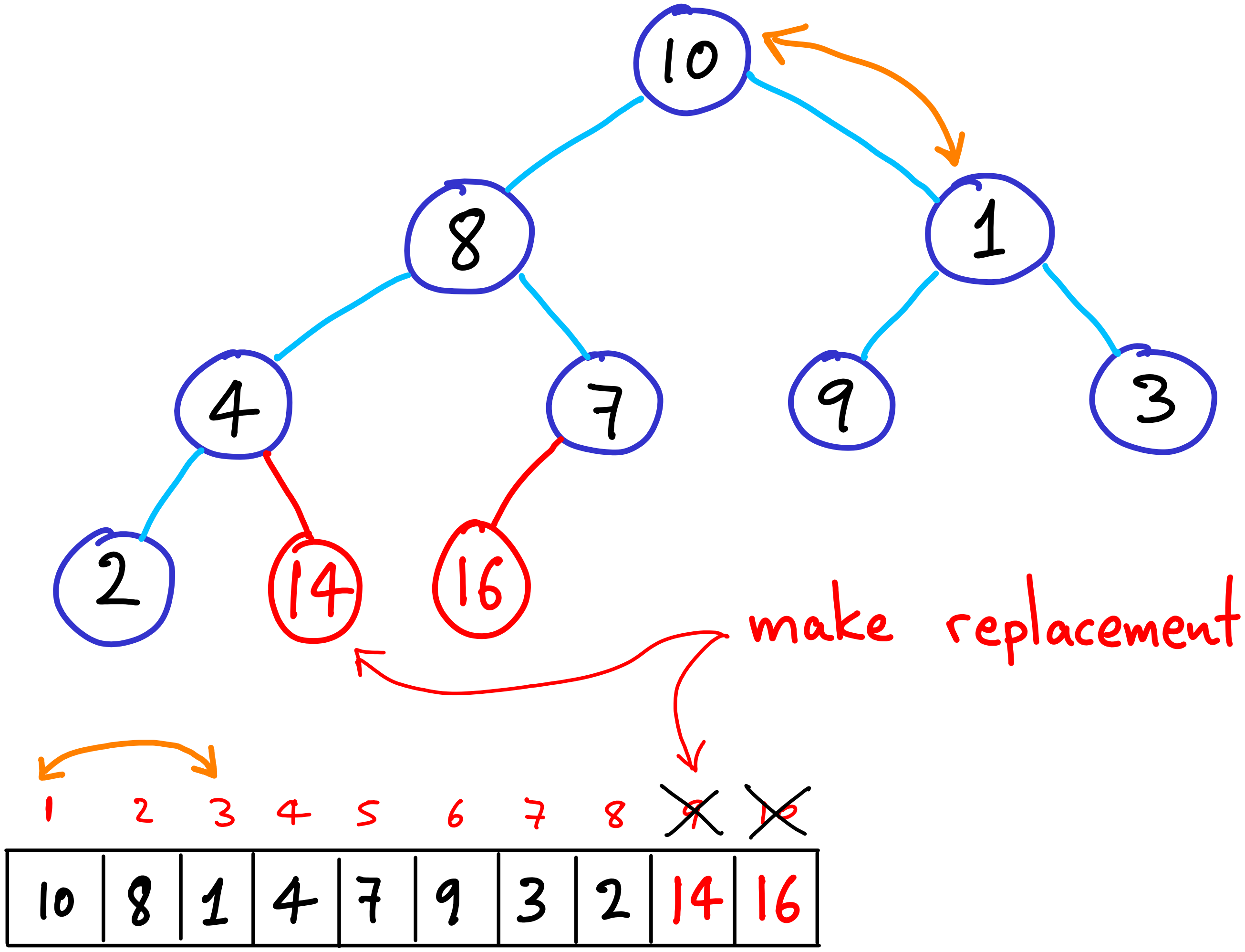

make replacement position inactive

1	2	3	4	5	6	7	8	~~9~~	~~10~~
10	8	1	4	7	9	3	2	14	16

Same as before

but we swap

max with replacement

How to sort data in a complete heap in place

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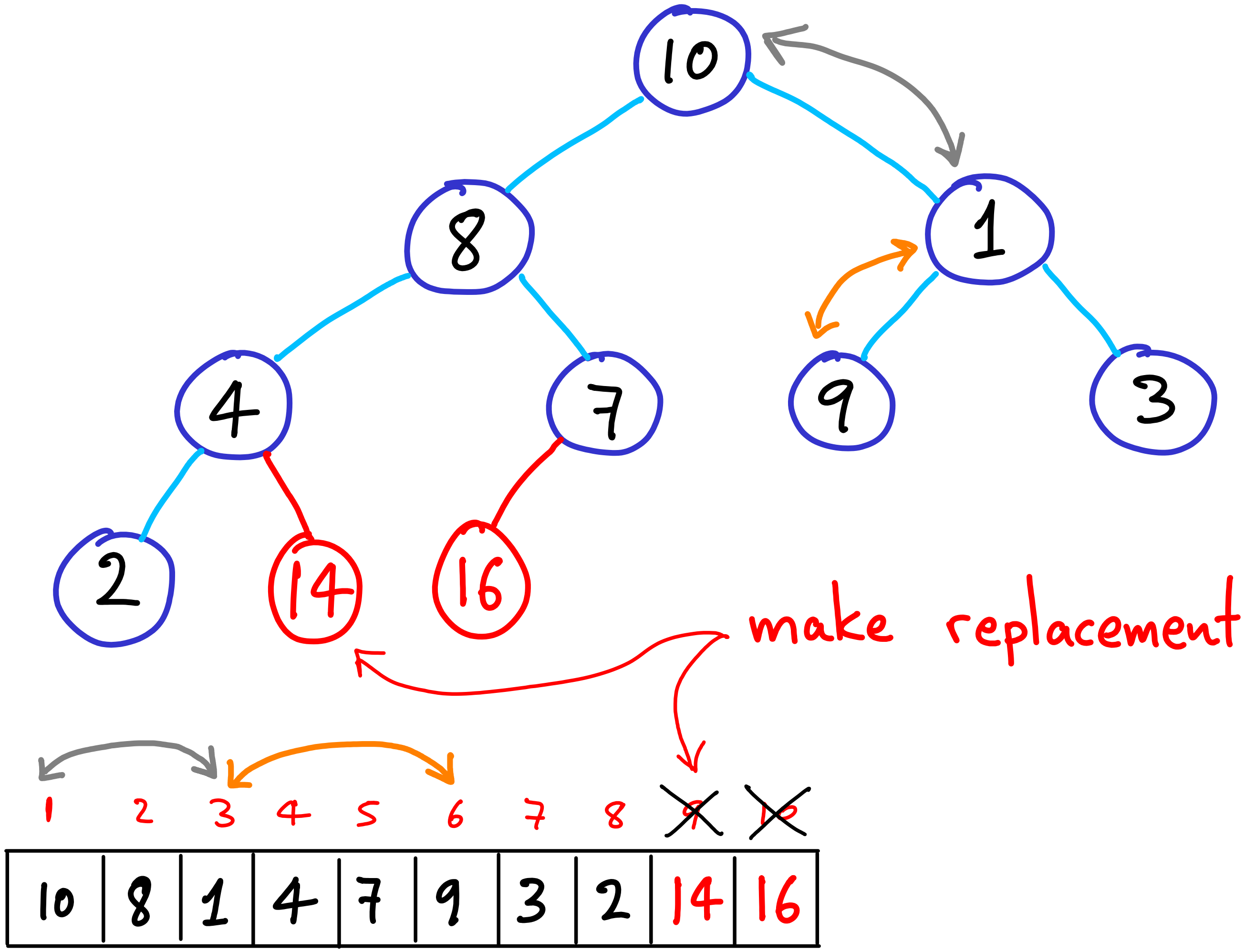

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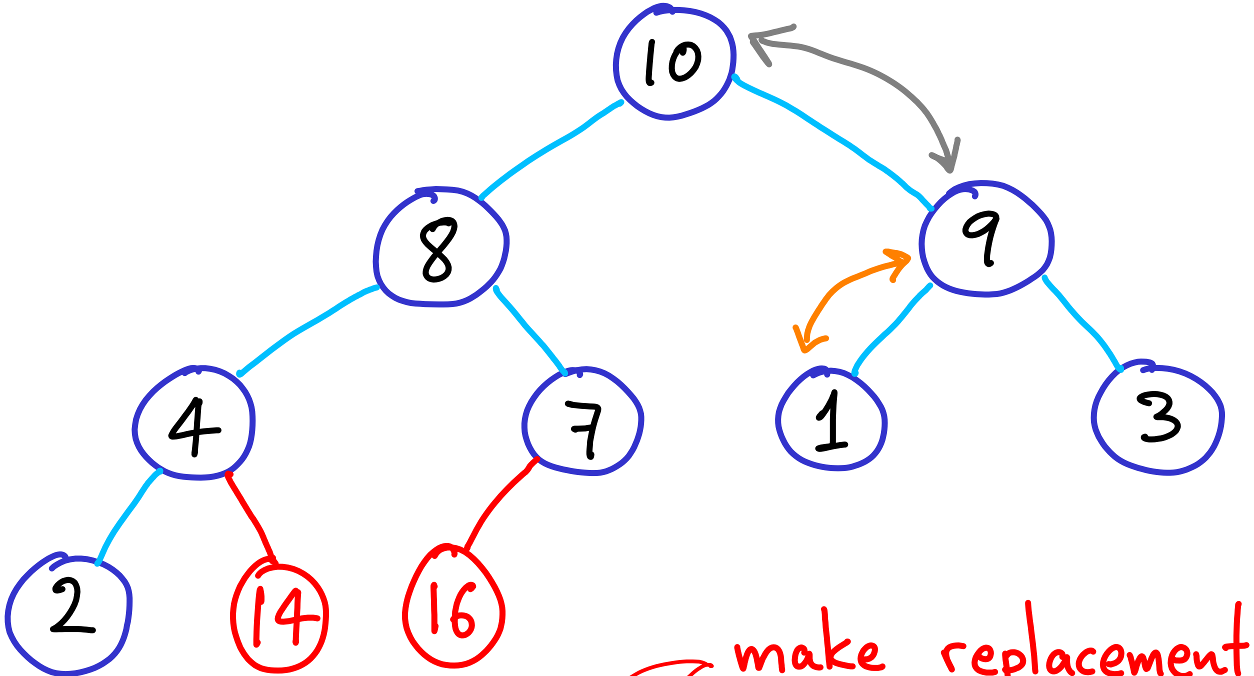

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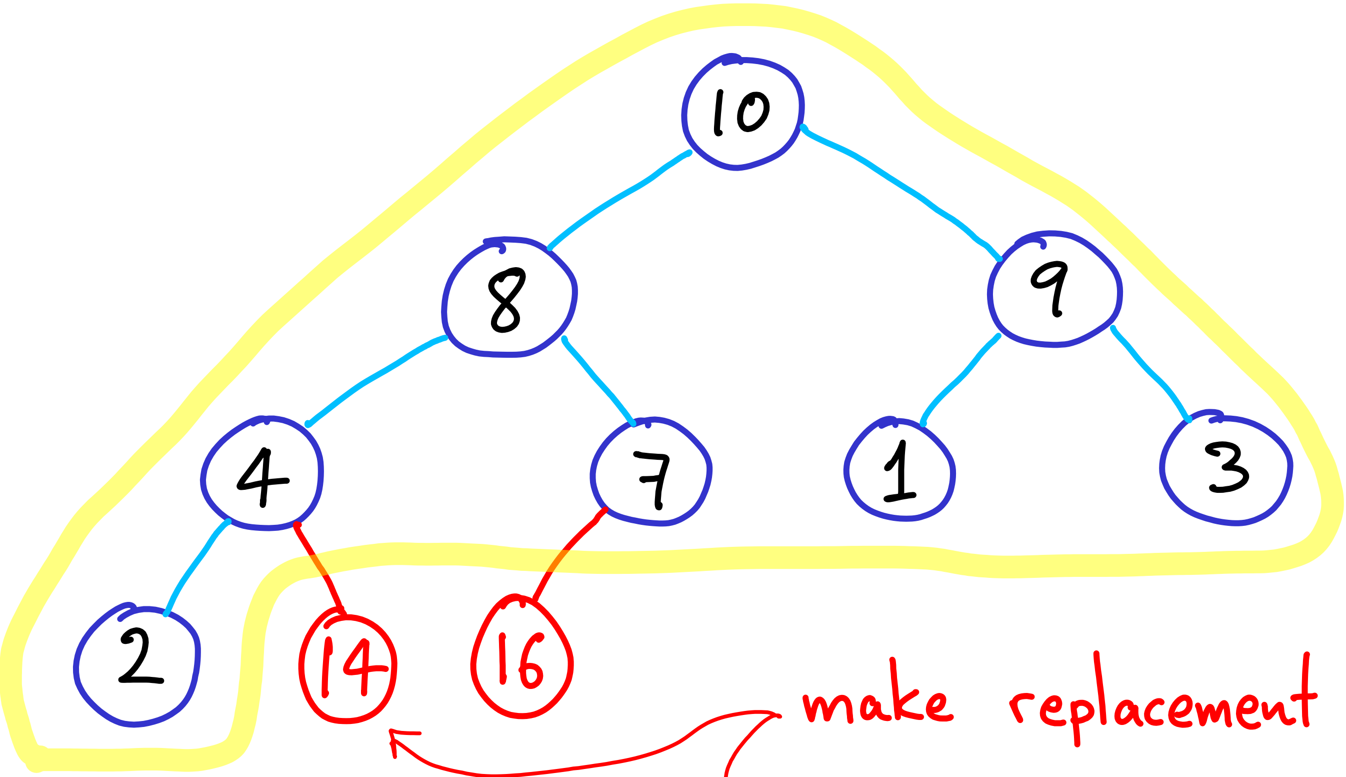

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1	2	3	4	5	6	7	8	~~9~~	~~10~~
10	8	9	4	7	1	3	2	14	16

valid heap

How to sort data in a complete heap in place

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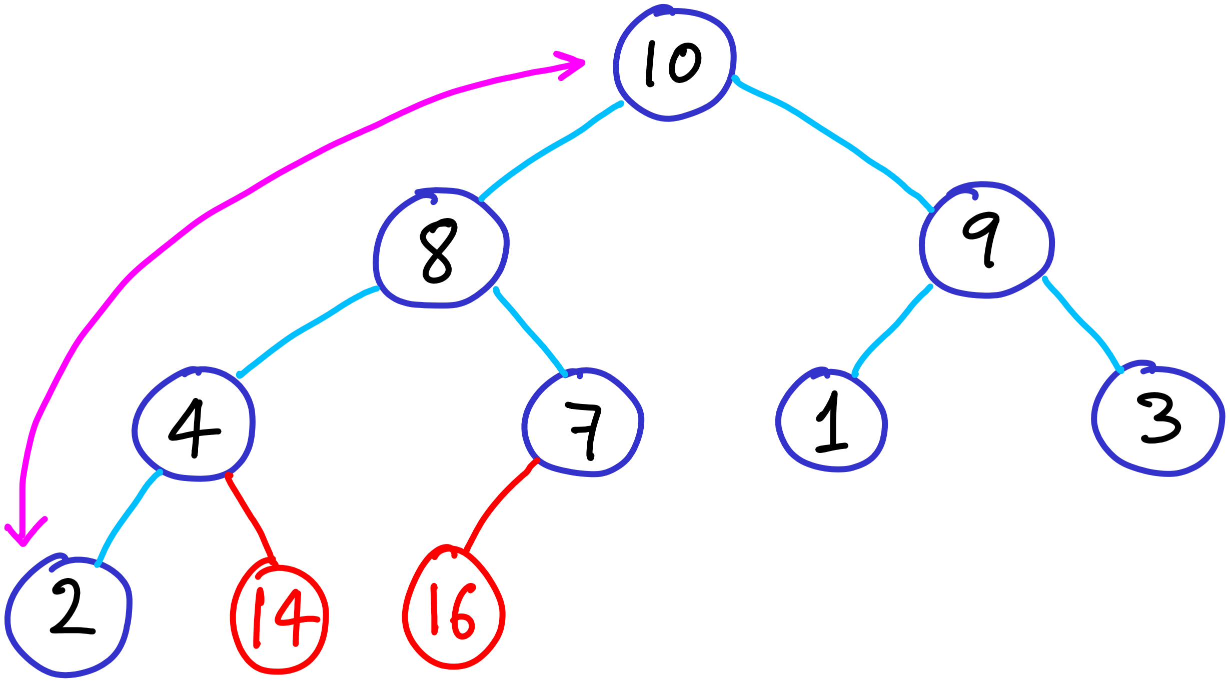

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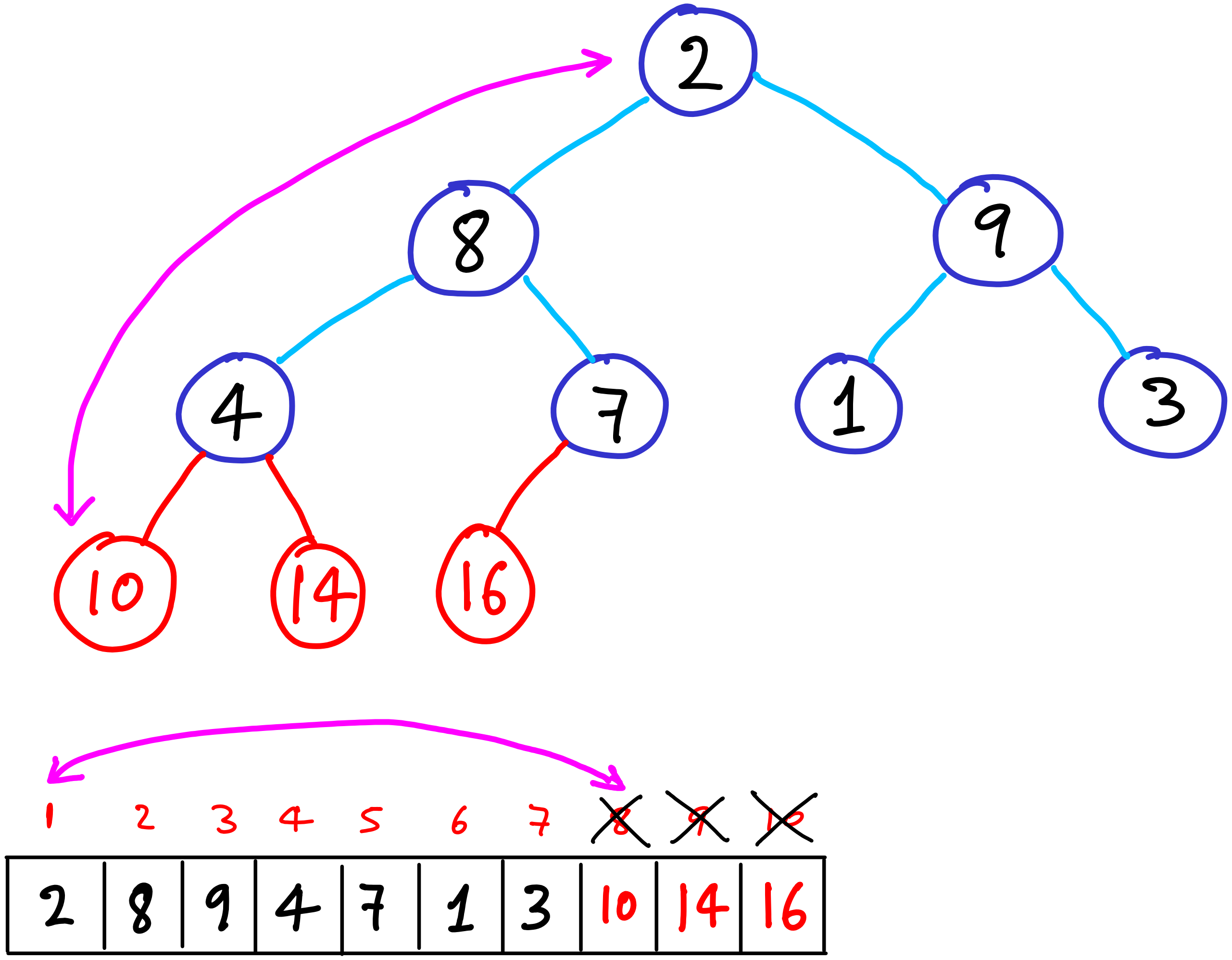

1	2	3	4	5	6	7	~~8~~	~~9~~	~~10~~
2	8	9	4	7	1	3	10	14	16

Same as before
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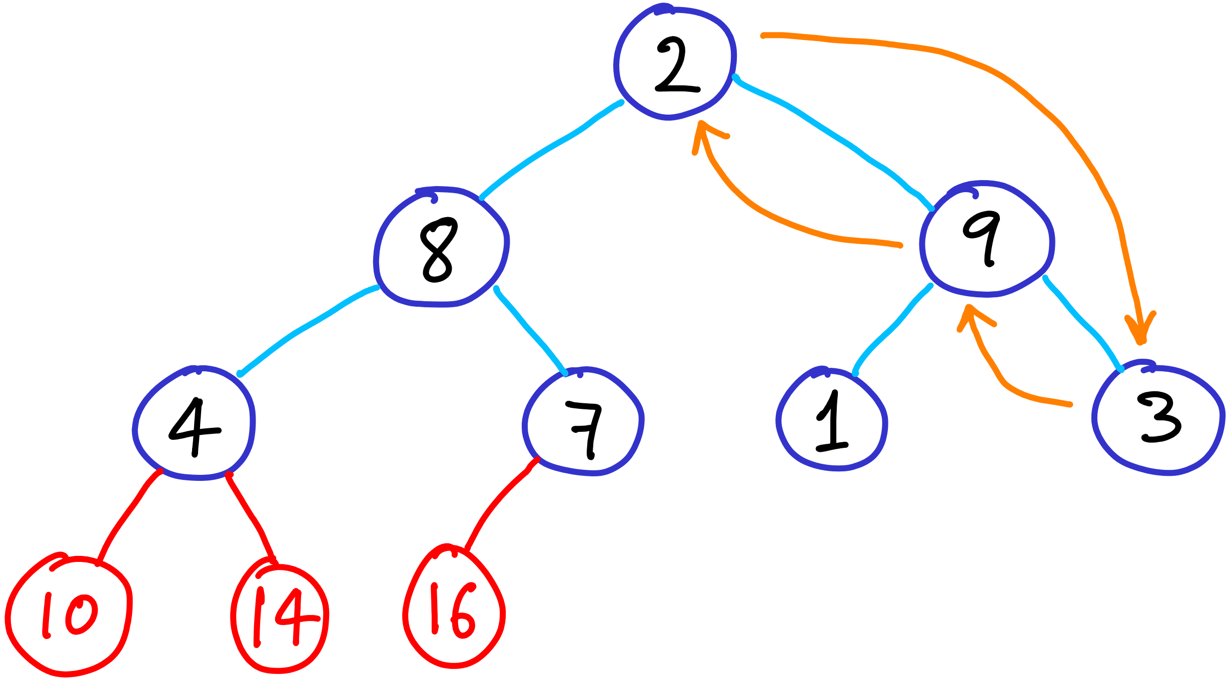

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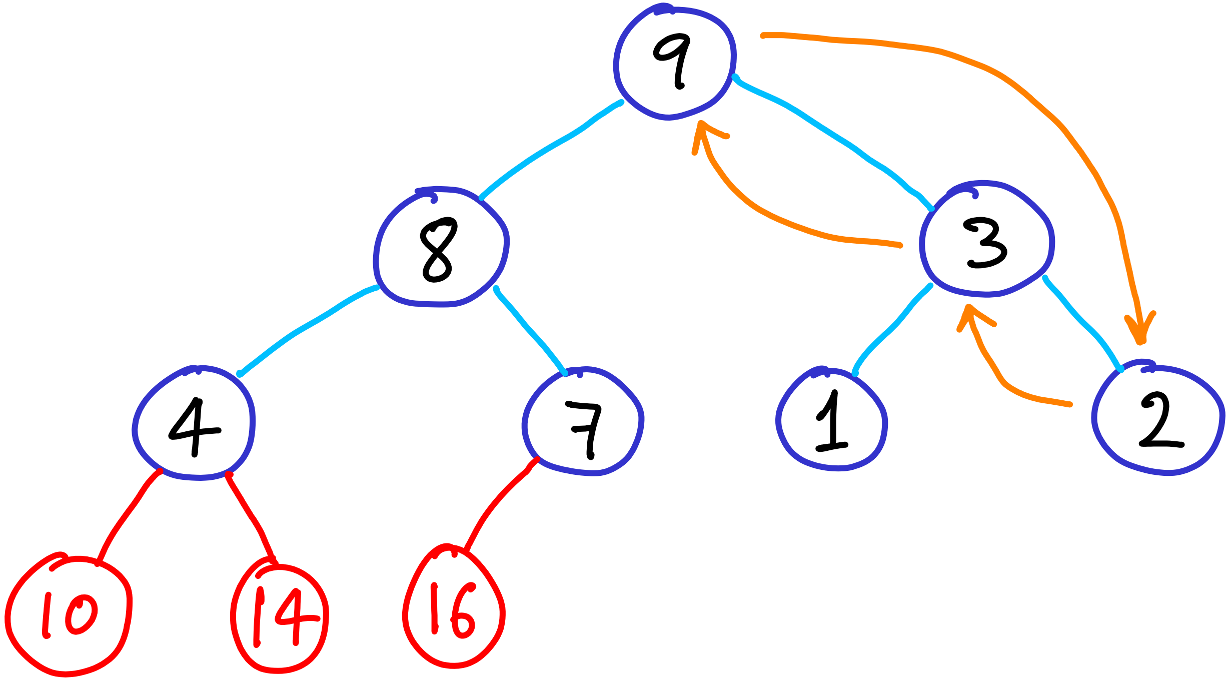

1	2	3	4	5	6	7	~~8~~	~~9~~	~~10~~
9	8	3	4	7	1	2	10	14	16

Same as before but we swap max with replacement

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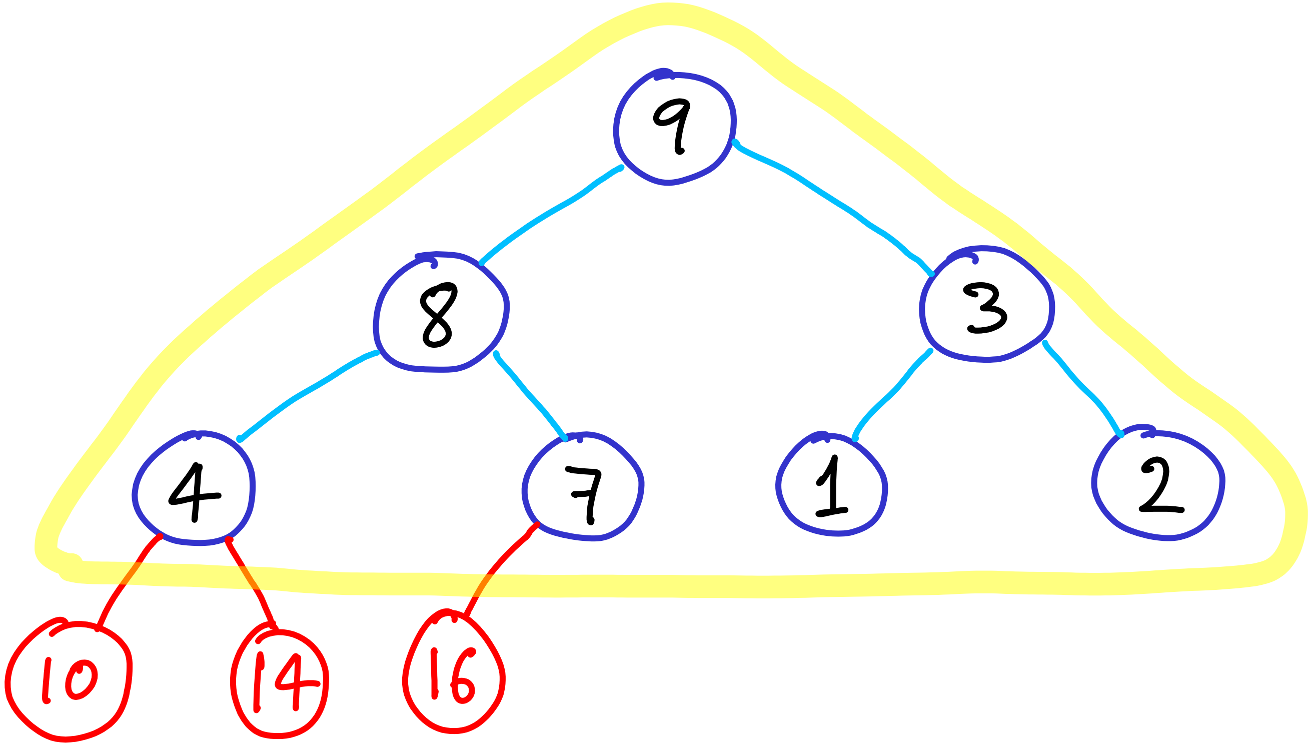

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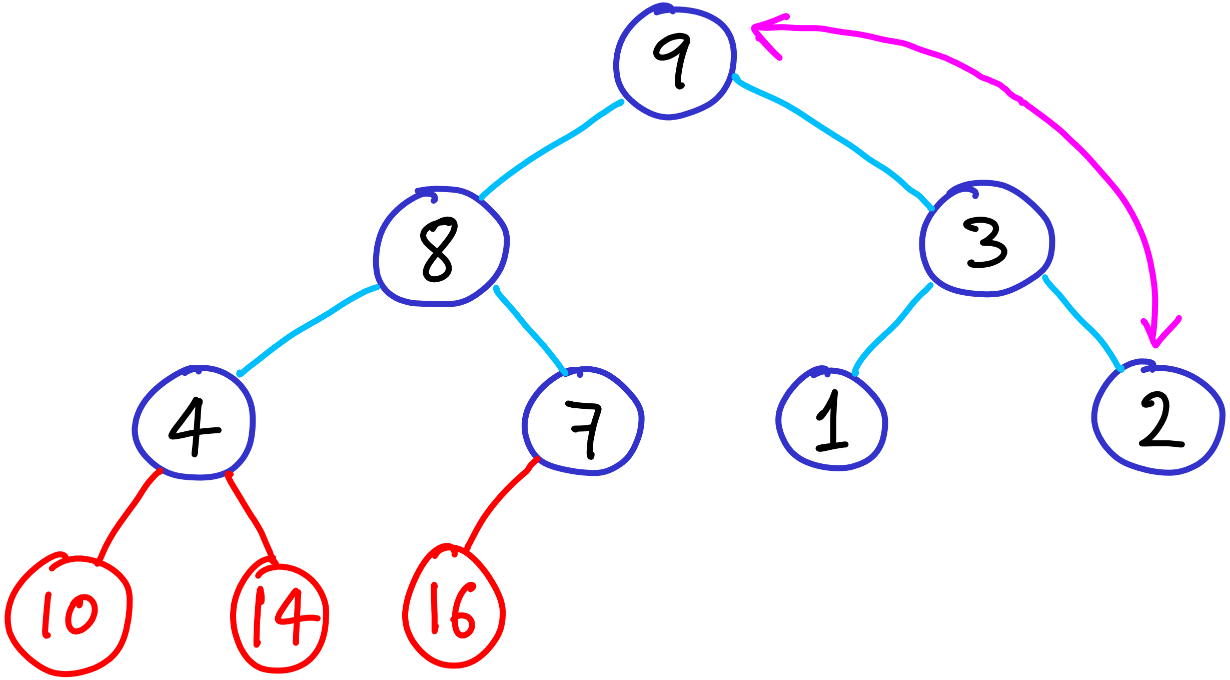

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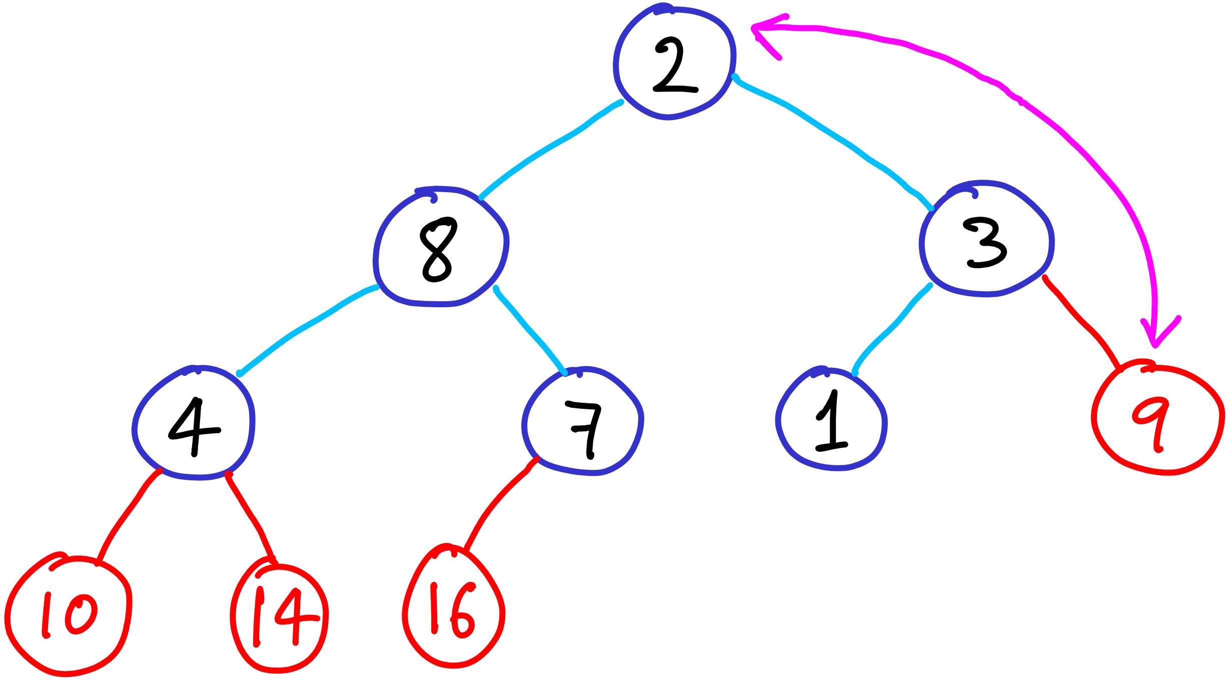

Same as before
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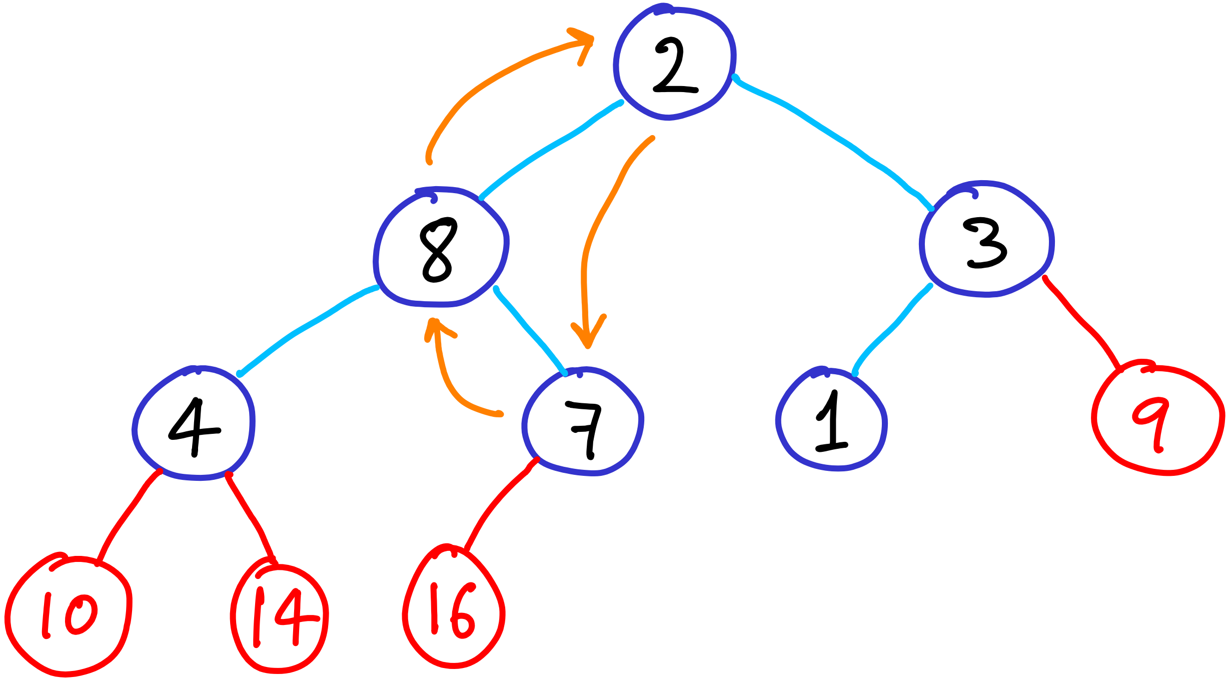

Same as before
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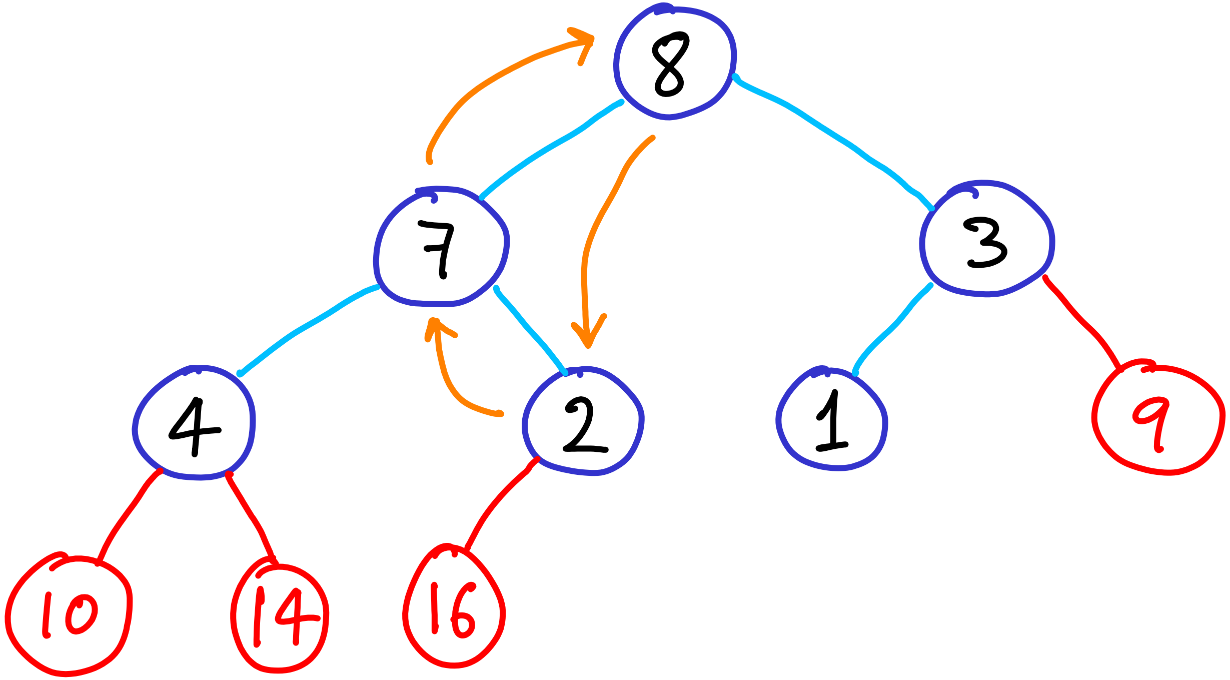

Same as before
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max with replacement

1	2	3	4	5	6	~~7~~	~~8~~	~~9~~	~~10~~
8	7	3	4	2	1	9	10	14	16

How to sort data in a complete heap in place

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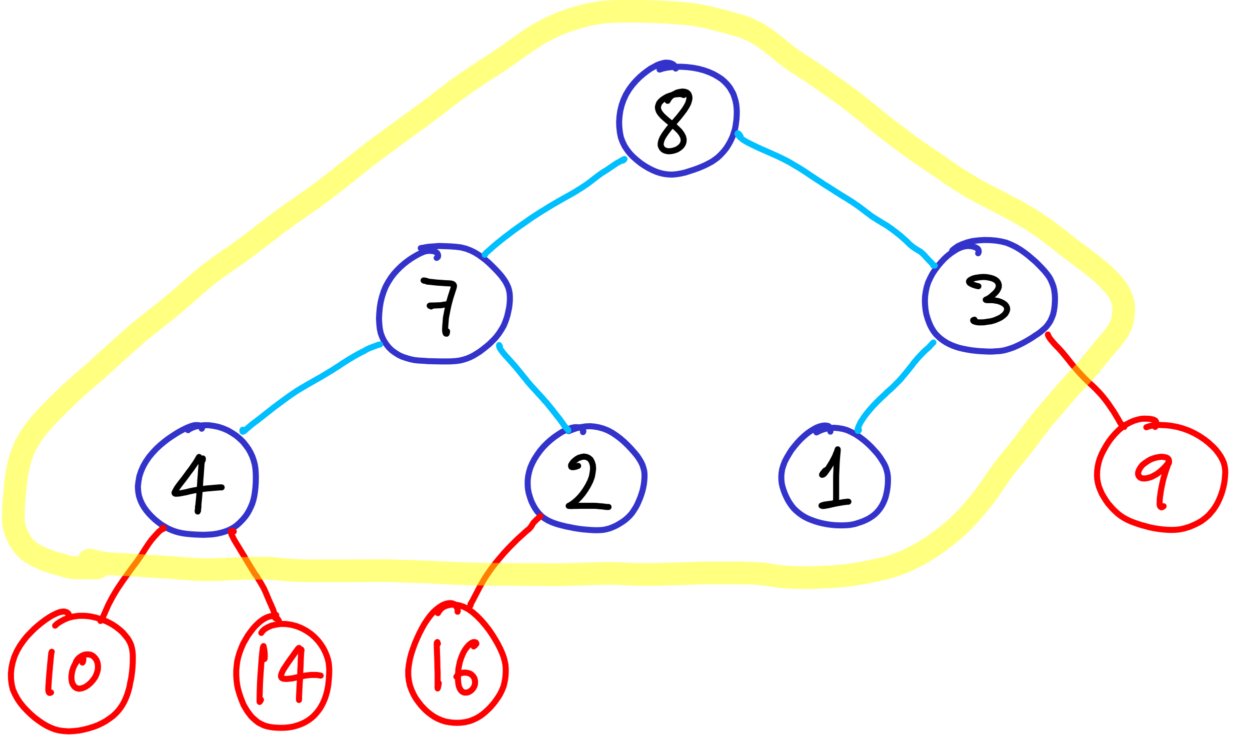
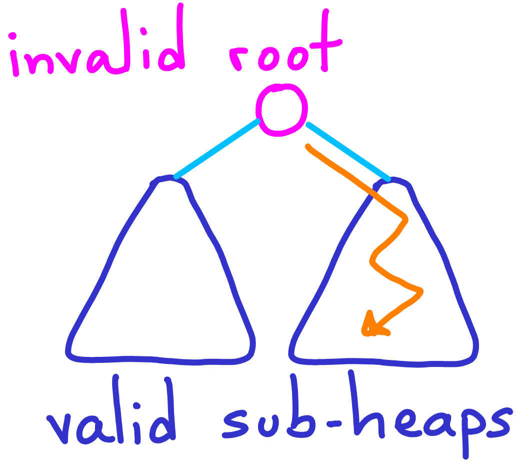


Same as before
but we swap
max with replacement

etc

1	2	3	4	5	6	~~7~~	~~8~~	~~9~~	~~10~~
8	7	3	4	2	1	9	10	14	16

Correctness of "heapify"



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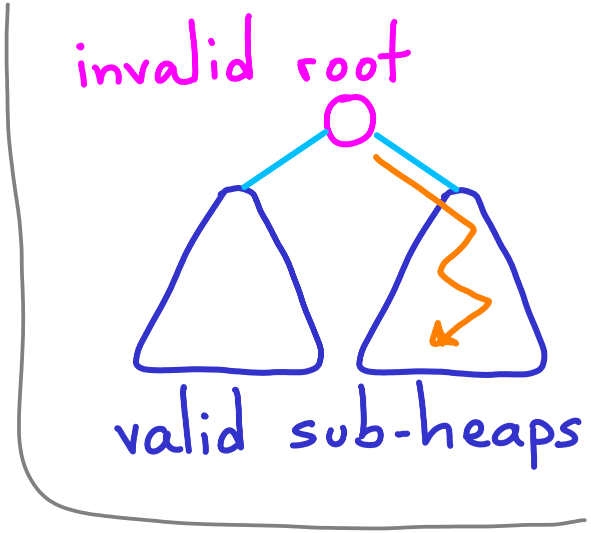
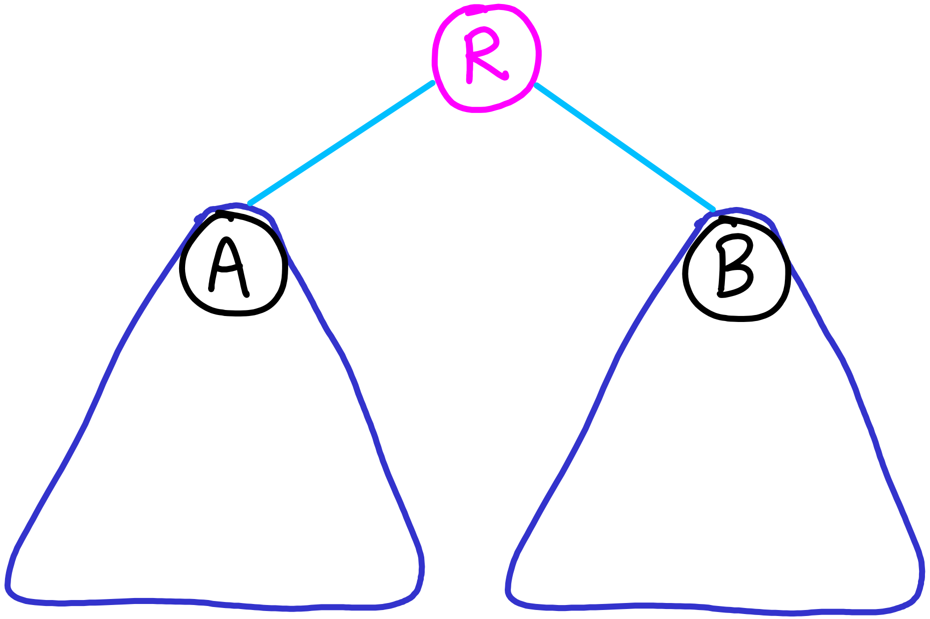
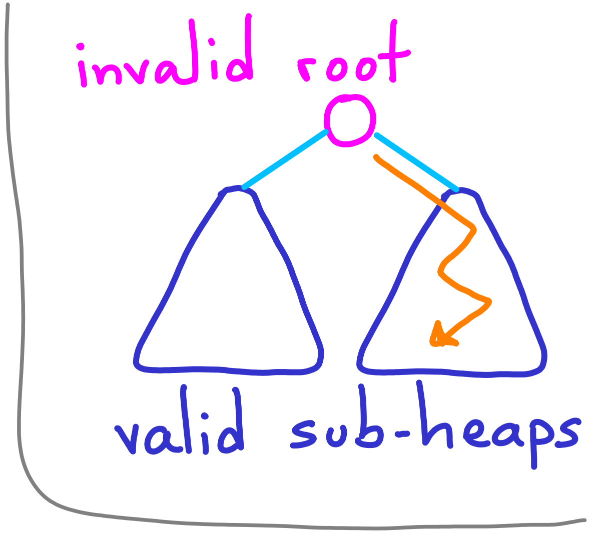
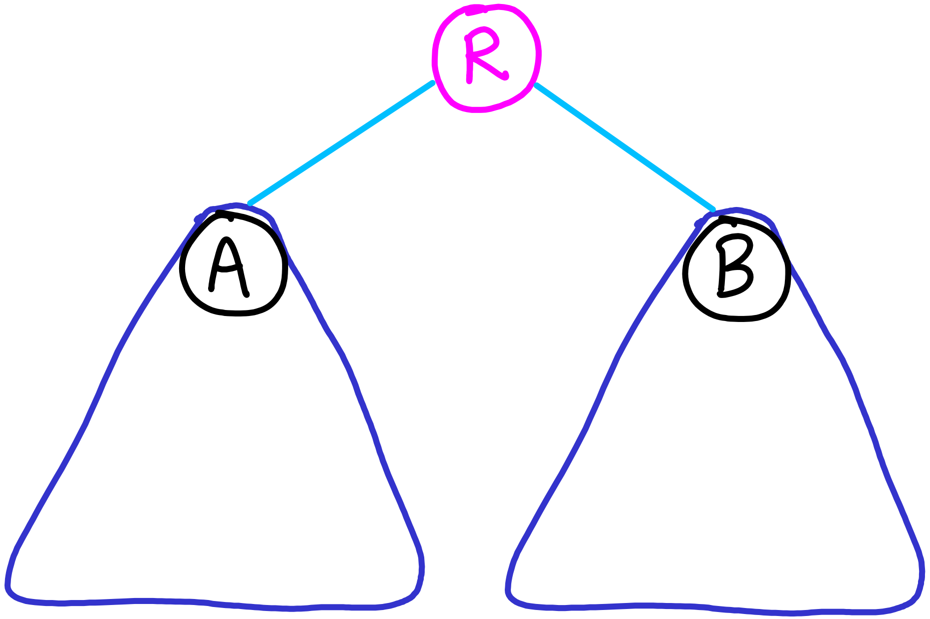




Assume $A < B$

Correctness of "heapify"

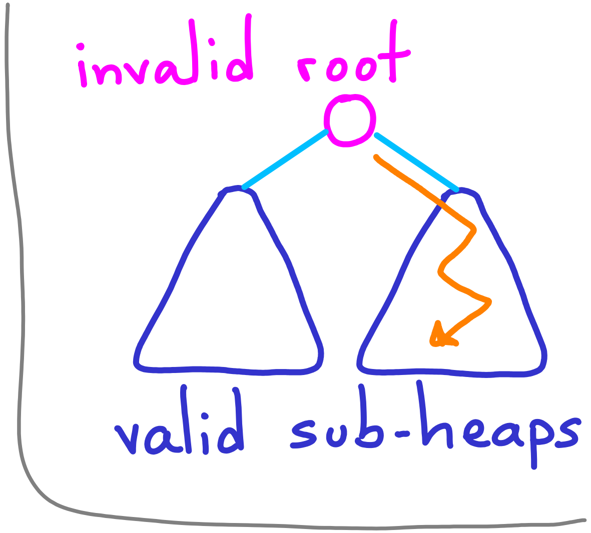


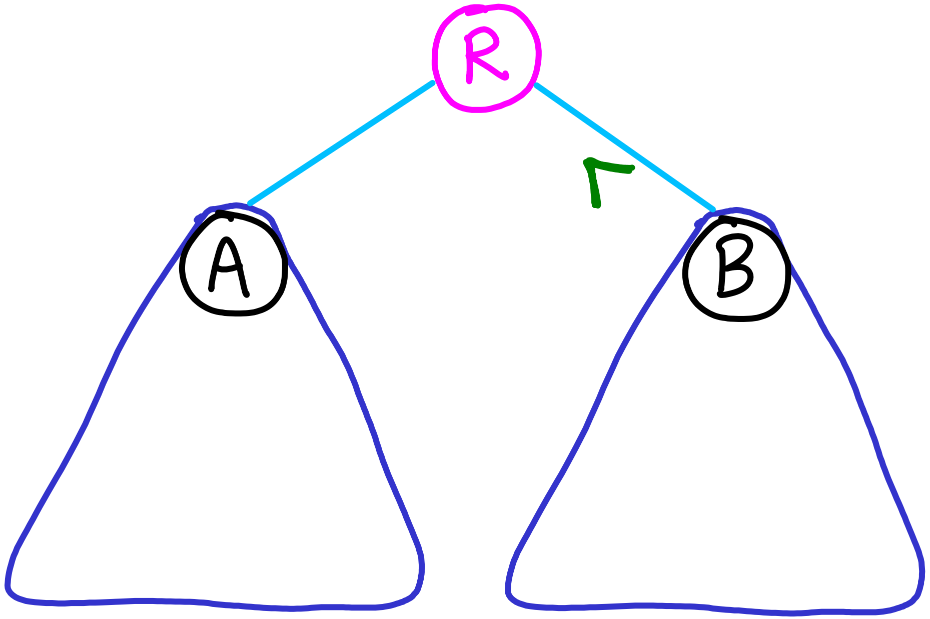


Assume $A < B$

if $R > B$, done. $(R > B > A)$

Correctness of "heapify"





Assume $A < B$

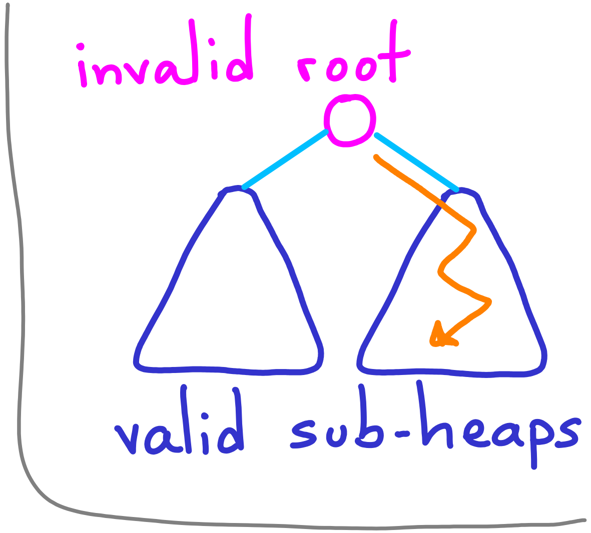
if $R > B$, done.

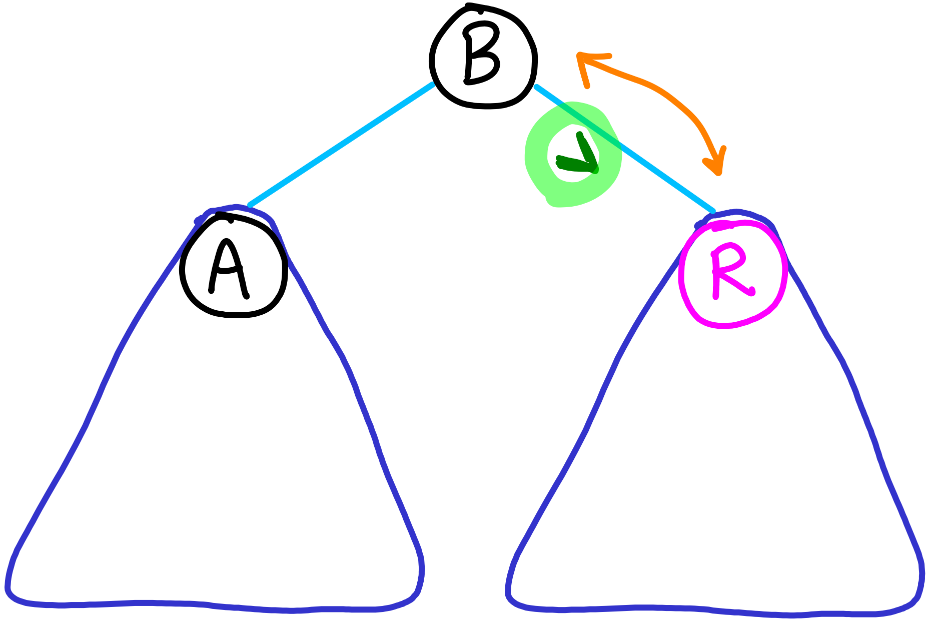
else ...

$(R > B > A)$

$(R < B)$

Correctness of "heapify"



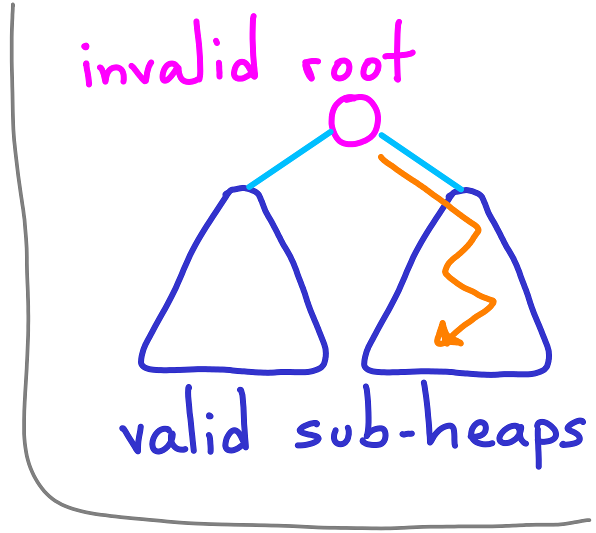


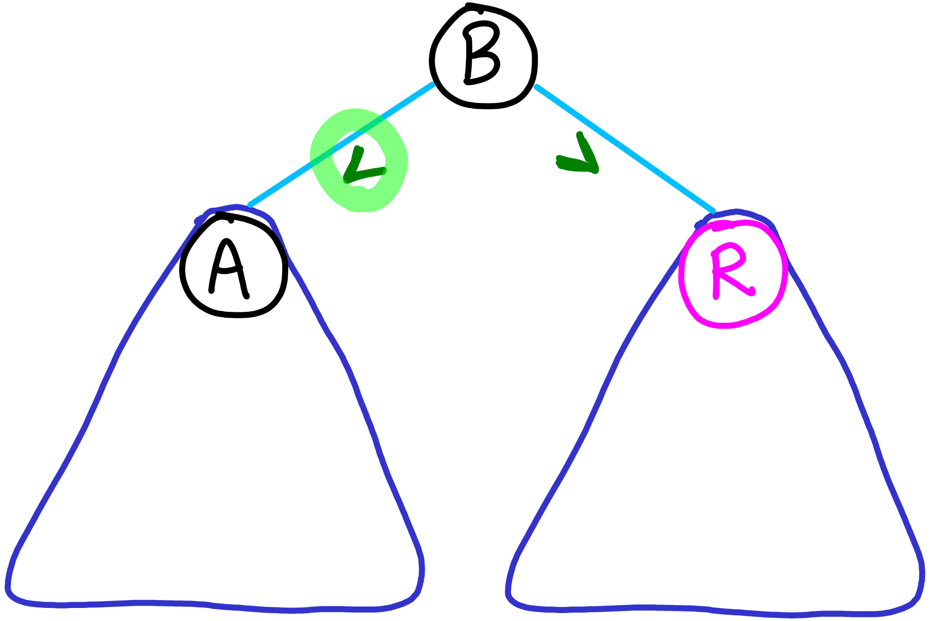
Assume $A < B$

if $R > B$, done. $(R > B > A)$

else swap B & R $(R < B)$

Correctness of "heapify"





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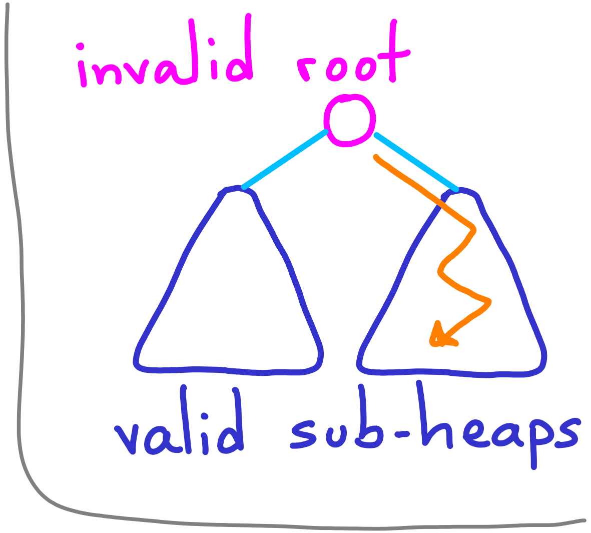
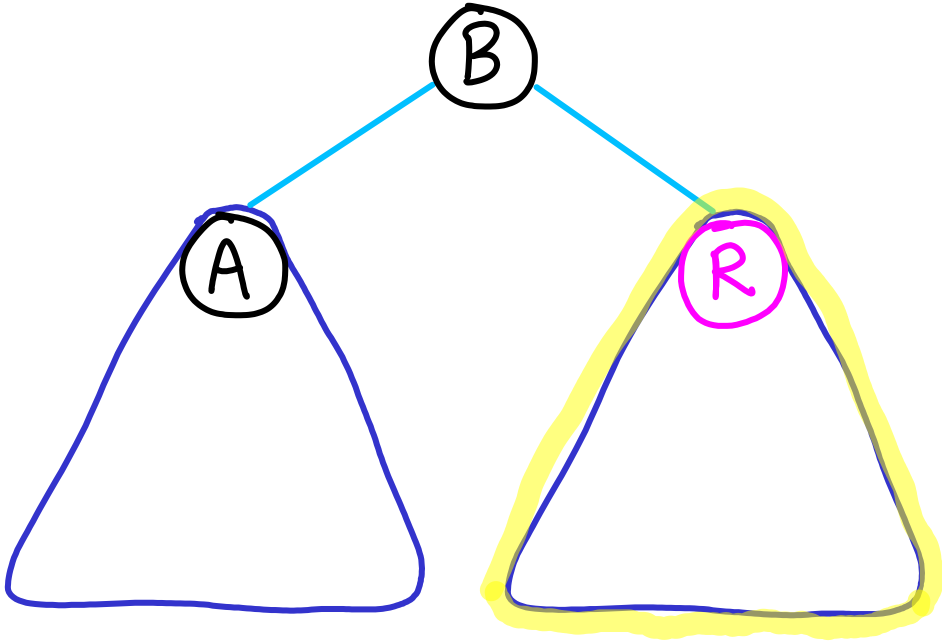
Correctness of "heapify"





Assume $A < B$

if $R > B$, done. $(R > B > A)$

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recurse

Summary

Given a heap we can extract max and heapify in $O(\log n)$ time.

↳ n rounds : $O(n \log n)$ to sort a heap

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↳ n rounds : $O(n\log n)$ to sort a heap

How do we construct a heap in the first place?

Heap building: the FORWARD METHOD

Heap building: the FORWARD METHOD (left to right)

1	2	3	4	5	6	7	8	9	10
3	9	7	10	8	4	14	2	16	1

→

Heap building: the FORWARD METHOD (left to right)

1	2	3	4	5	6	7	8	9	10
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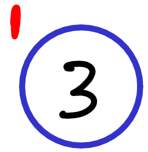

Heap building: the FORWARD METHOD (left to right)

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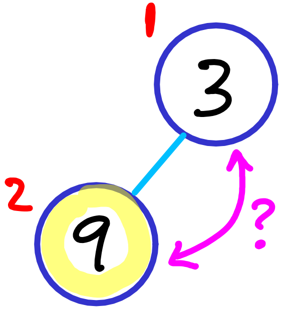

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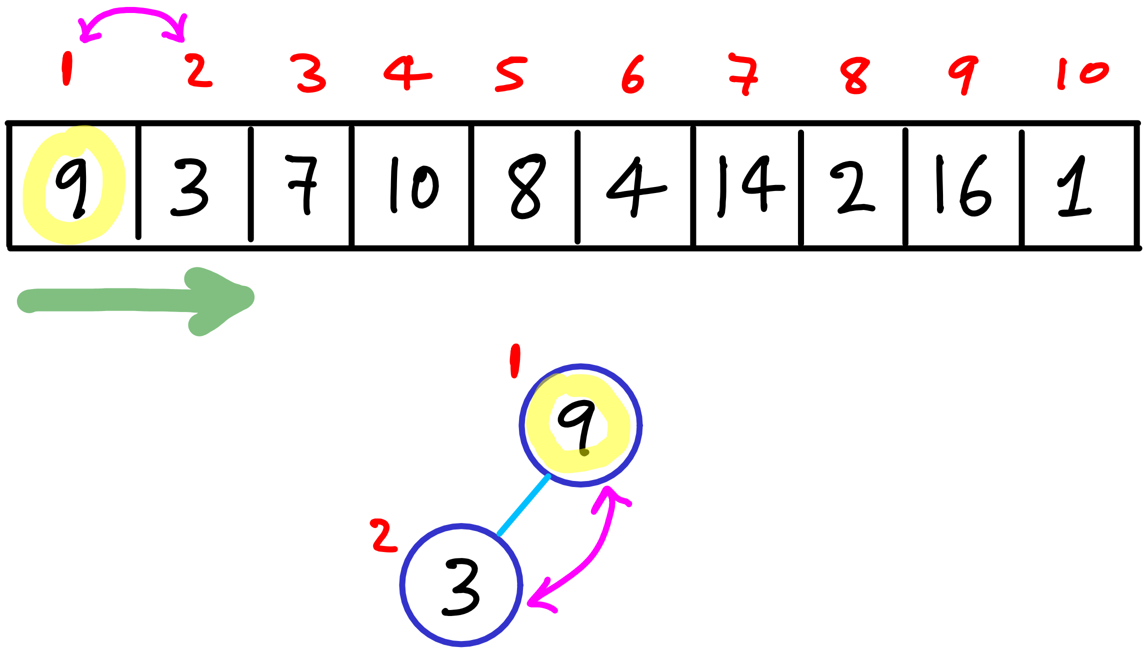

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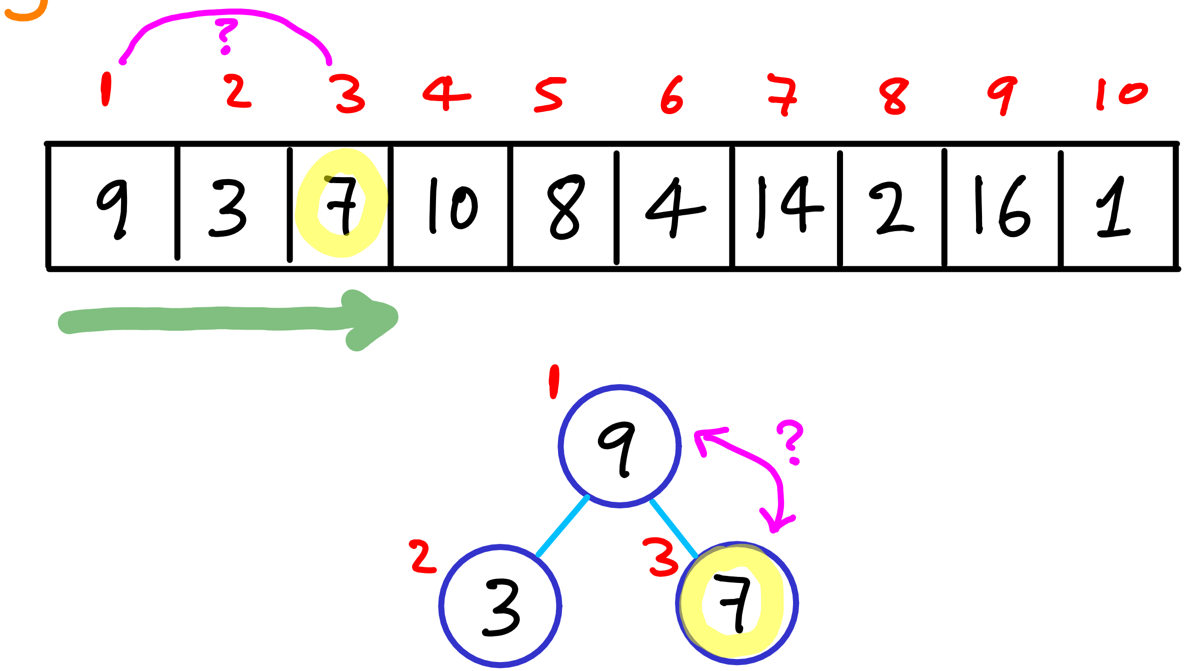

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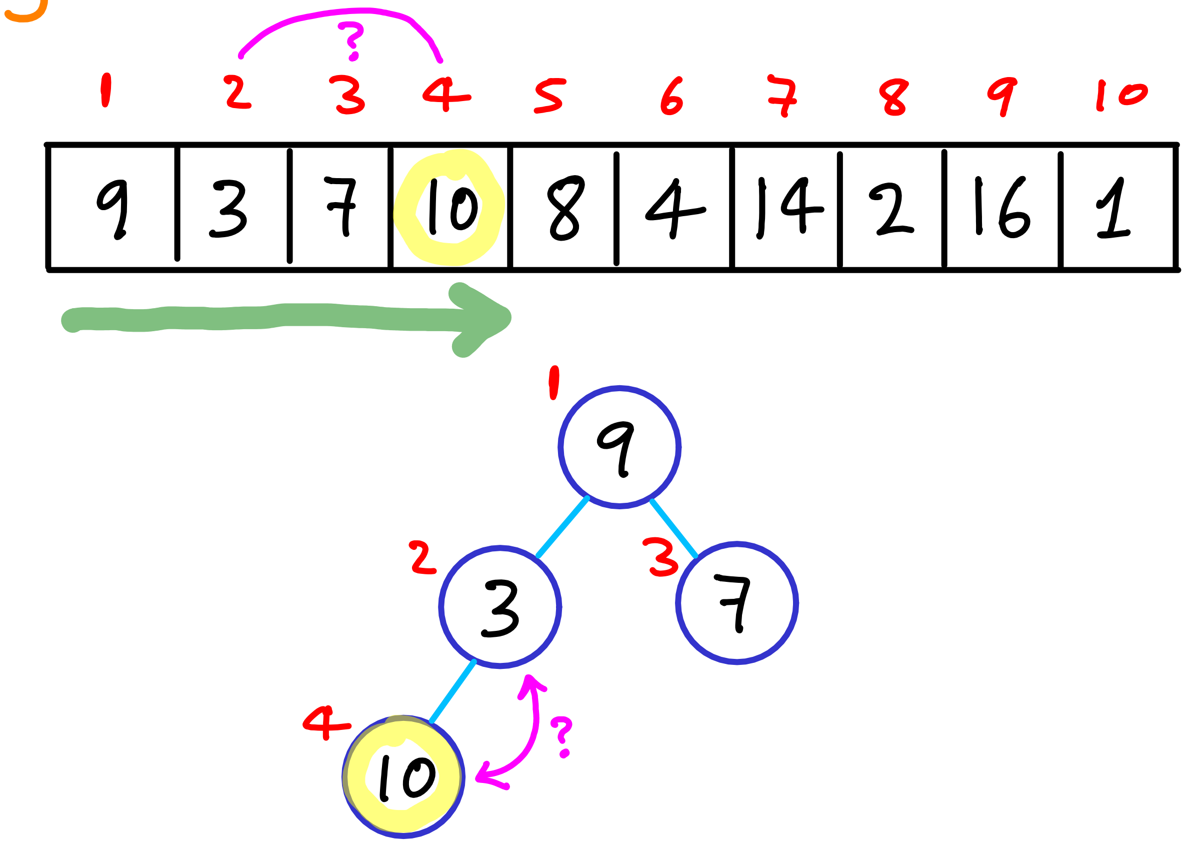

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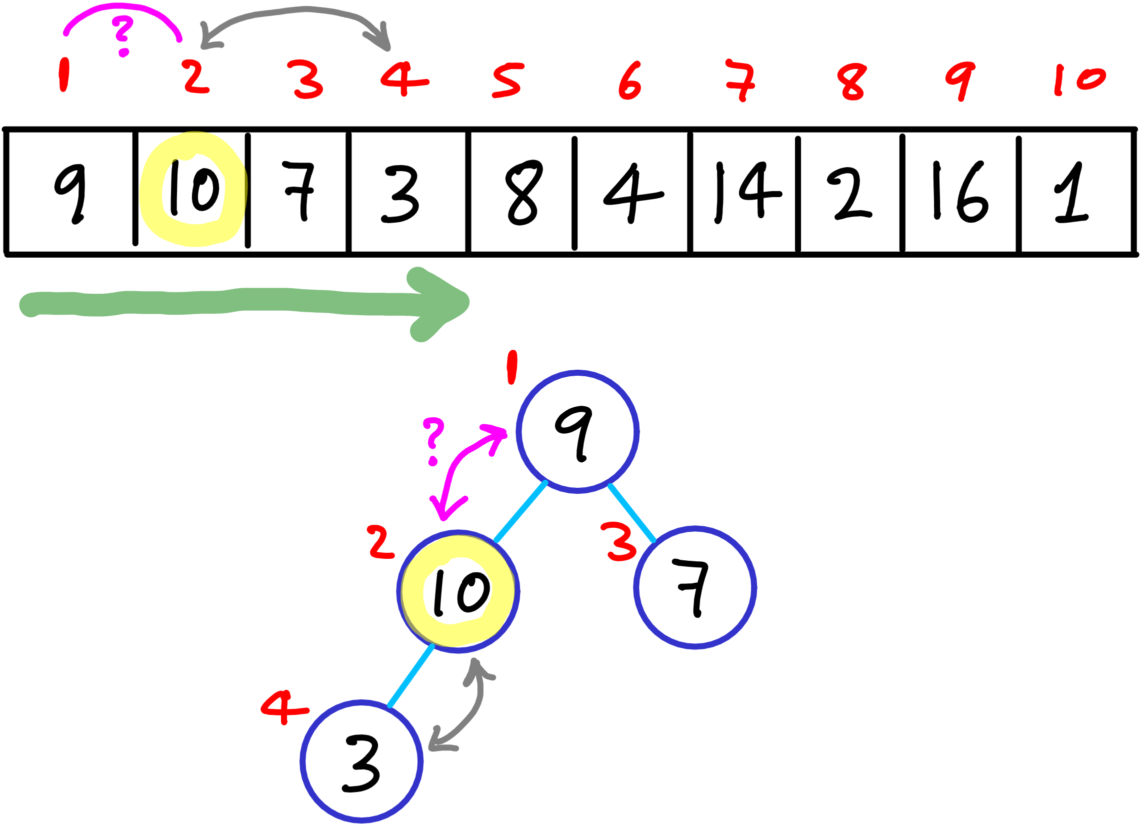

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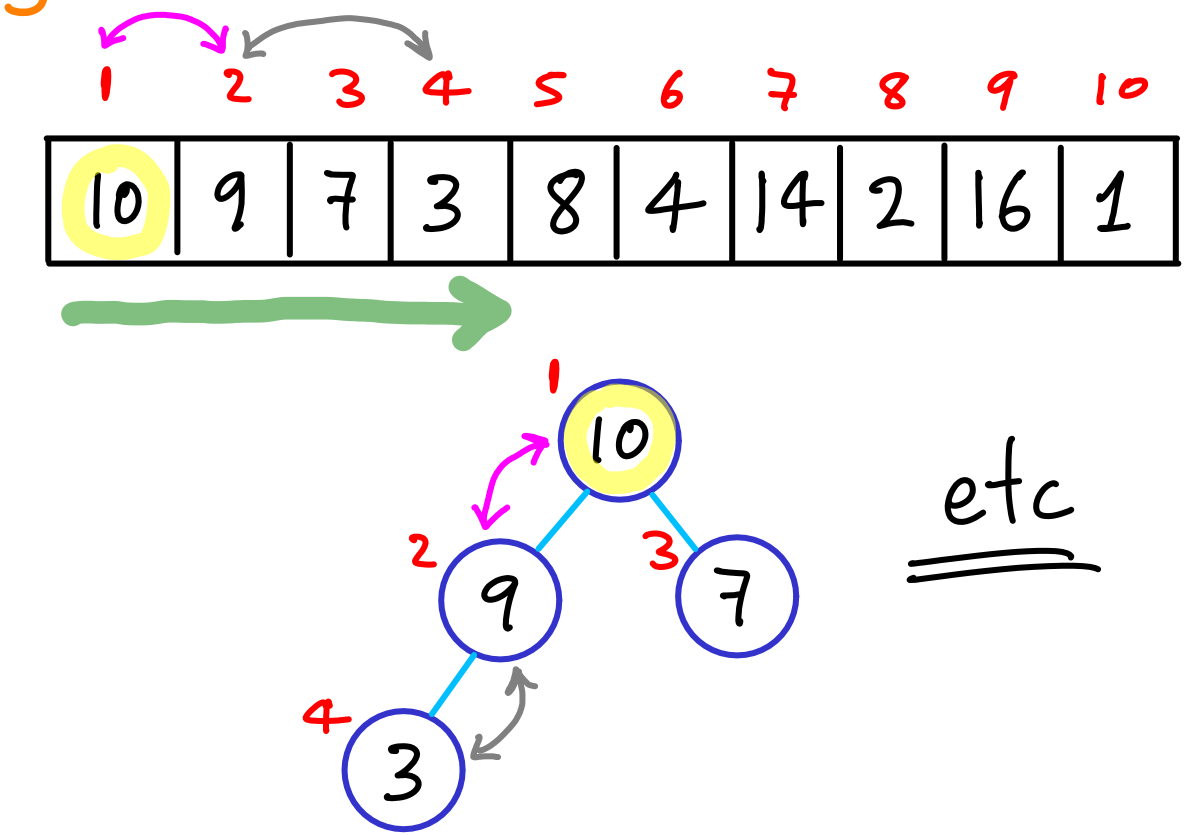

1	2	3	4	5	6	7	8	9	10
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etc

Heap building: the FORWARD METHOD (left to right)

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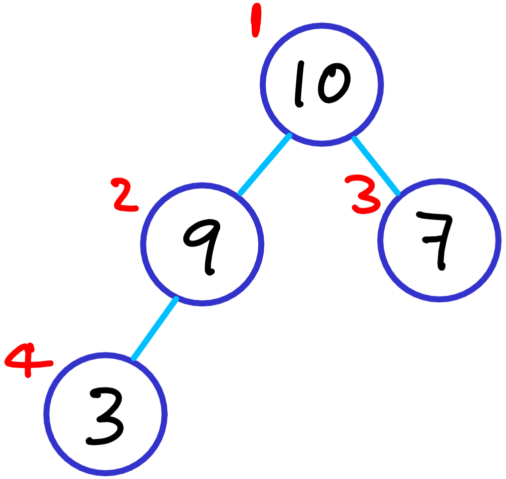

time?

Heap building: the FORWARD METHOD (left to right)

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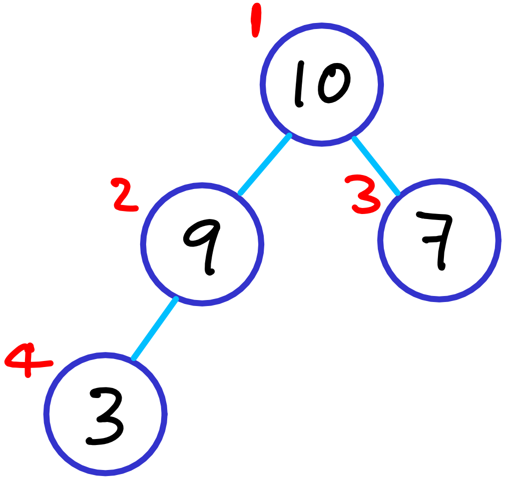
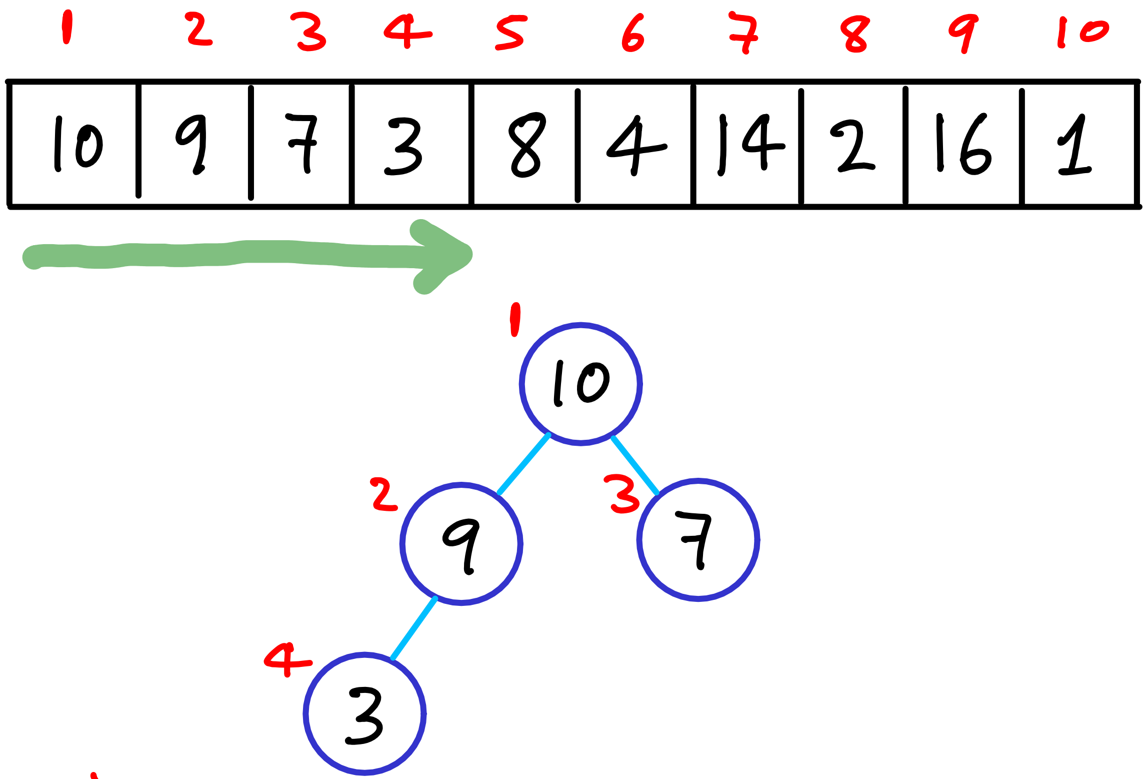


time = $O(n \log n)$

$O(\log n)$ per insertion

Heap building: the FORWARD METHOD (left to right)



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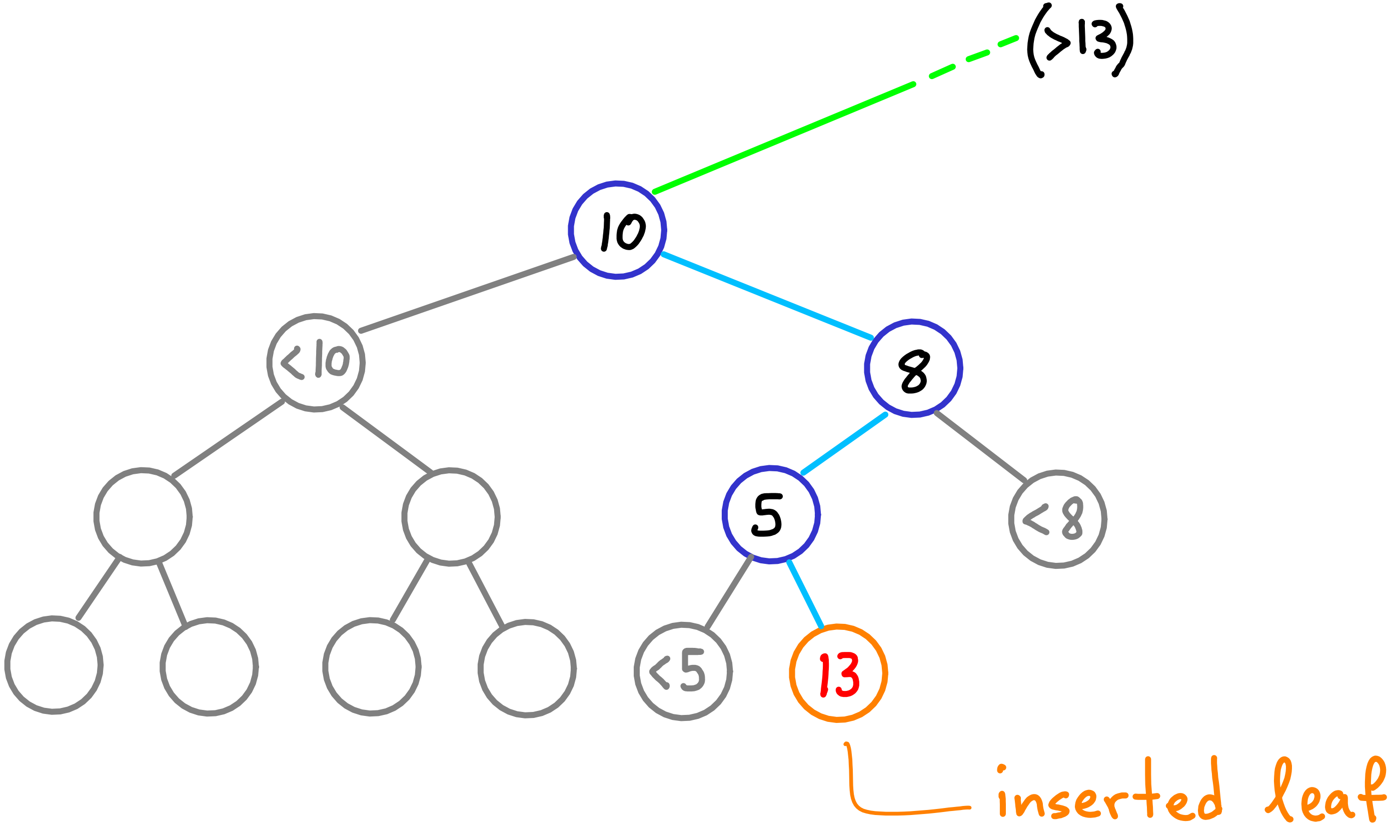
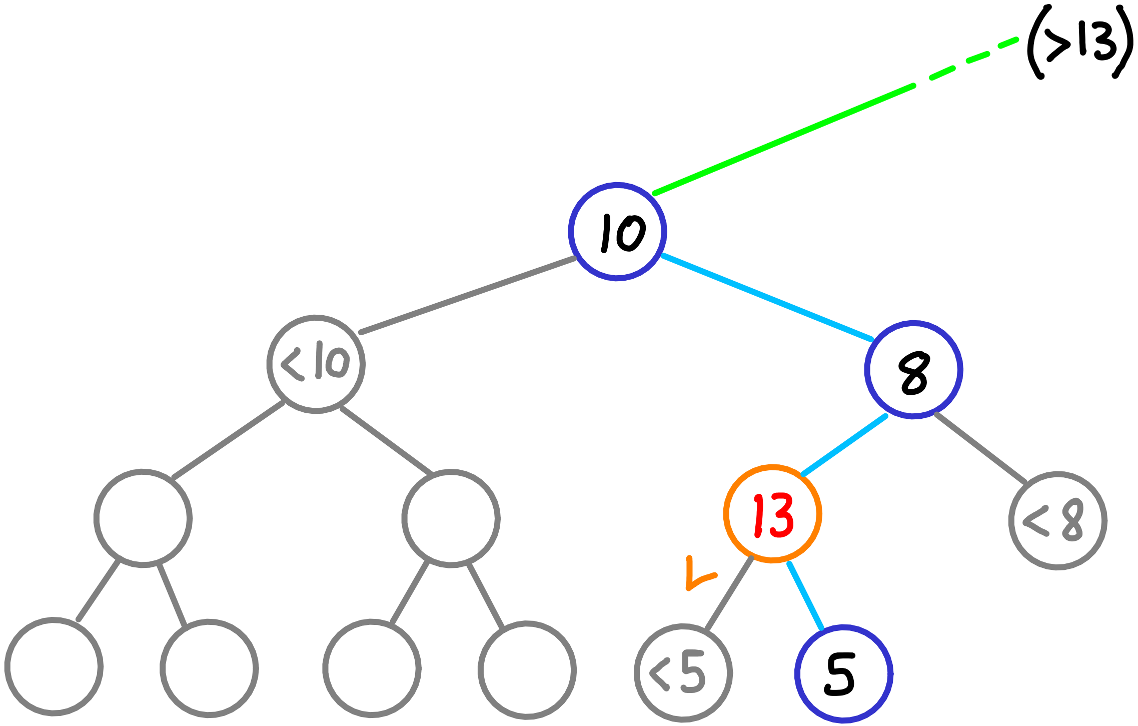
time = O(n log n)

O(log n) per insertion

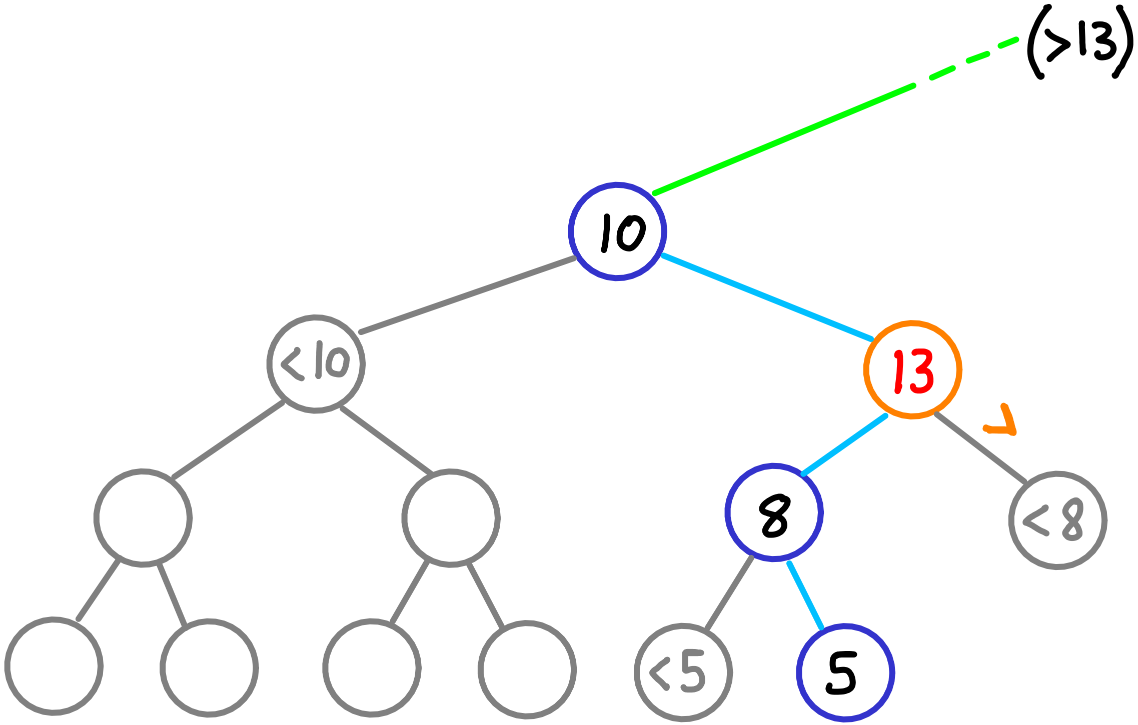
Works for streaming data

Correctness (sketch)





Correctness (sketch)



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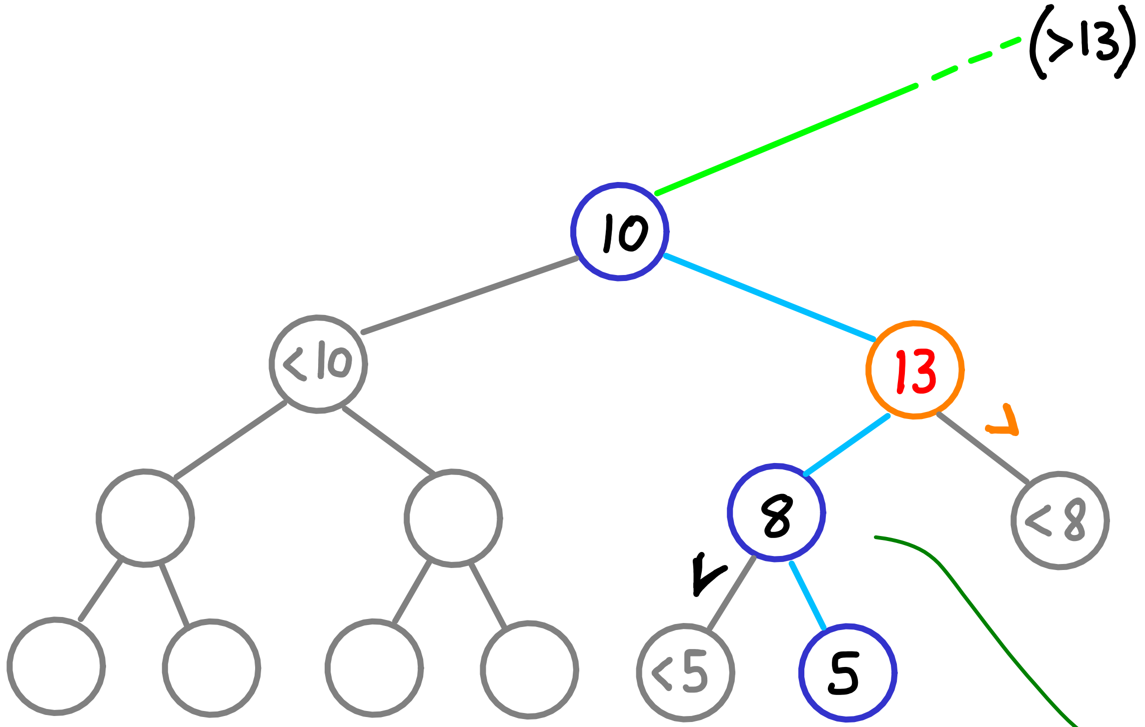
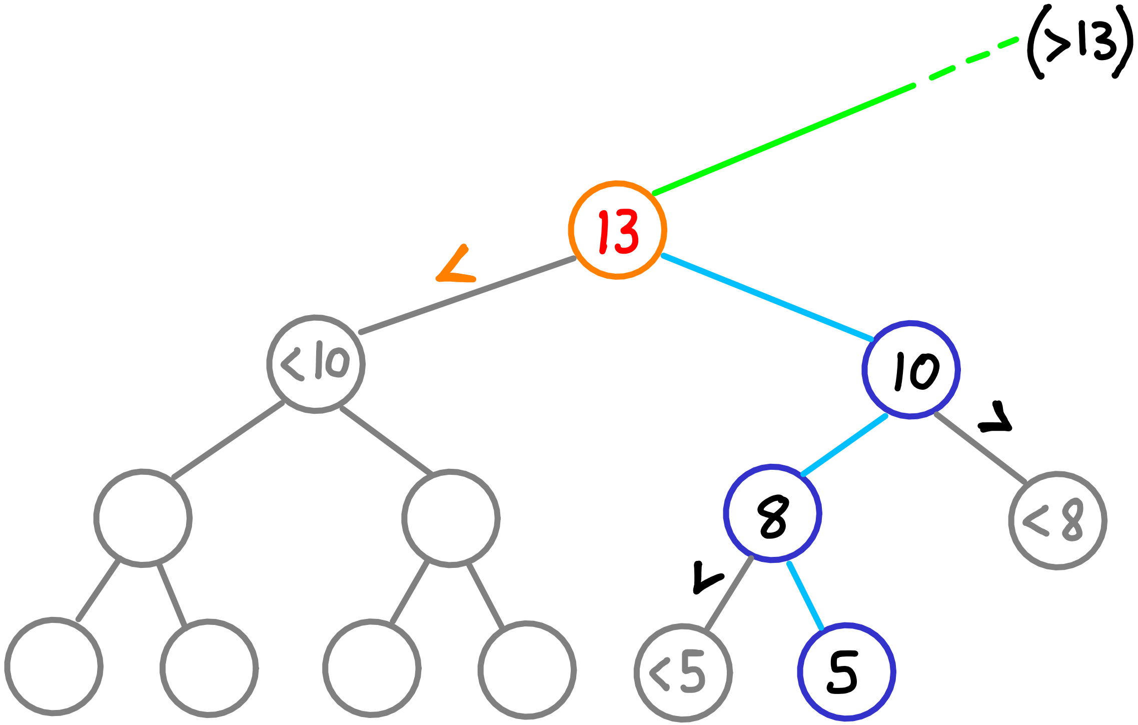


was 5, initially

replaced by parent: no problem



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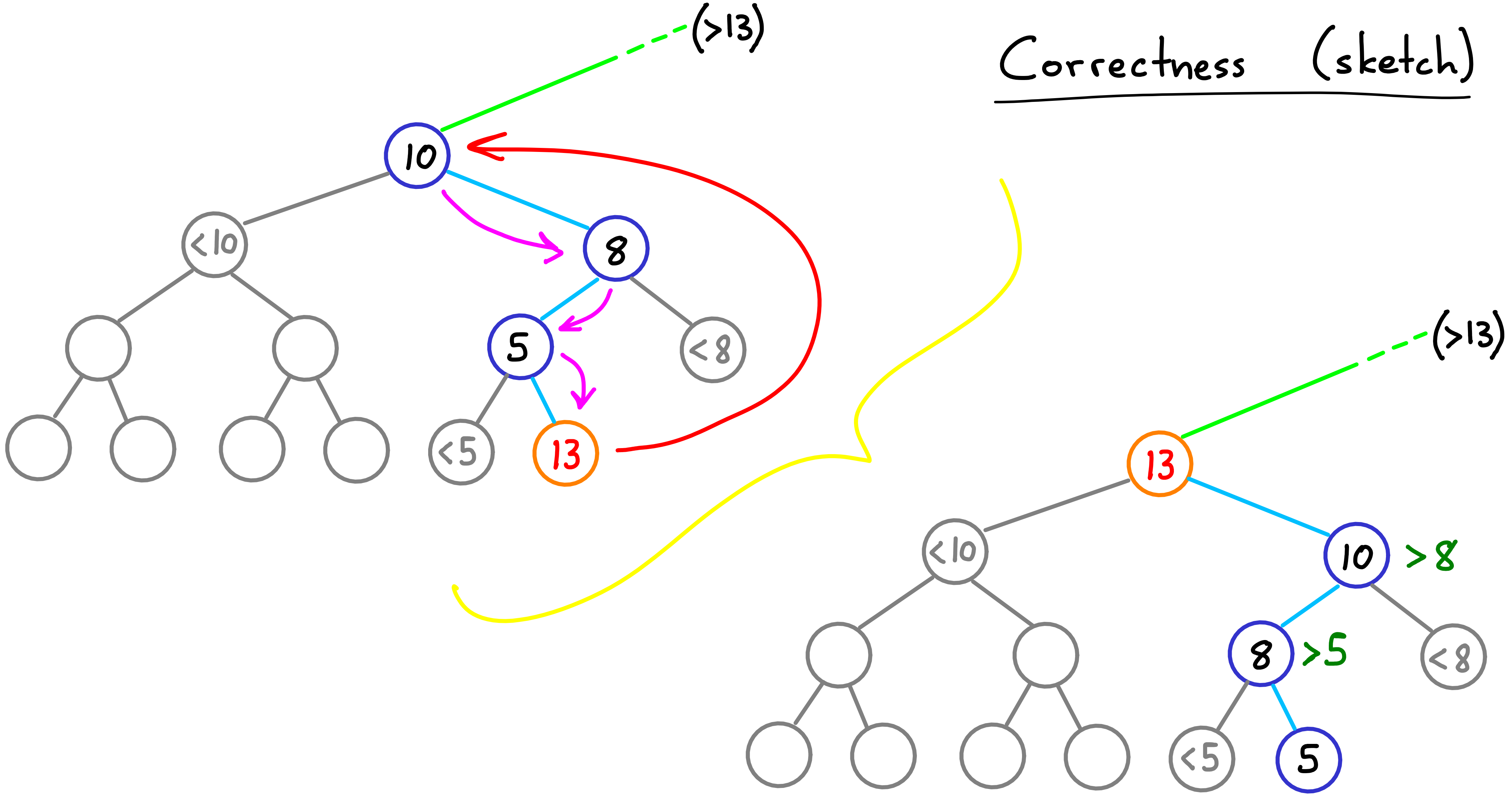
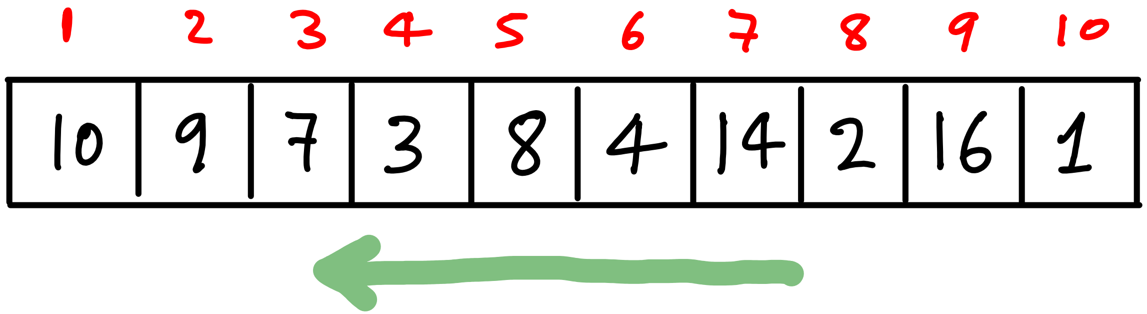


Heap building: the REVERSE METHOD (right to left)



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10	9	7	3	8	4	14	2	16	1

CLRS 6.3, p156

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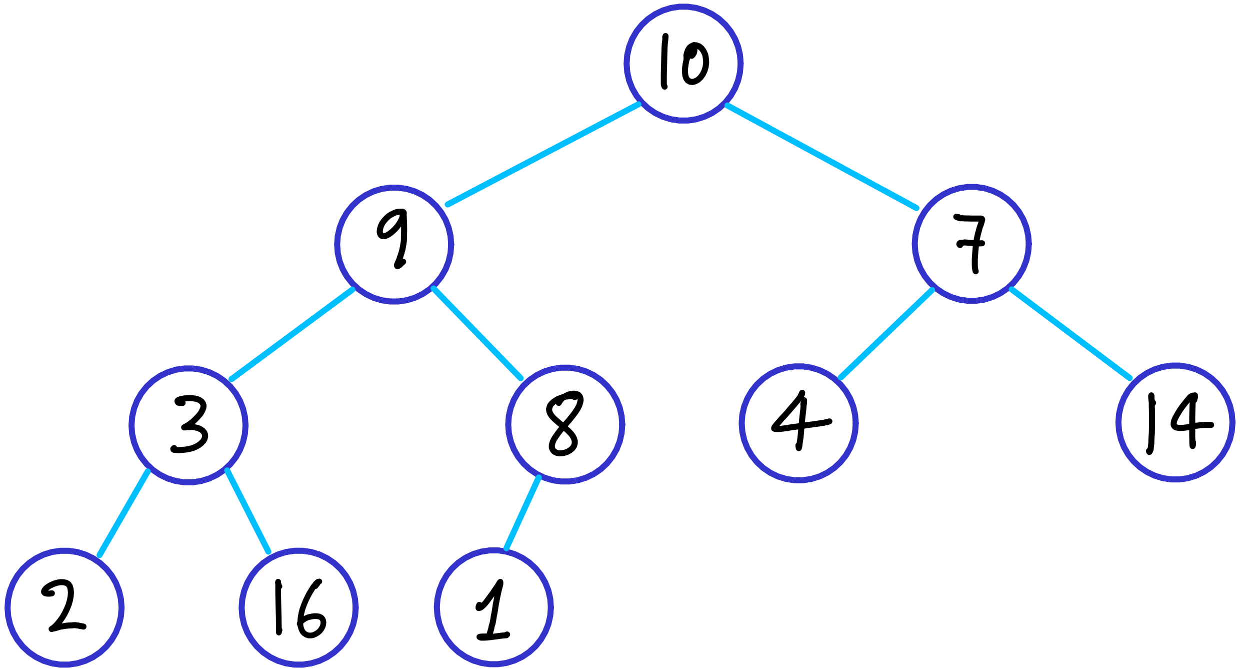

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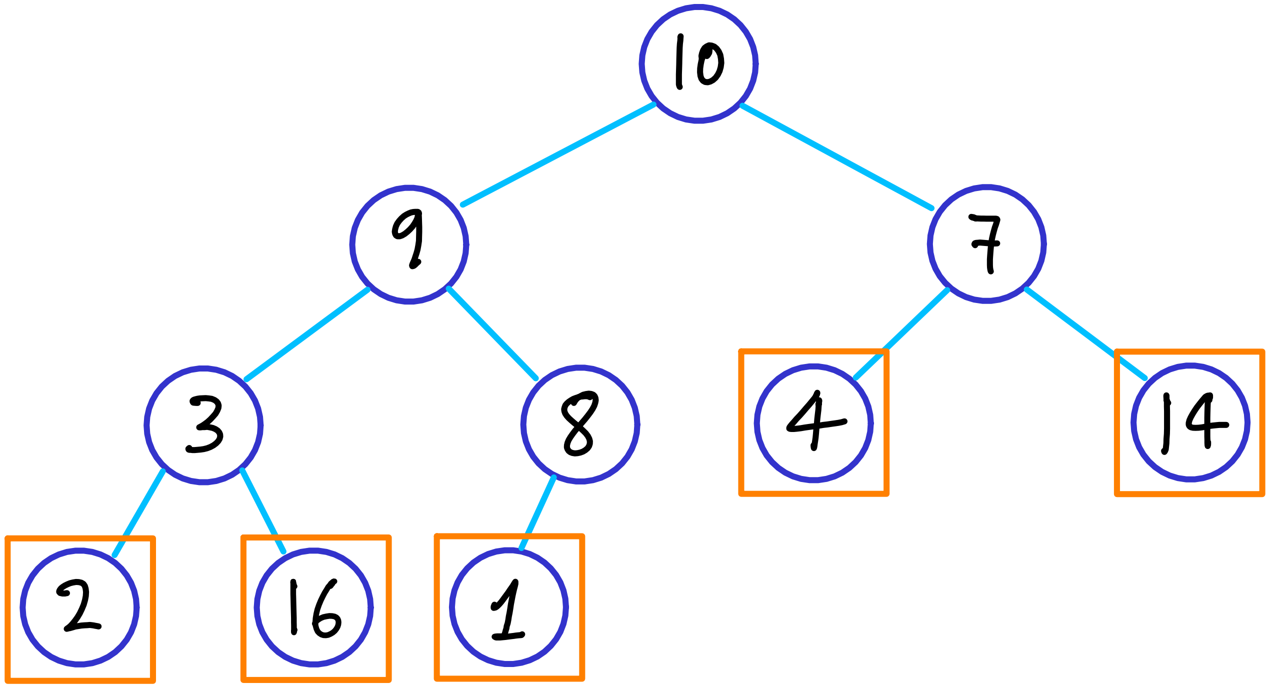

already heaps

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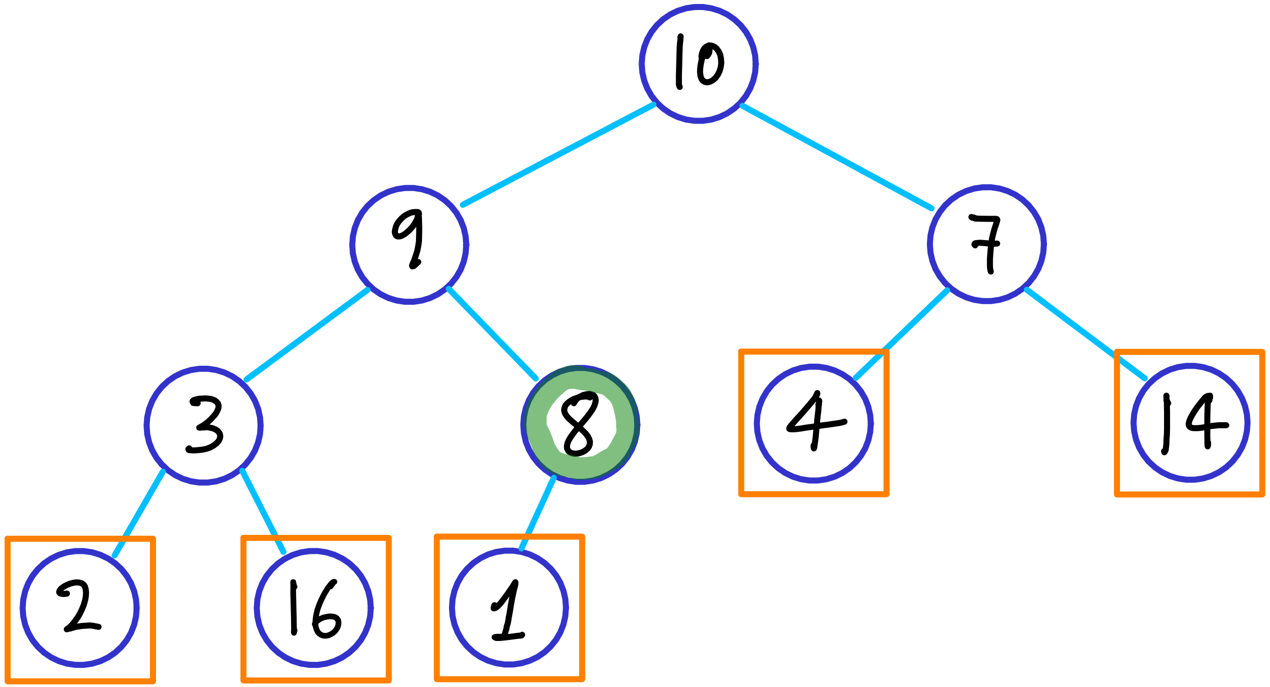

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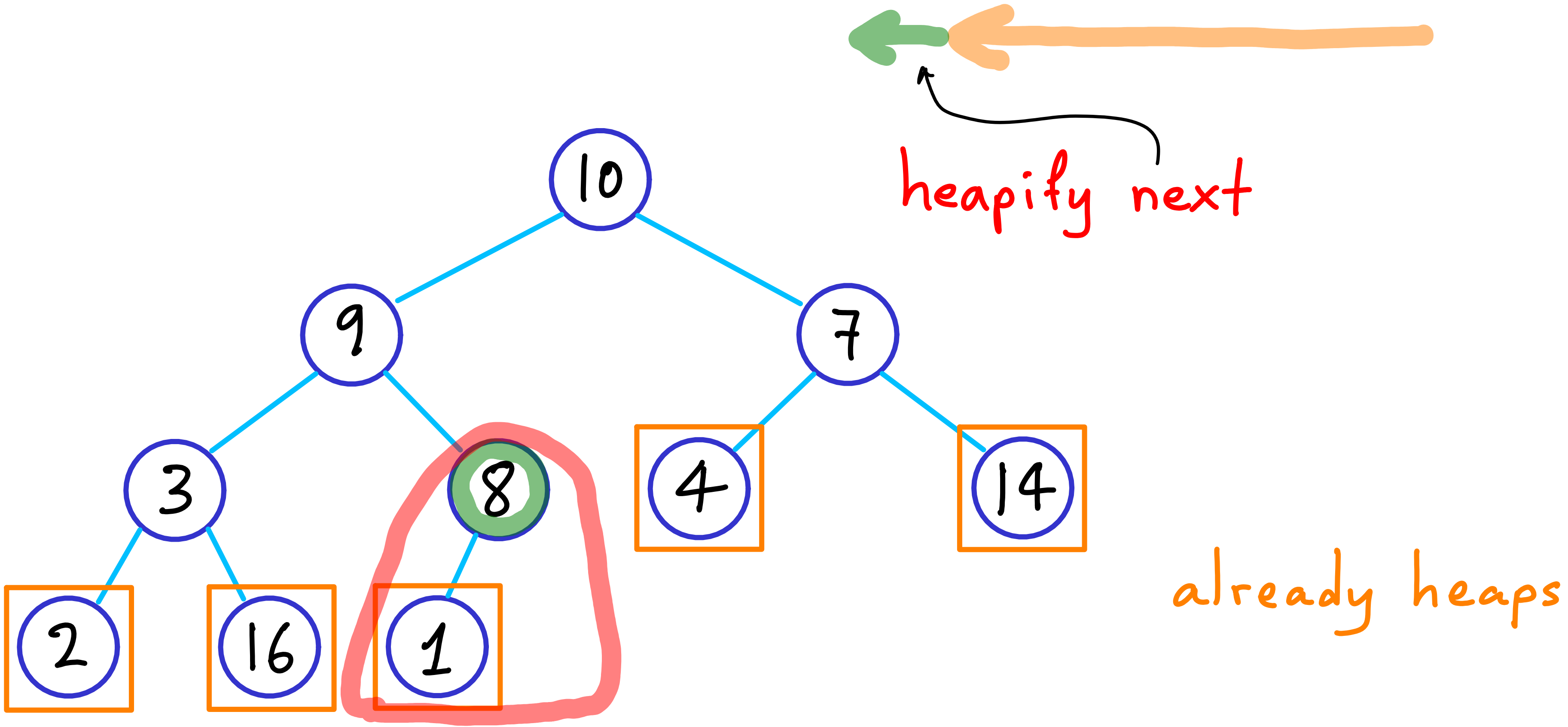
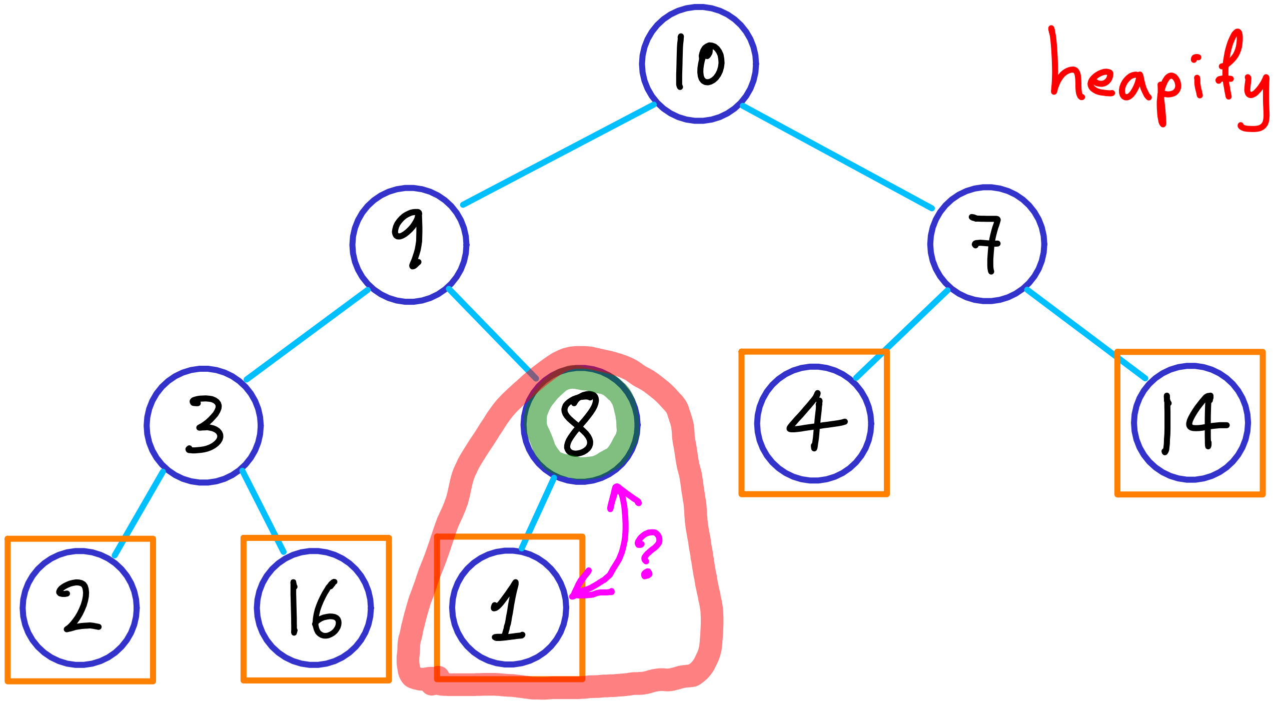


Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	3	8	4	14	2	16	1

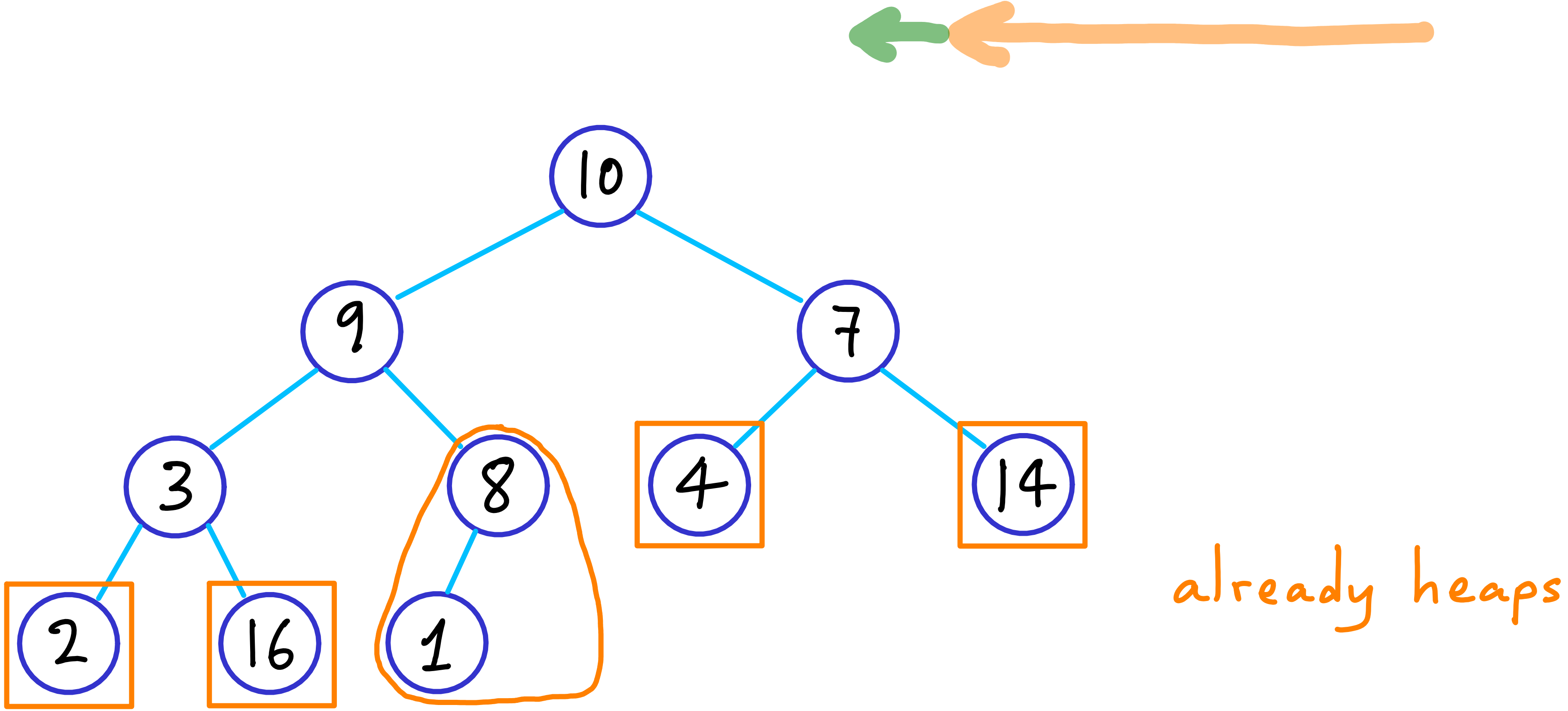
heapify next



already heaps

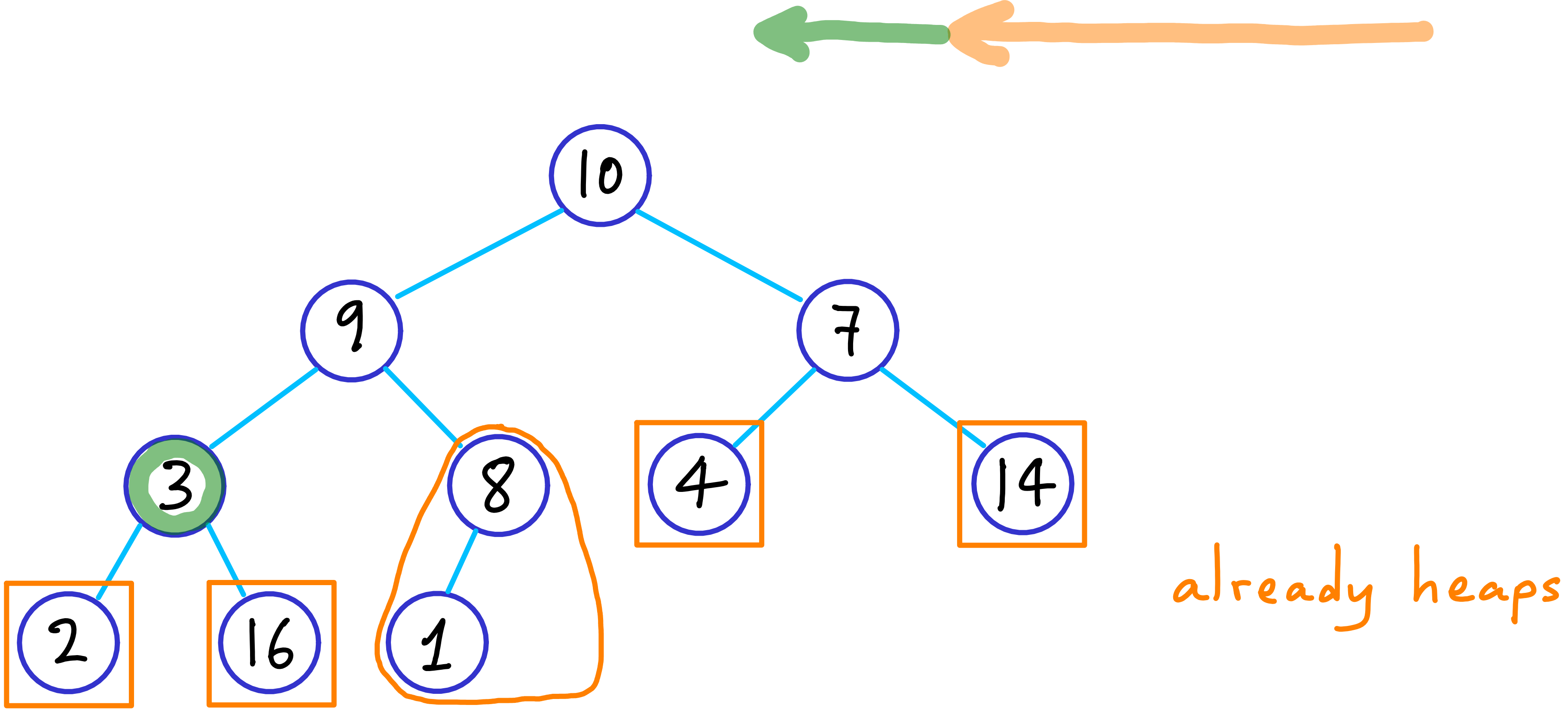
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	3	8	4	14	2	16	1



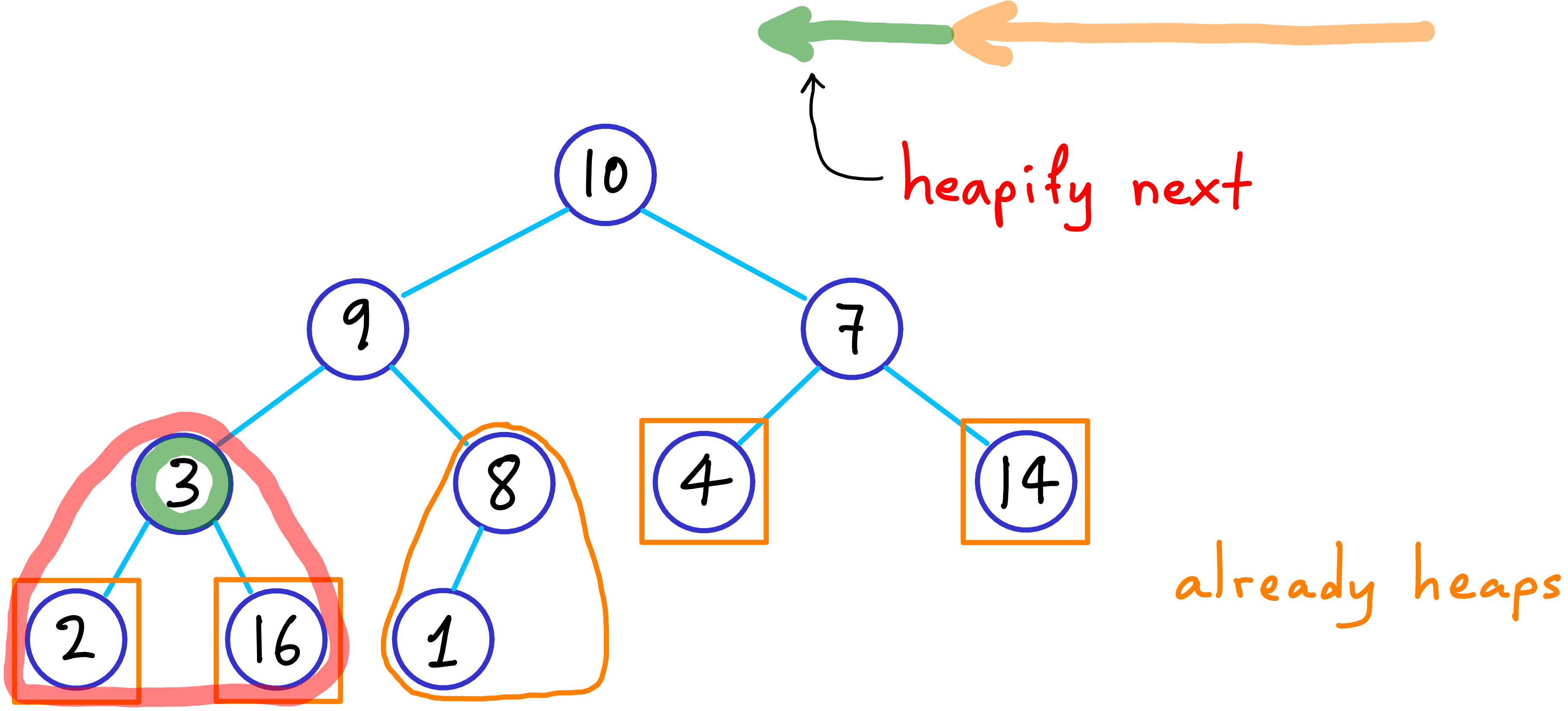
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	3	8	4	14	2	16	1

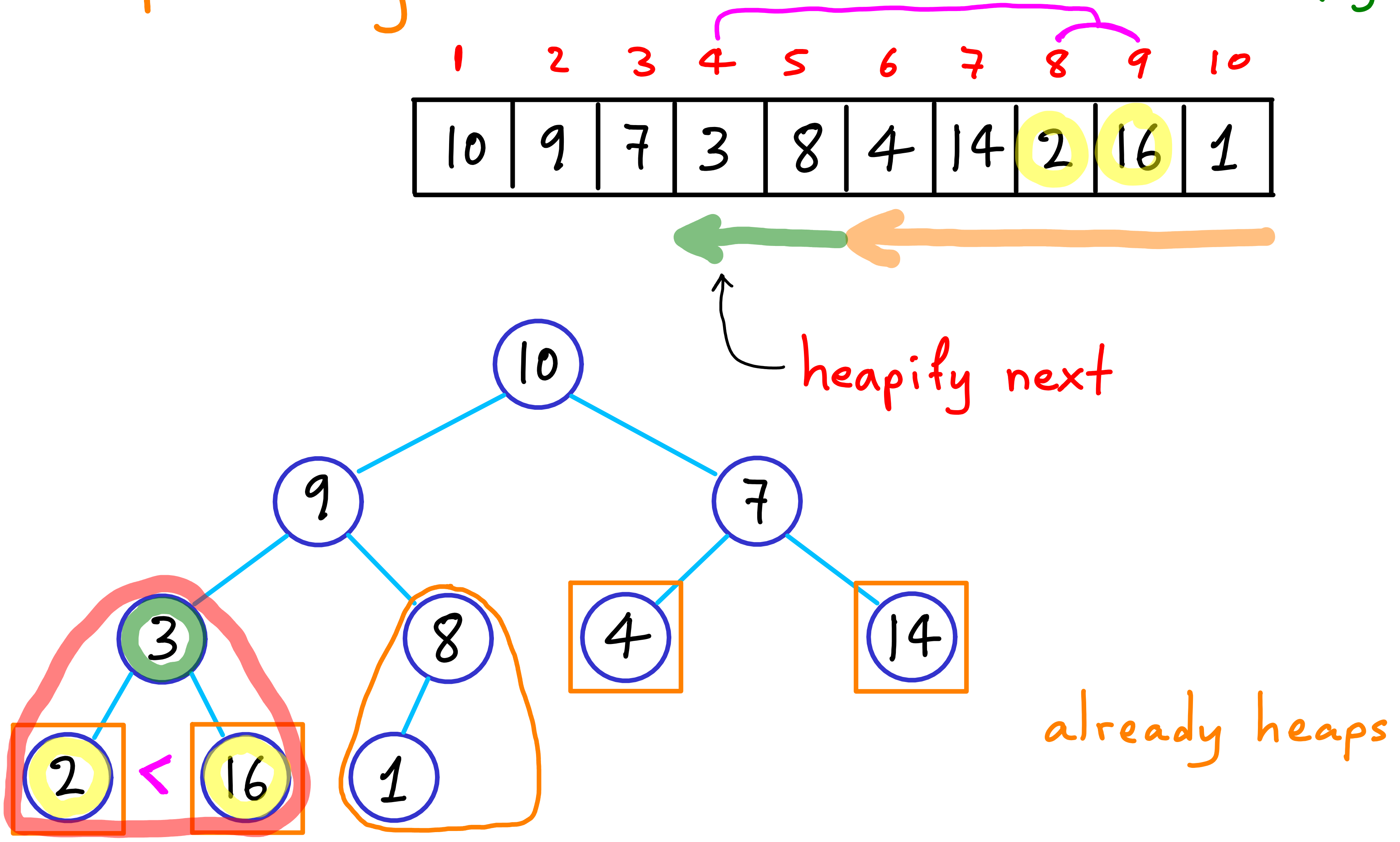


Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	3	8	4	14	2	16	1



Heap building: the REVERSE METHOD (right to left)

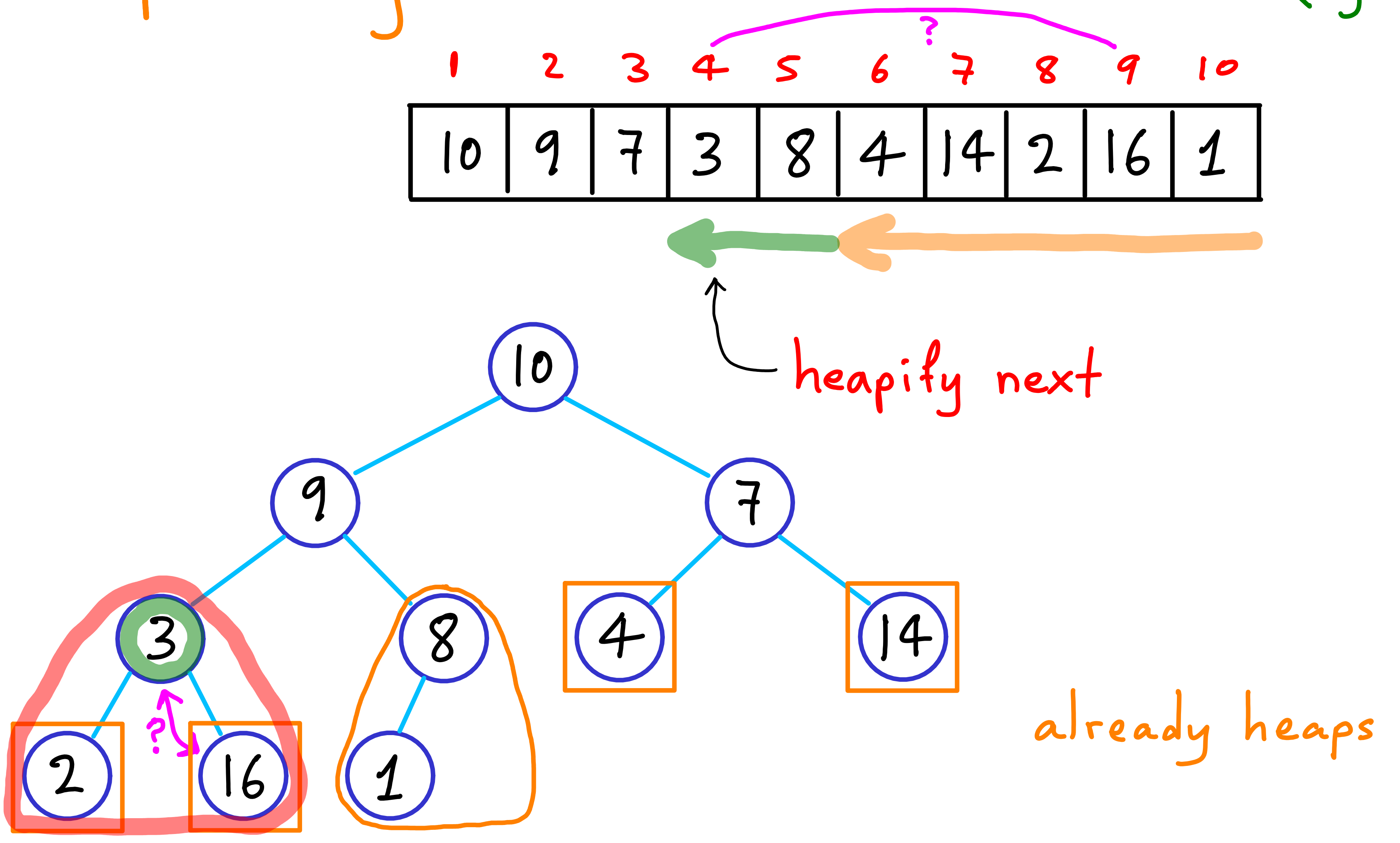


1	2	3	4	5	6	7	8	9	10
10	9	7	3	8	4	14	2	16	1

heapify next

already heaps

Heap building: the REVERSE METHOD (right to left)



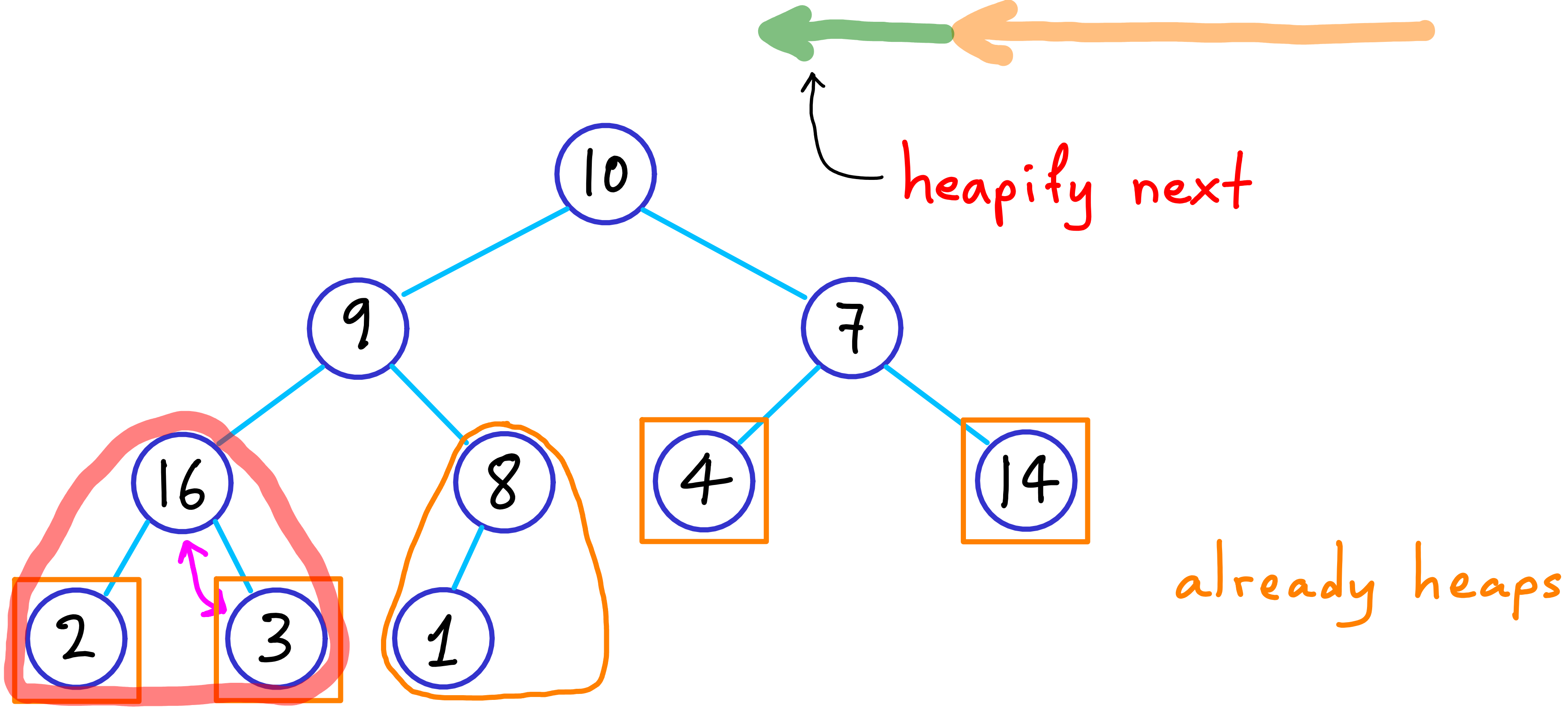
1	2	3	4	5	6	7	8	9	10
10	9	7	3	8	4	14	2	16	1

heapify next

already heaps

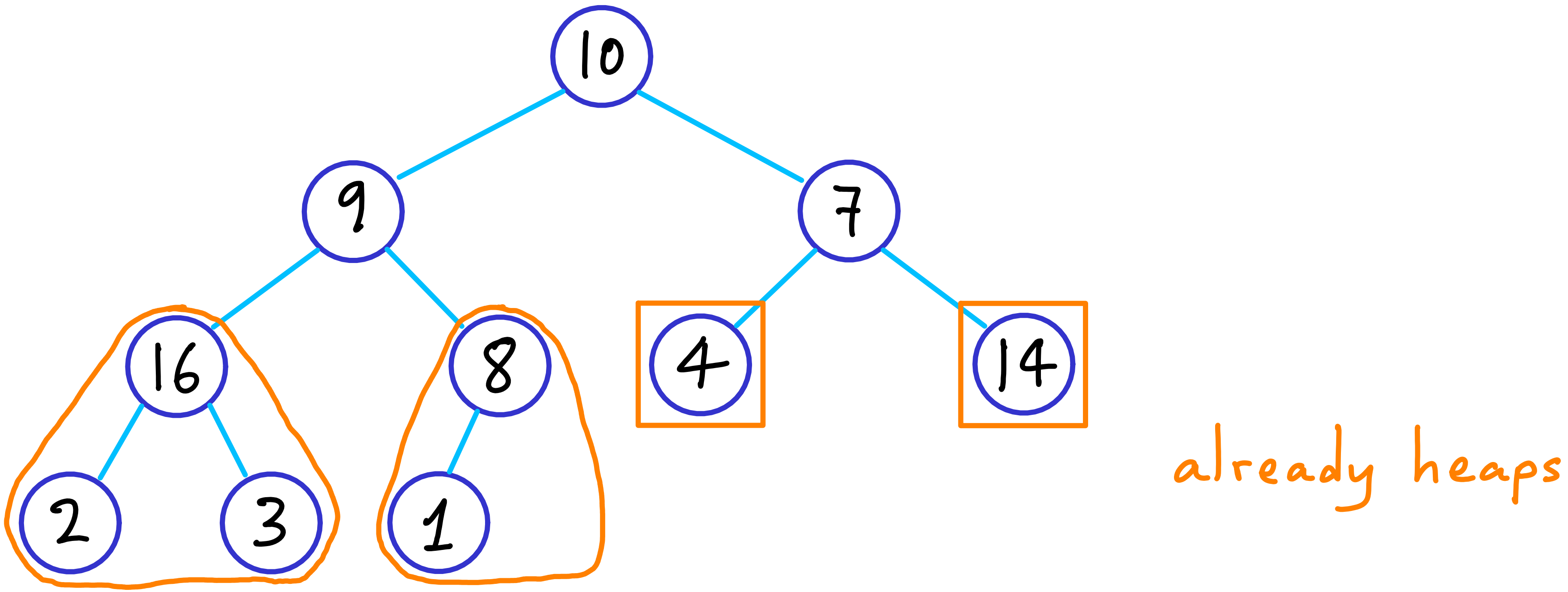
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	16	8	4	14	2	3	1



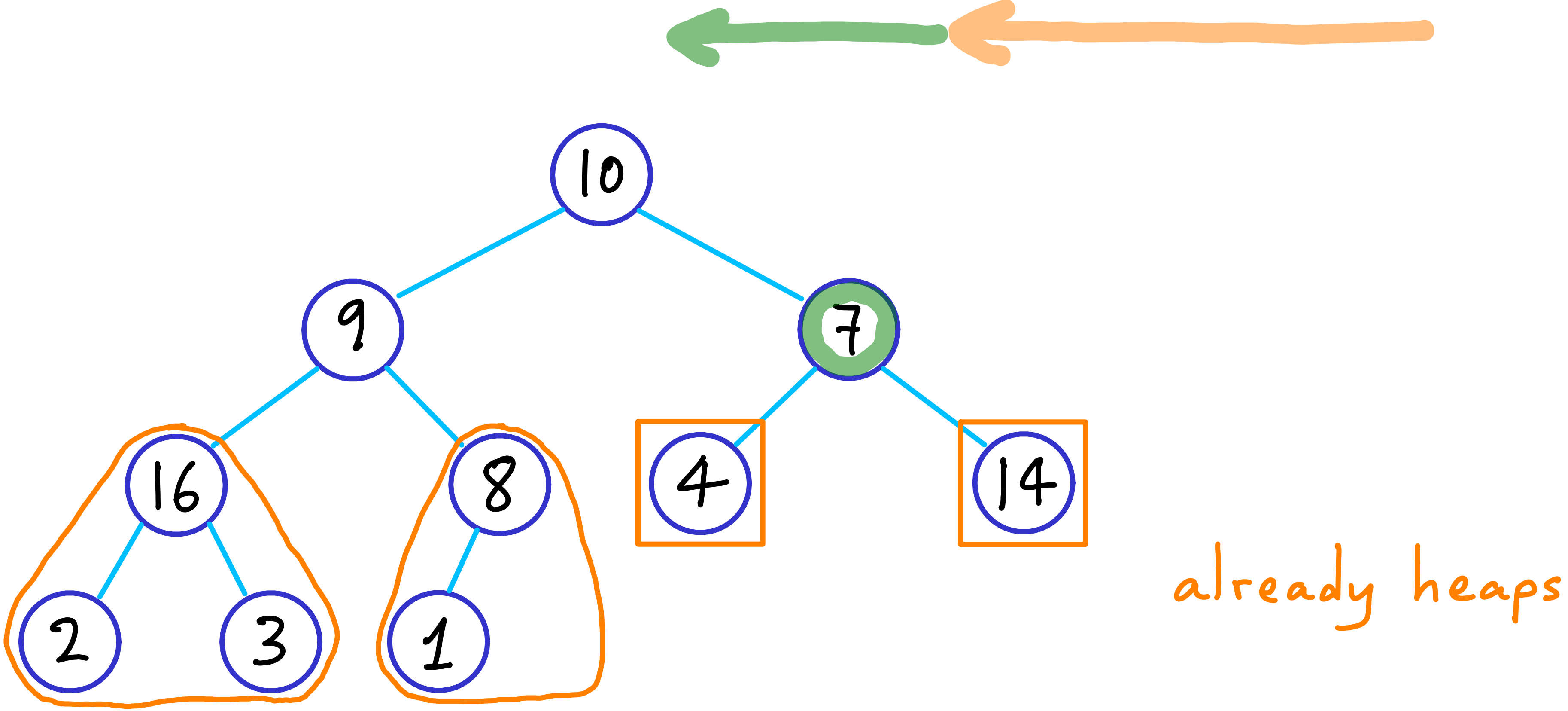
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	16	8	4	14	2	3	1



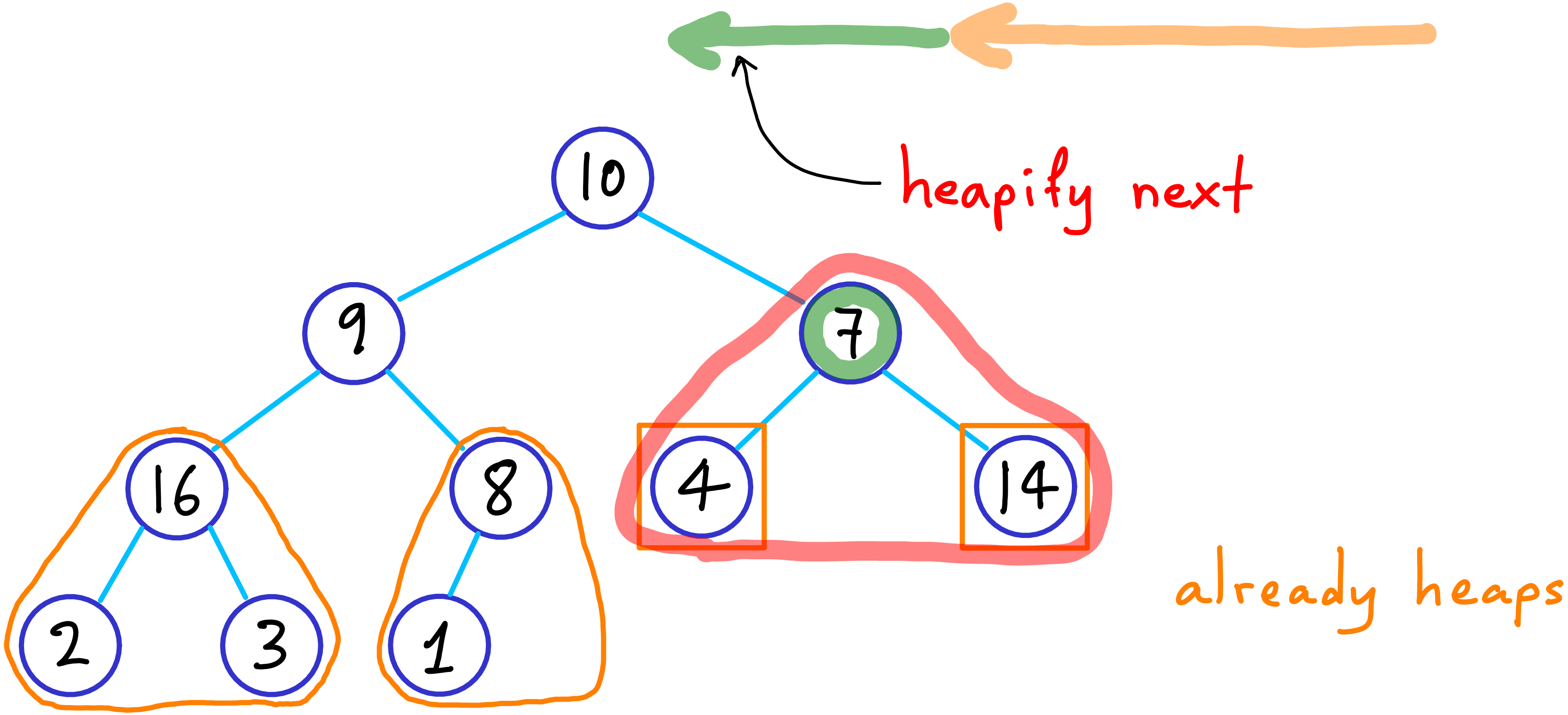
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	16	8	4	14	2	3	1



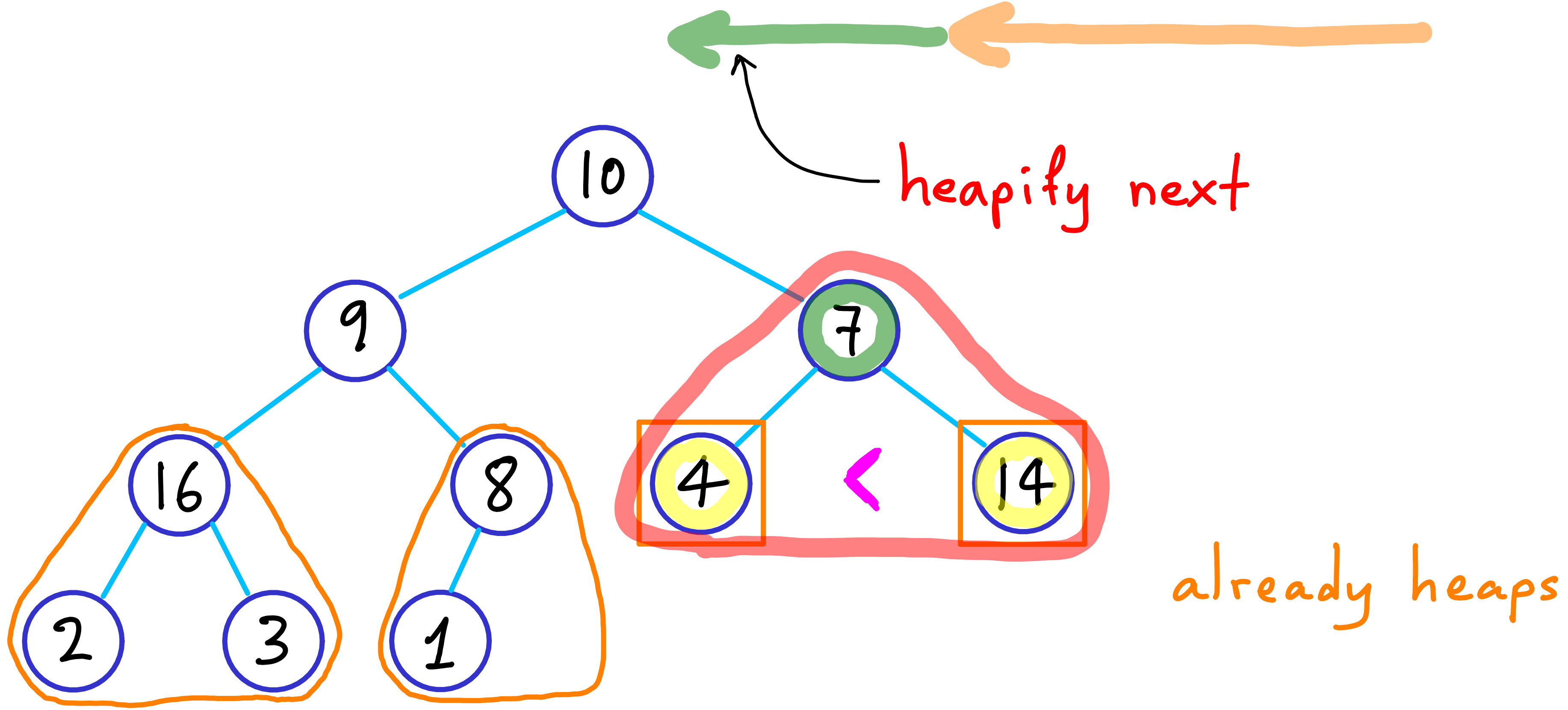
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	7	16	8	4	14	2	3	1



Heap building: the REVERSE METHOD (right to left)

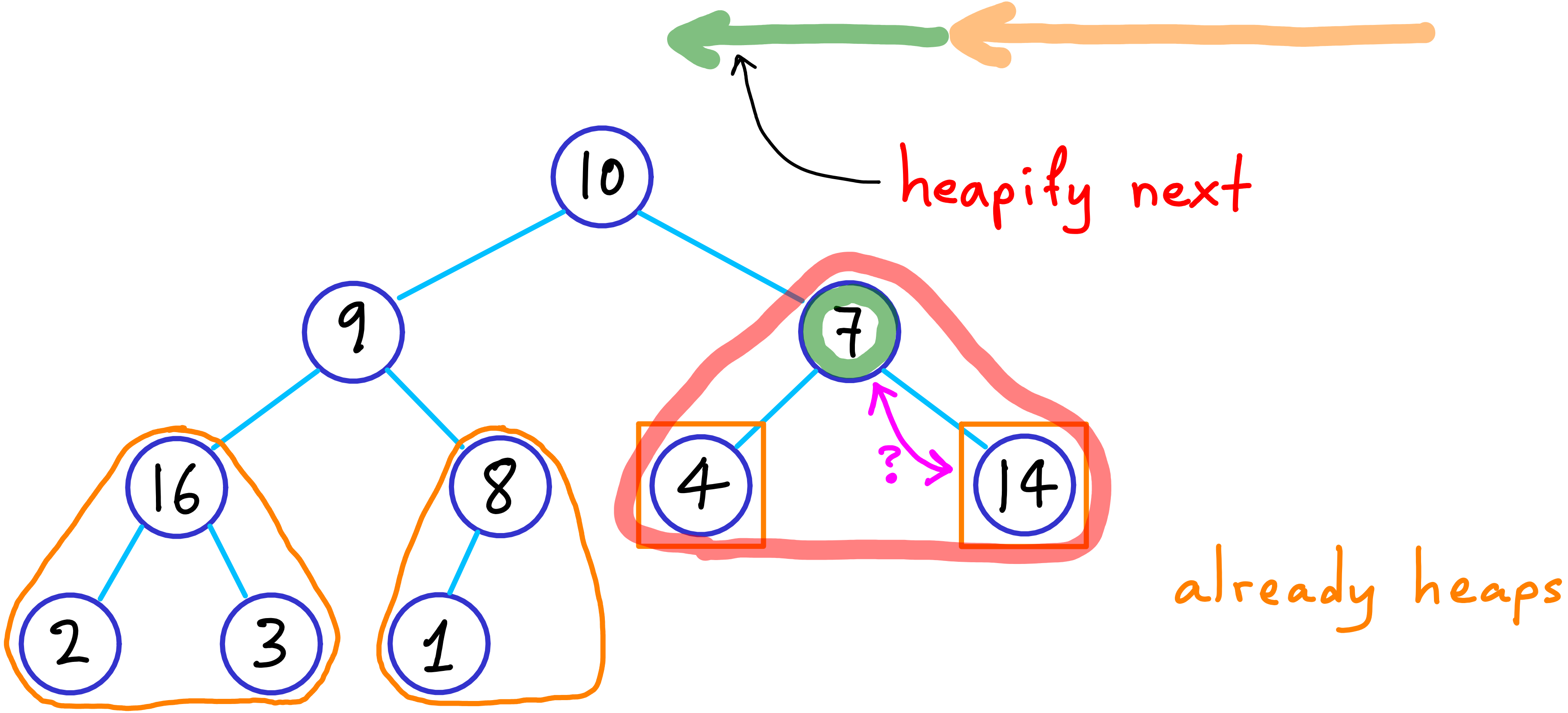
1	2	3	4	5	6	7	8	9	10
10	9	7	16	8	4	14	2	3	1



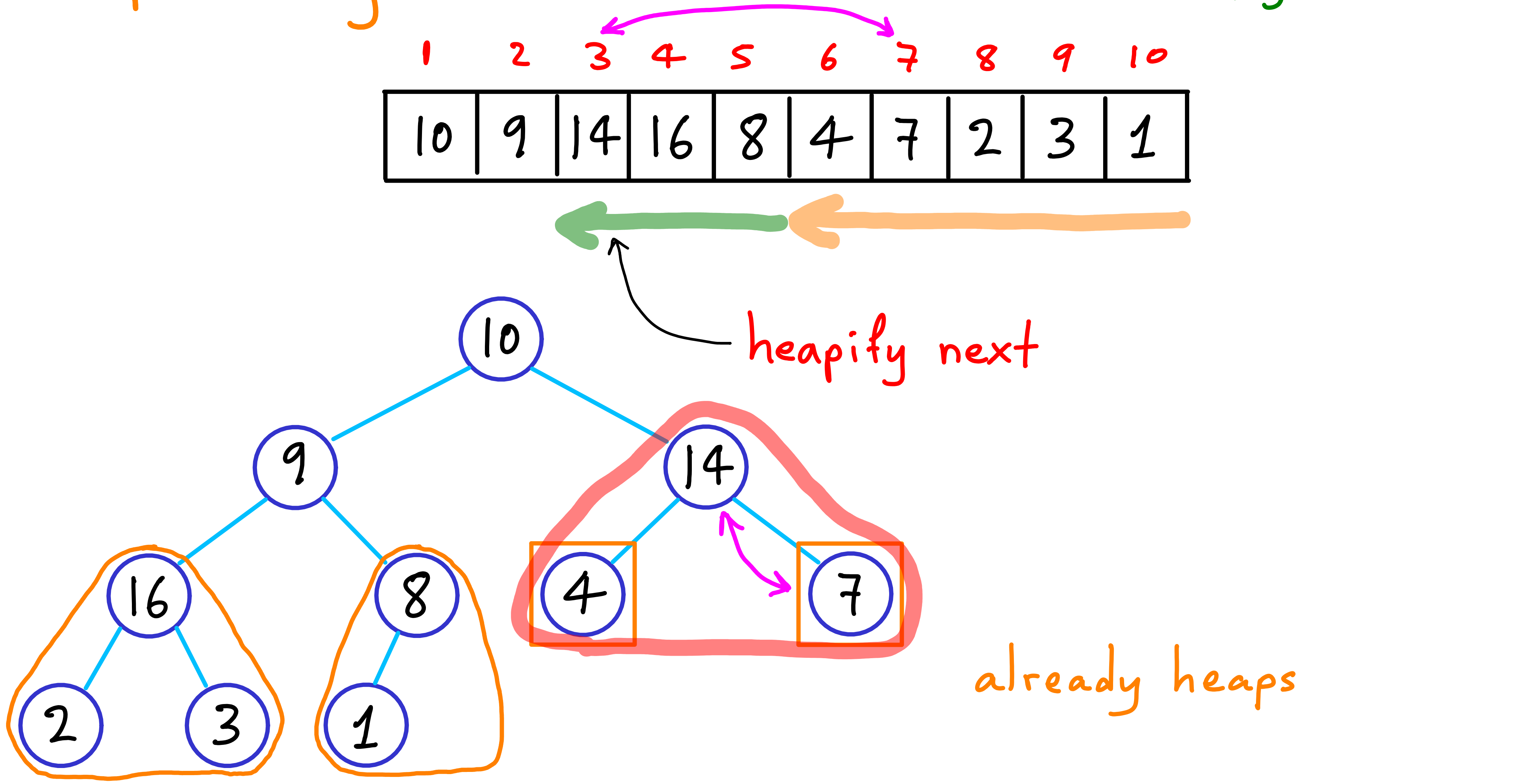
Heap building: the REVERSE METHOD (right to left)

?

1	2	3	4	5	6	7	8	9	10
10	9	7	16	8	4	14	2	3	1



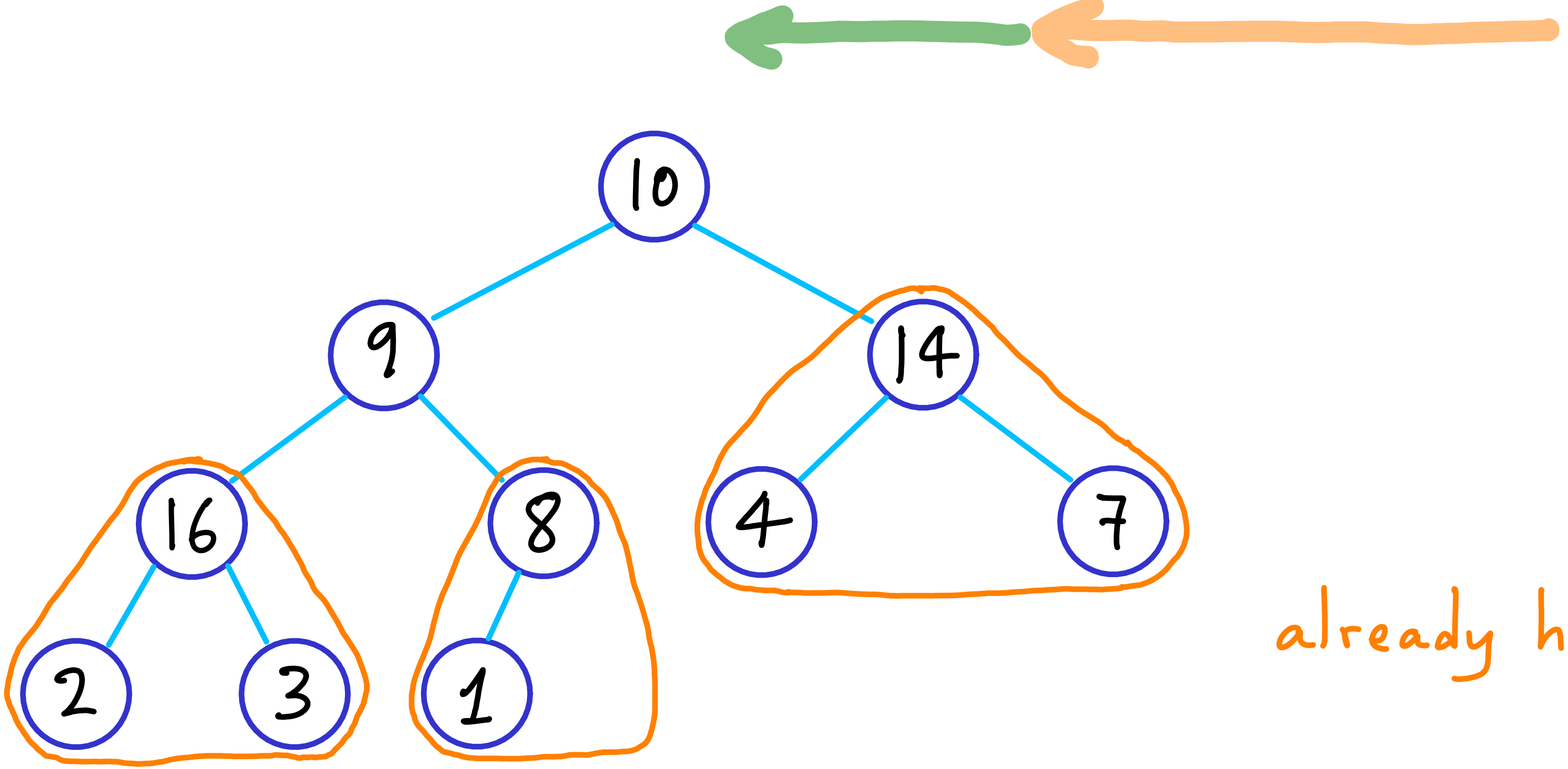
Heap building: the REVERSE METHOD (right to left)



1	2	3	4	5	6	7	8	9	10
10	9	14	16	8	4	7	2	3	1

Heap building: the REVERSE METHOD (right to left)

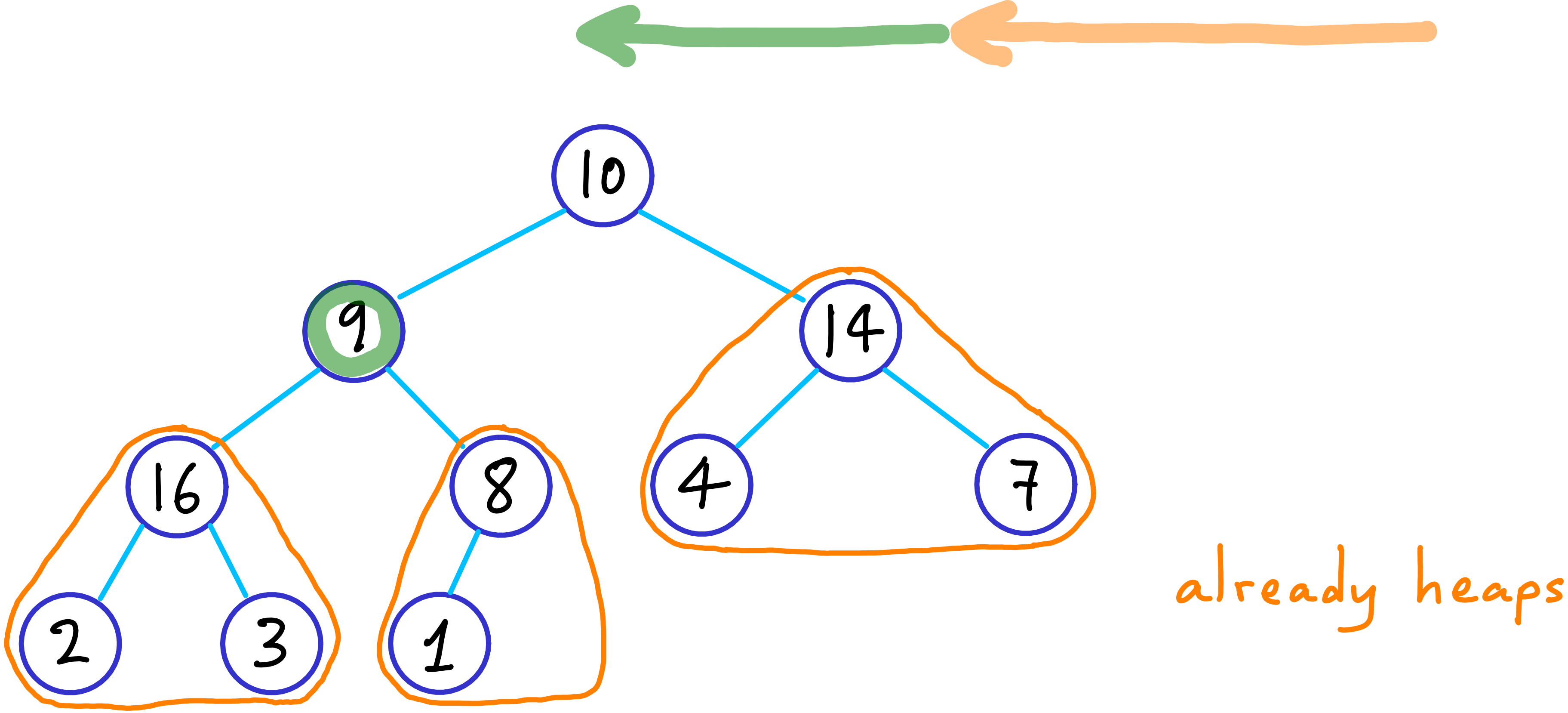
1	2	3	4	5	6	7	8	9	10
10	9	14	16	8	4	7	2	3	1



already heaps

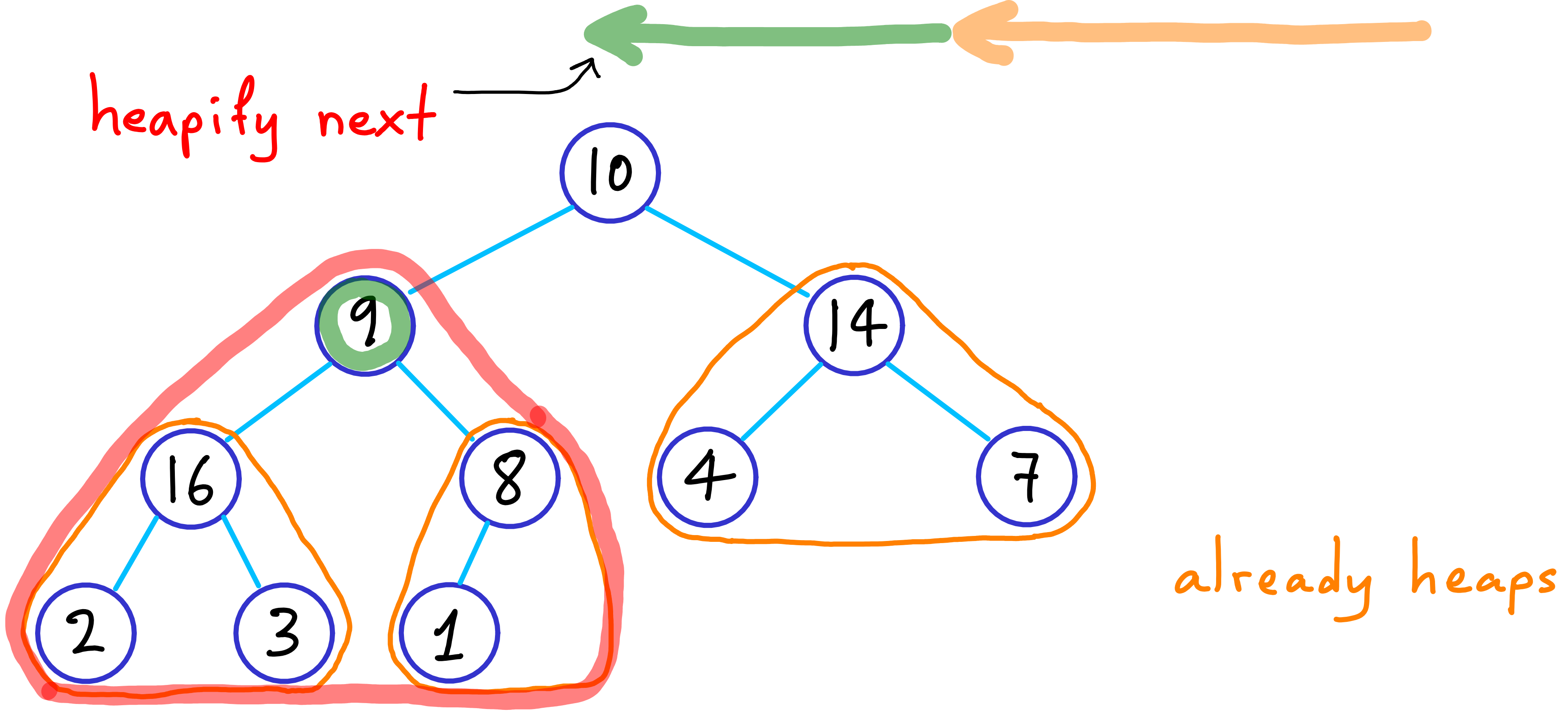
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	14	16	8	4	7	2	3	1



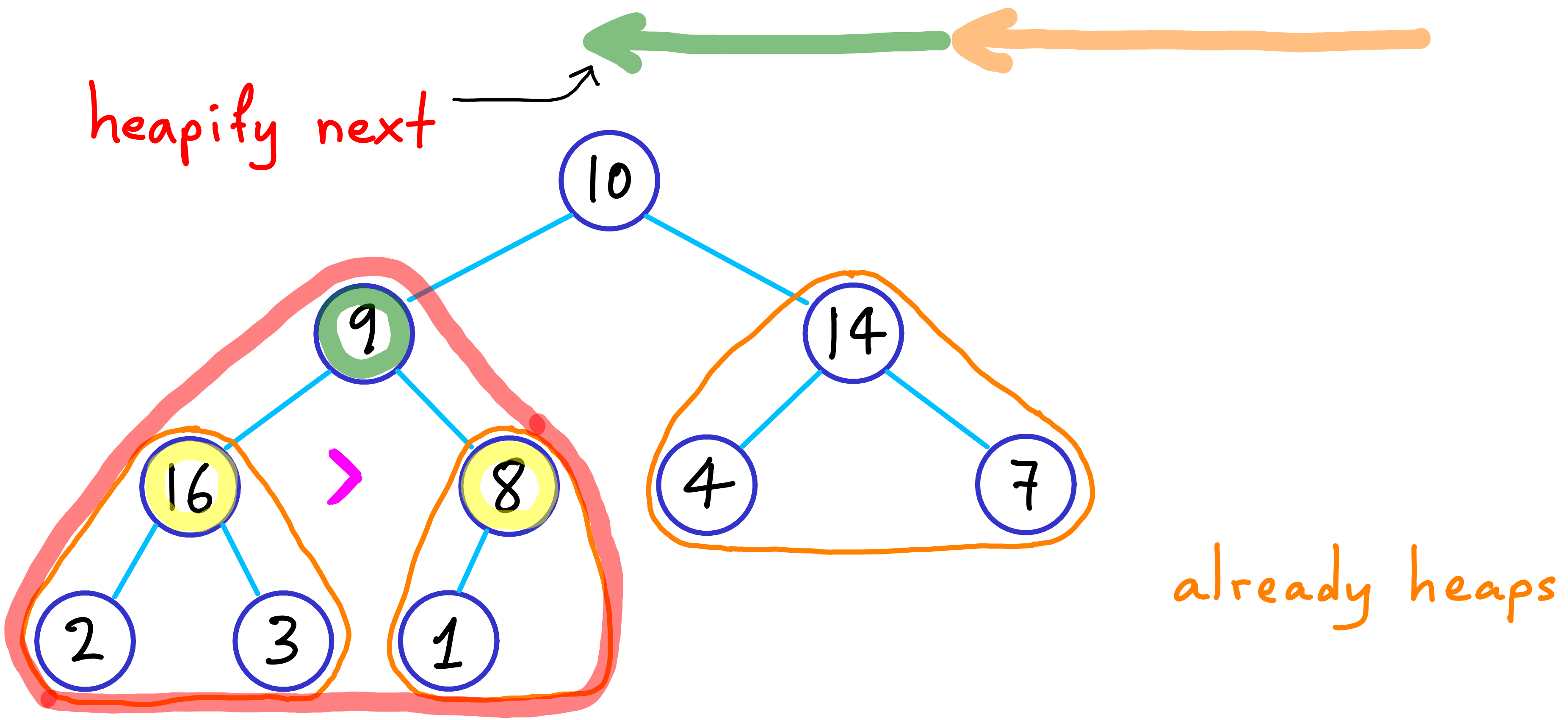
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	14	16	8	4	7	2	3	1



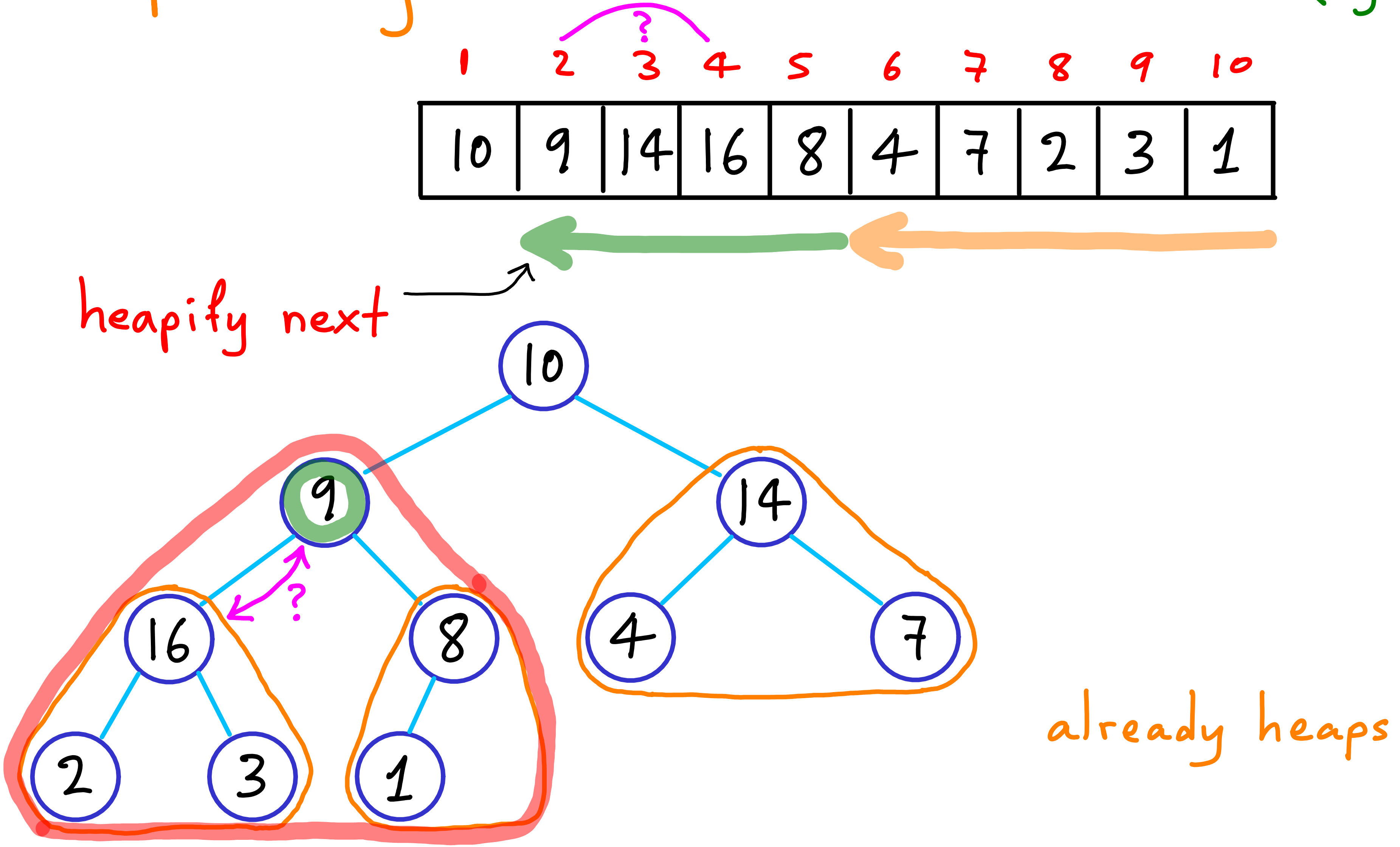
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	14	16	8	4	7	2	3	1



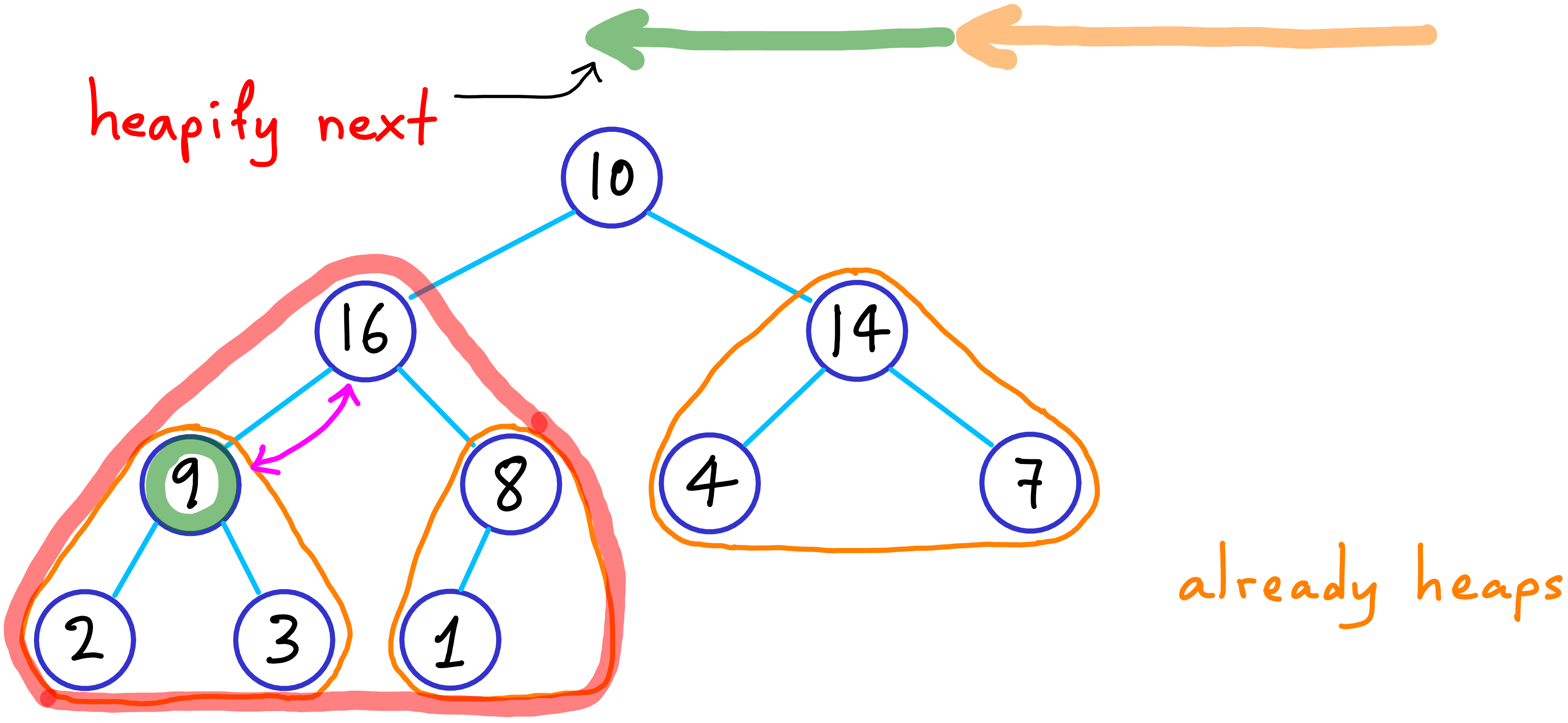
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	9	14	16	8	4	7	2	3	1

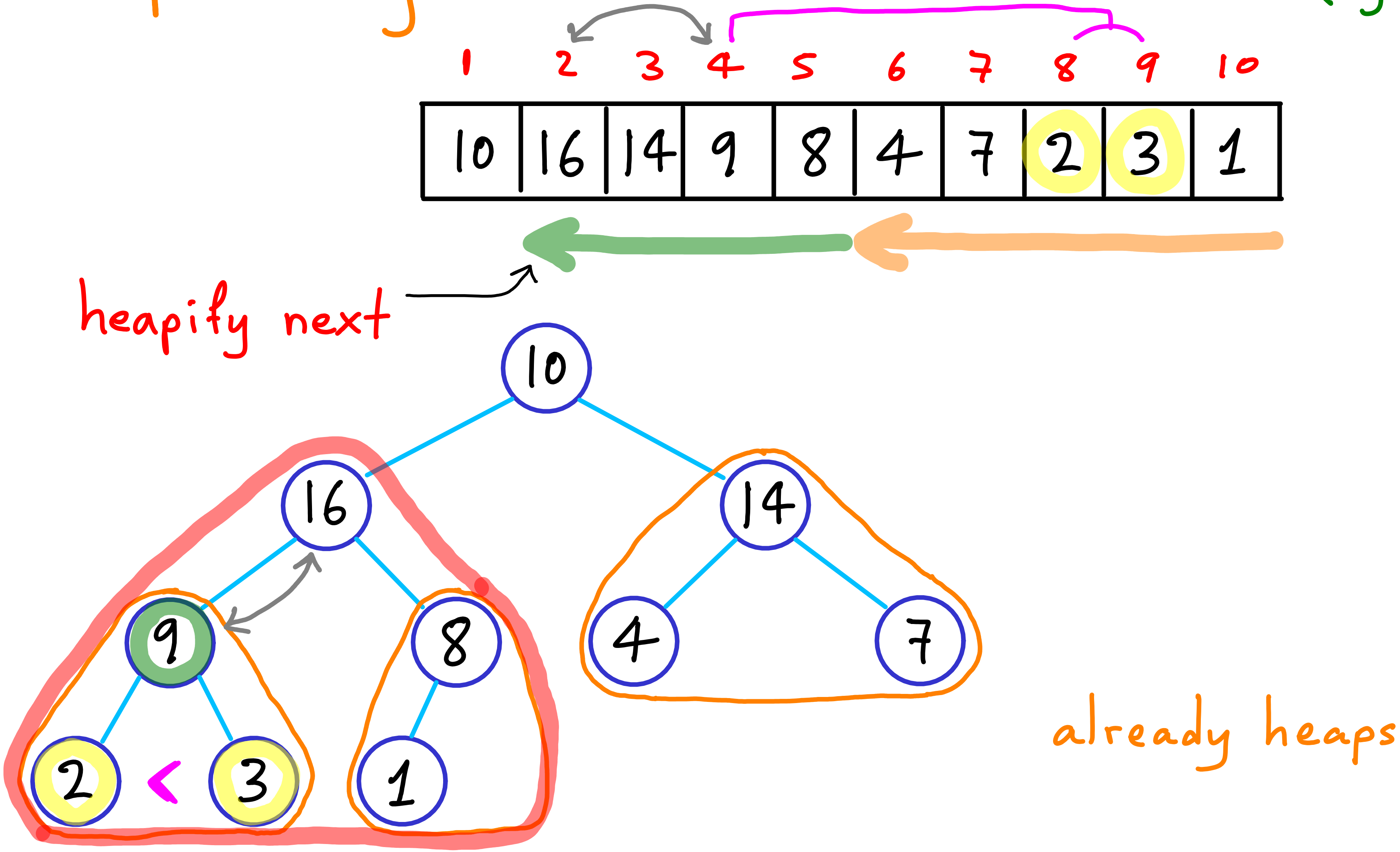


Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1

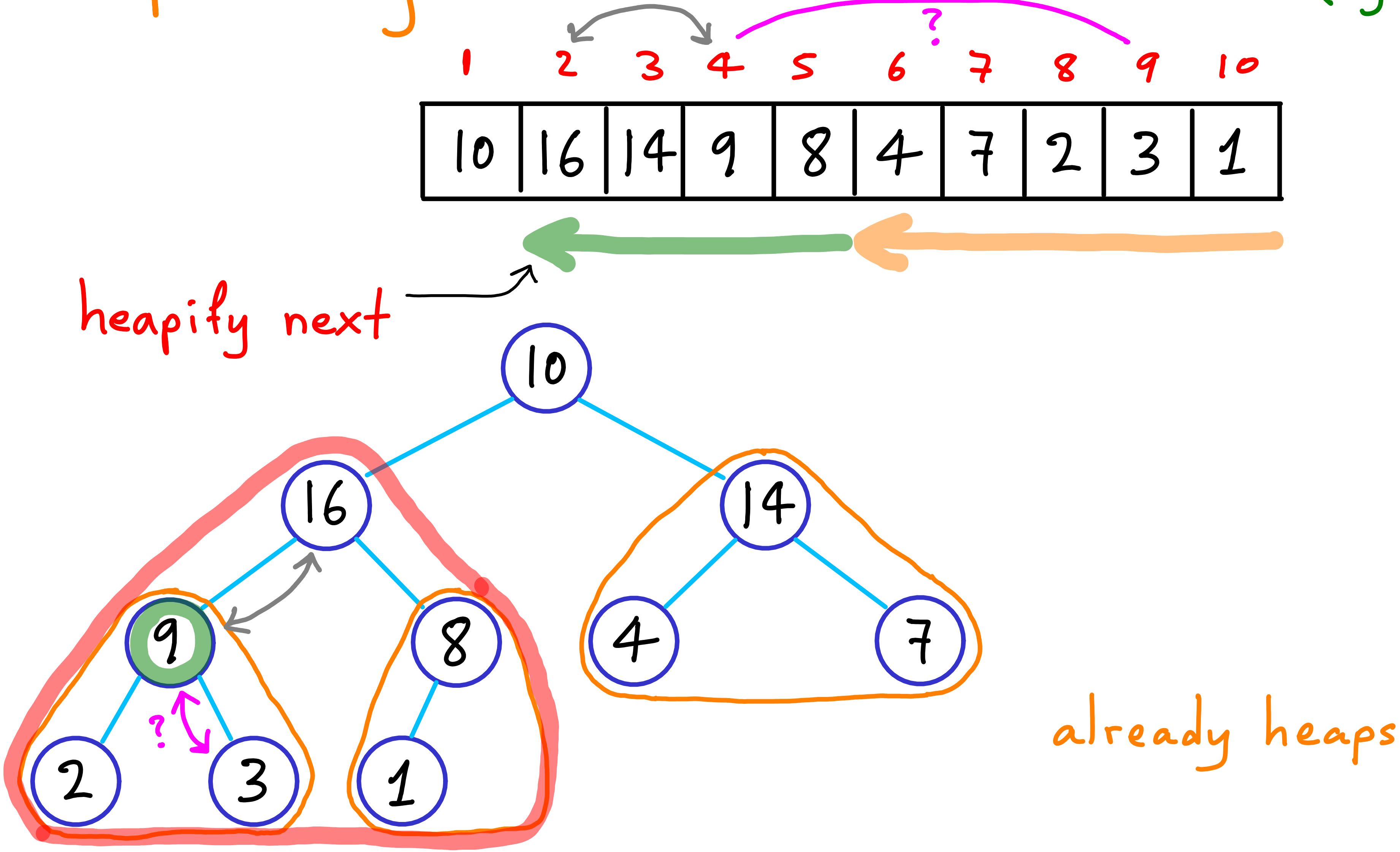


Heap building: the REVERSE METHOD (right to left)



1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1

Heap building: the REVERSE METHOD (right to left)



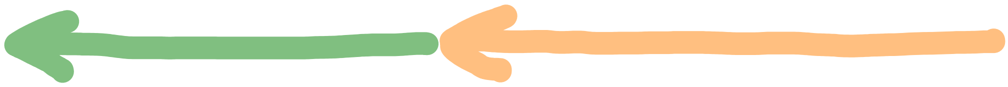
1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1

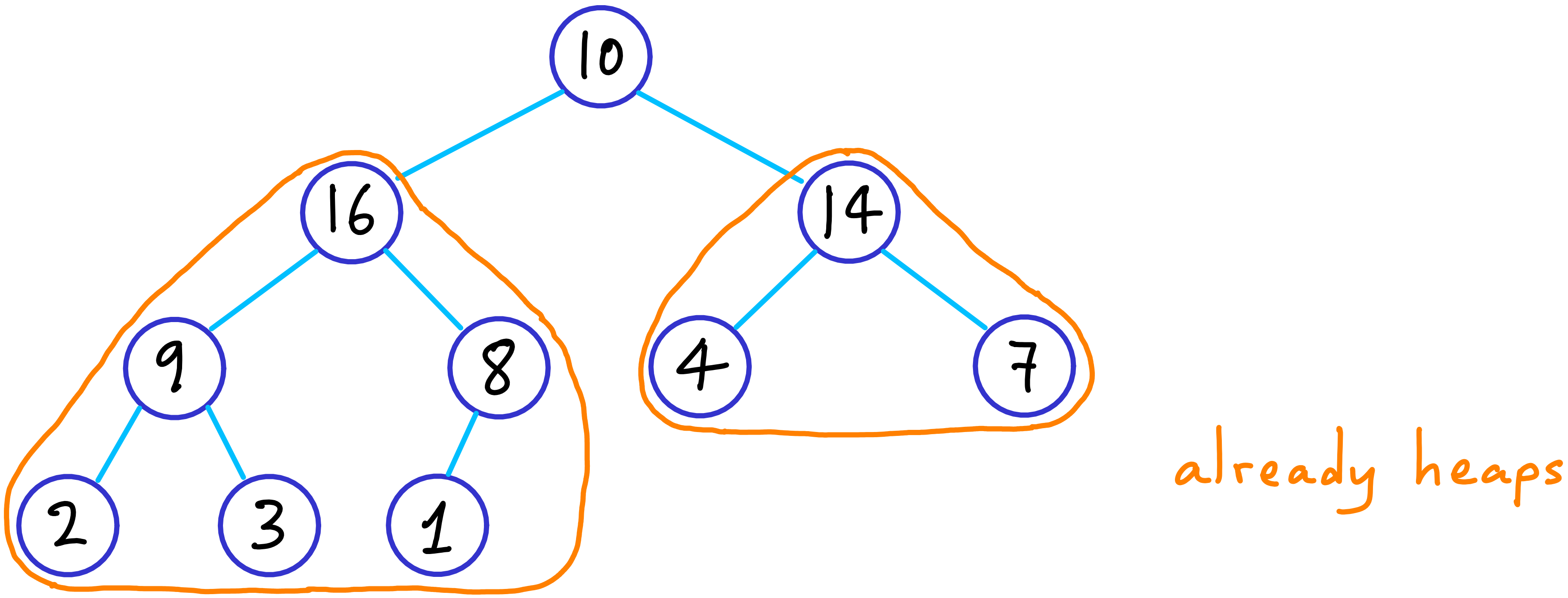
heapify next

already heaps

Heap building: the REVERSE METHOD (right to left)

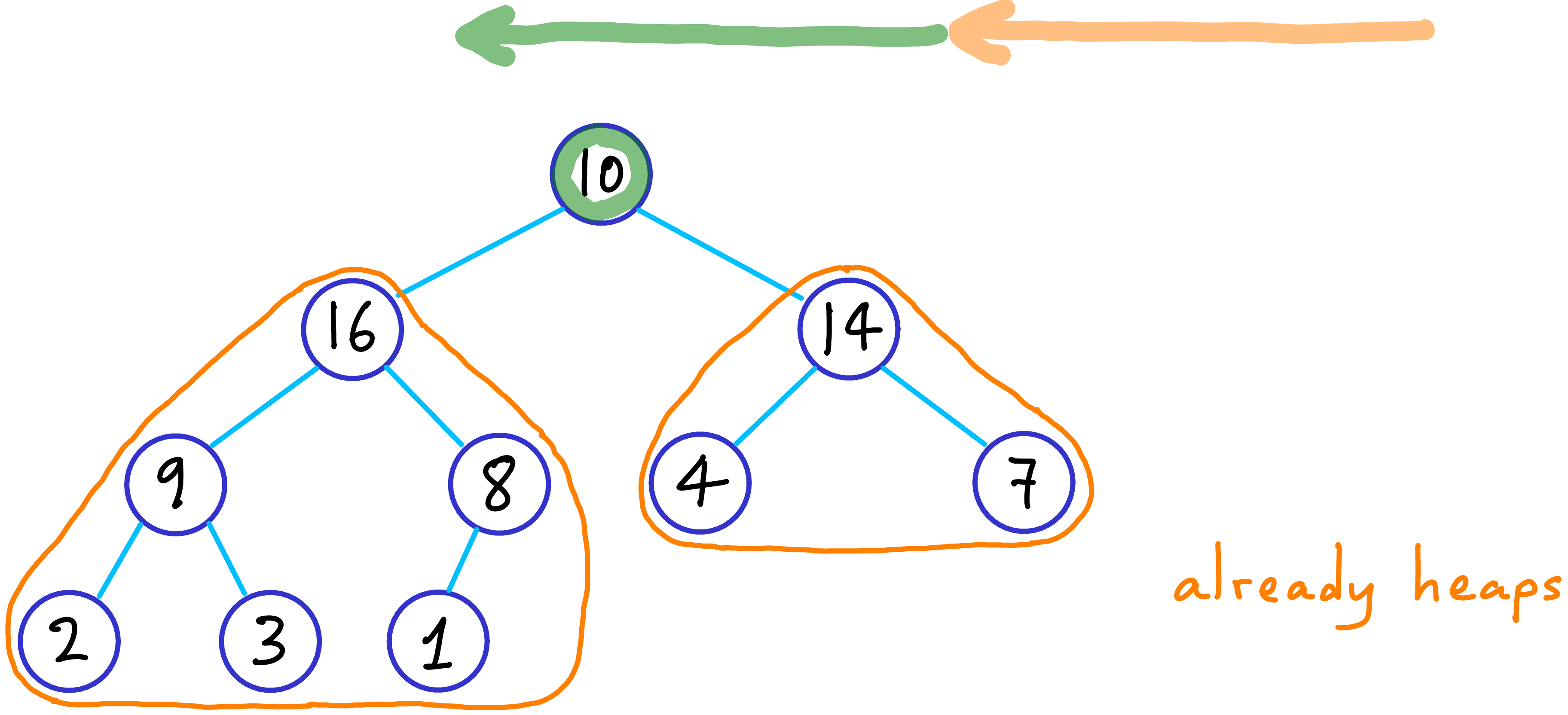
1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1





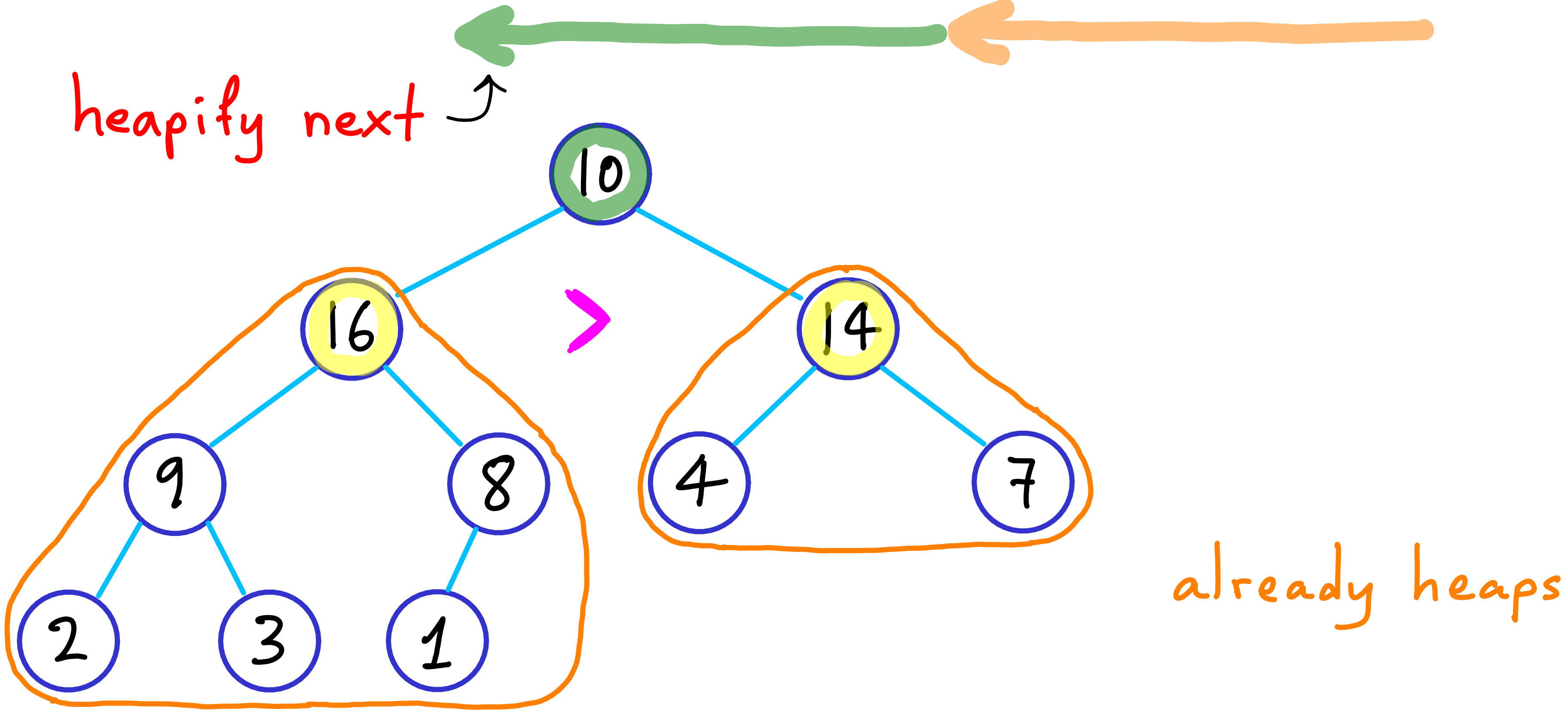
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1



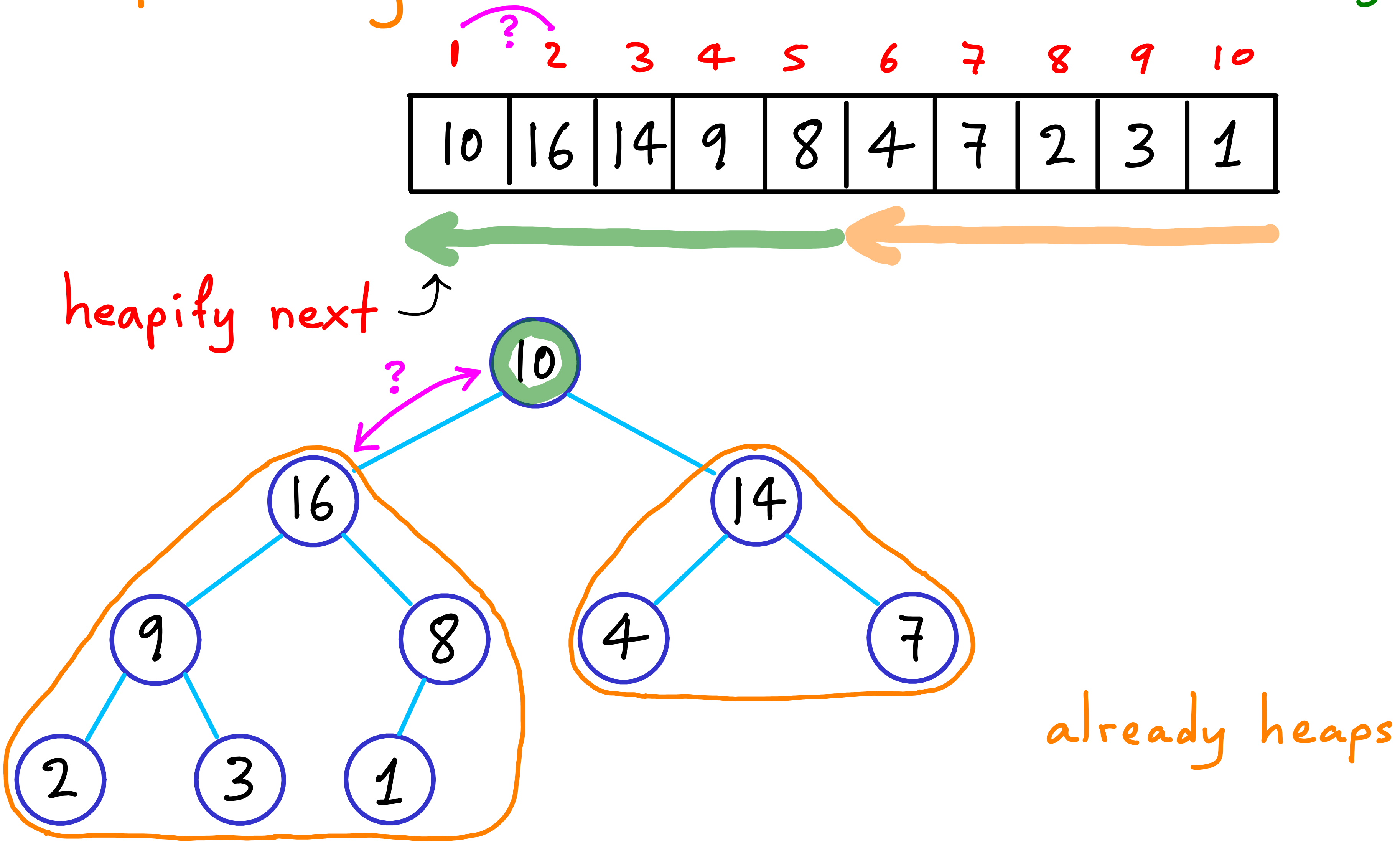
Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1

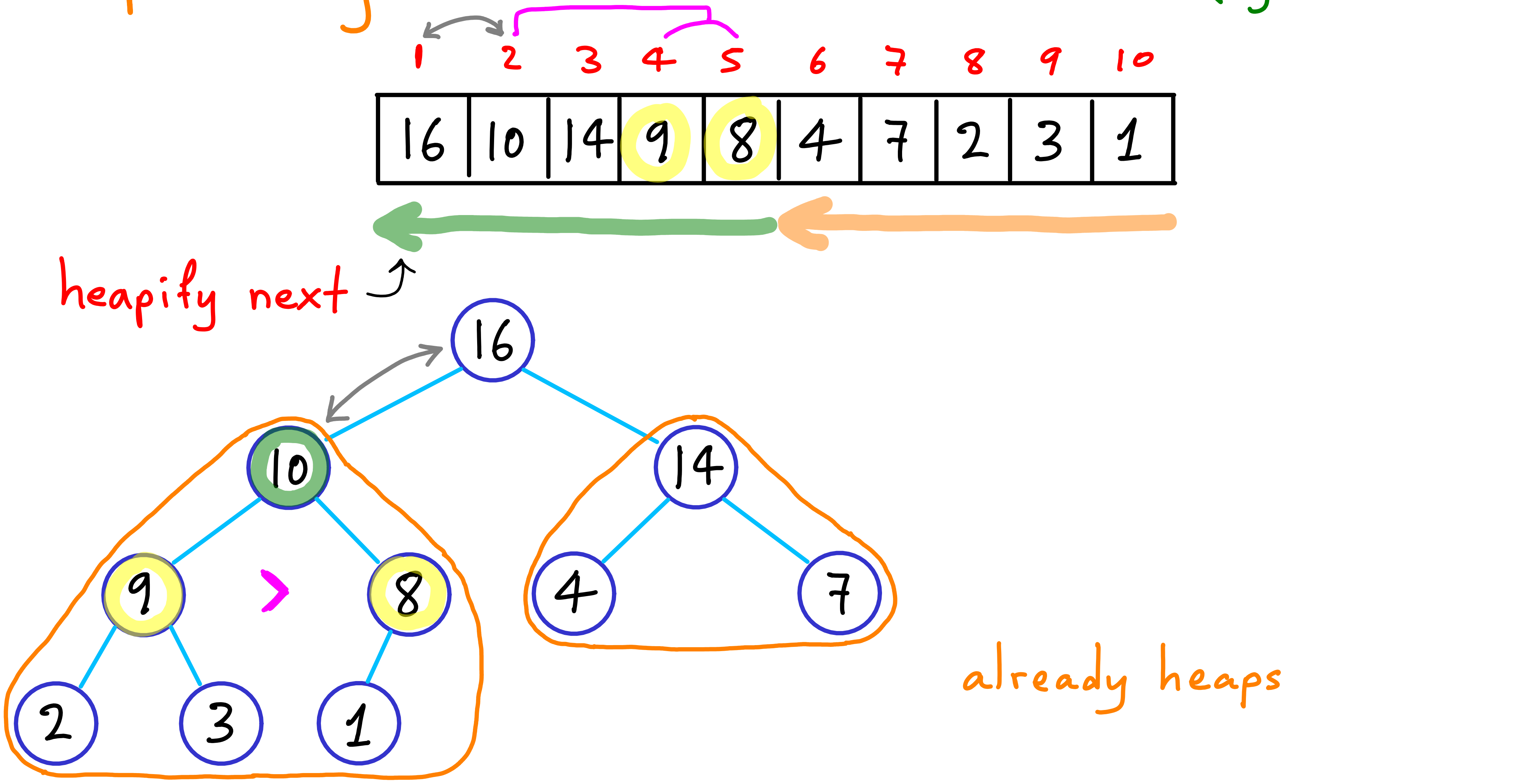


Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
10	16	14	9	8	4	7	2	3	1

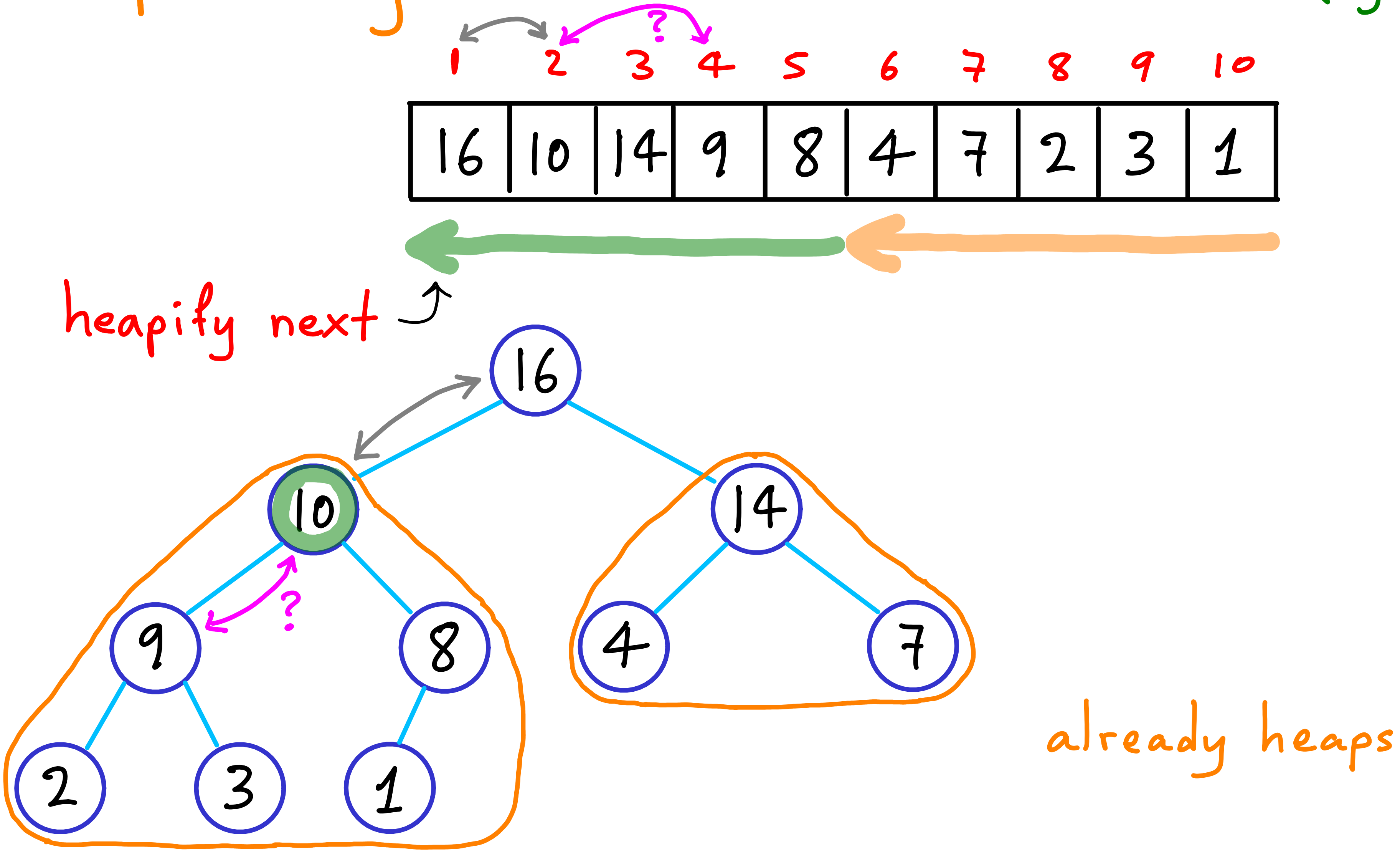


Heap building: the REVERSE METHOD (right to left)



1	2	3	4	5	6	7	8	9	10
16	10	14	9	8	4	7	2	3	1

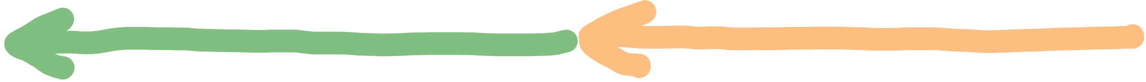
Heap building: the REVERSE METHOD (right to left)

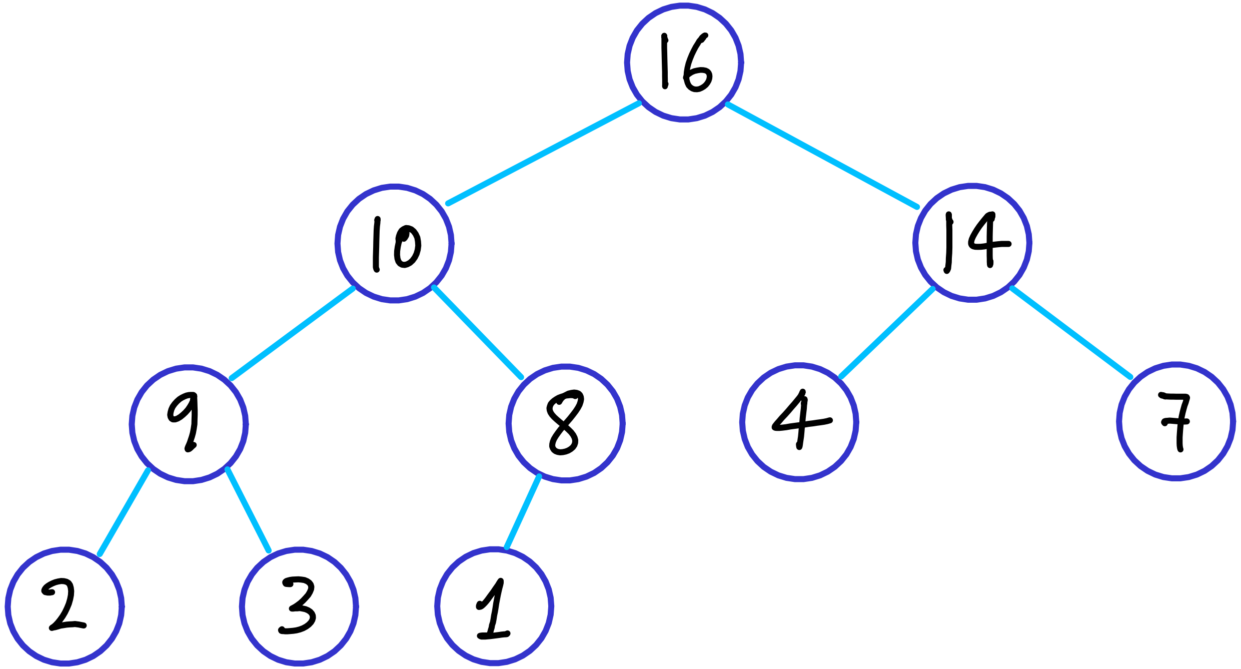


1	2	3	4	5	6	7	8	9	10
16	10	14	9	8	4	7	2	3	1

Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
16	10	14	9	8	4	7	2	3	1

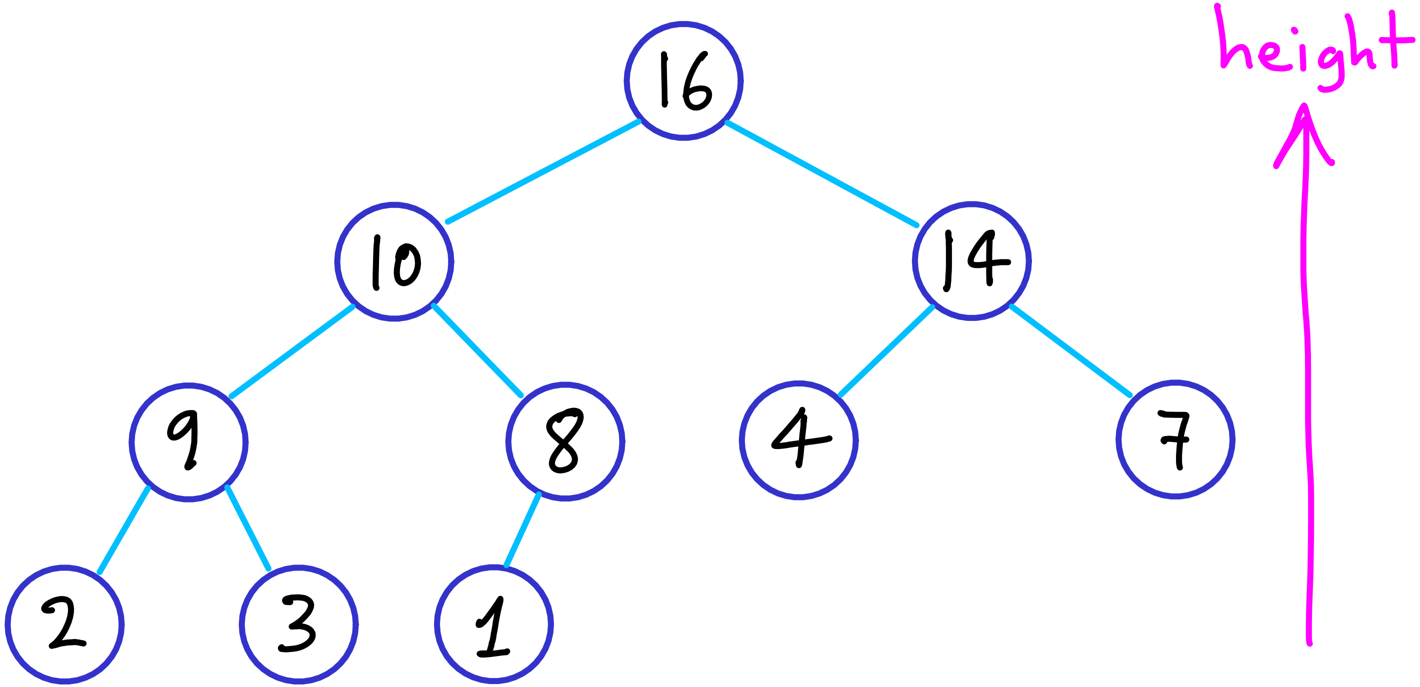




Time ?

Heap building: the REVERSE METHOD (right to left)

1	2	3	4	5	6	7	8	9	10
16	10	14	9	8	4	7	2	3	1

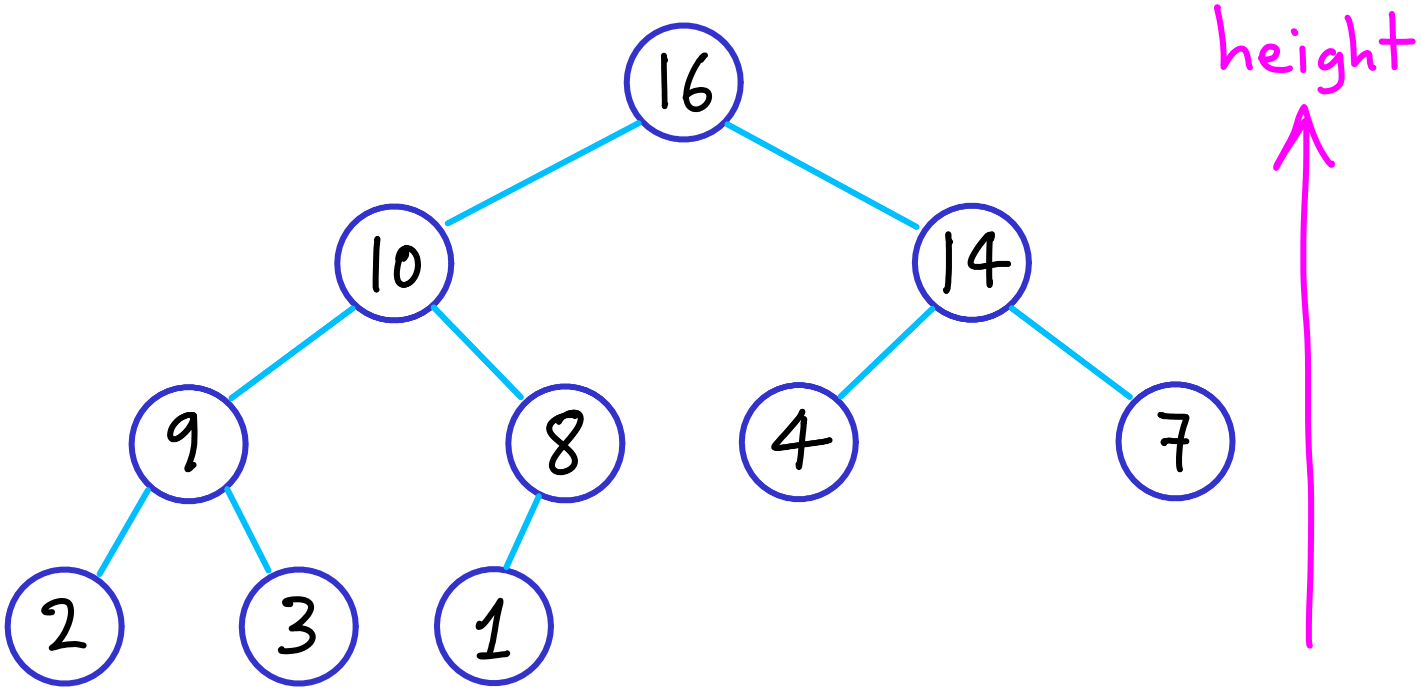


Time ?

$\text{heapify}(x) = O(\text{height}(x))$

Heap building: the REVERSE METHOD (right to left)

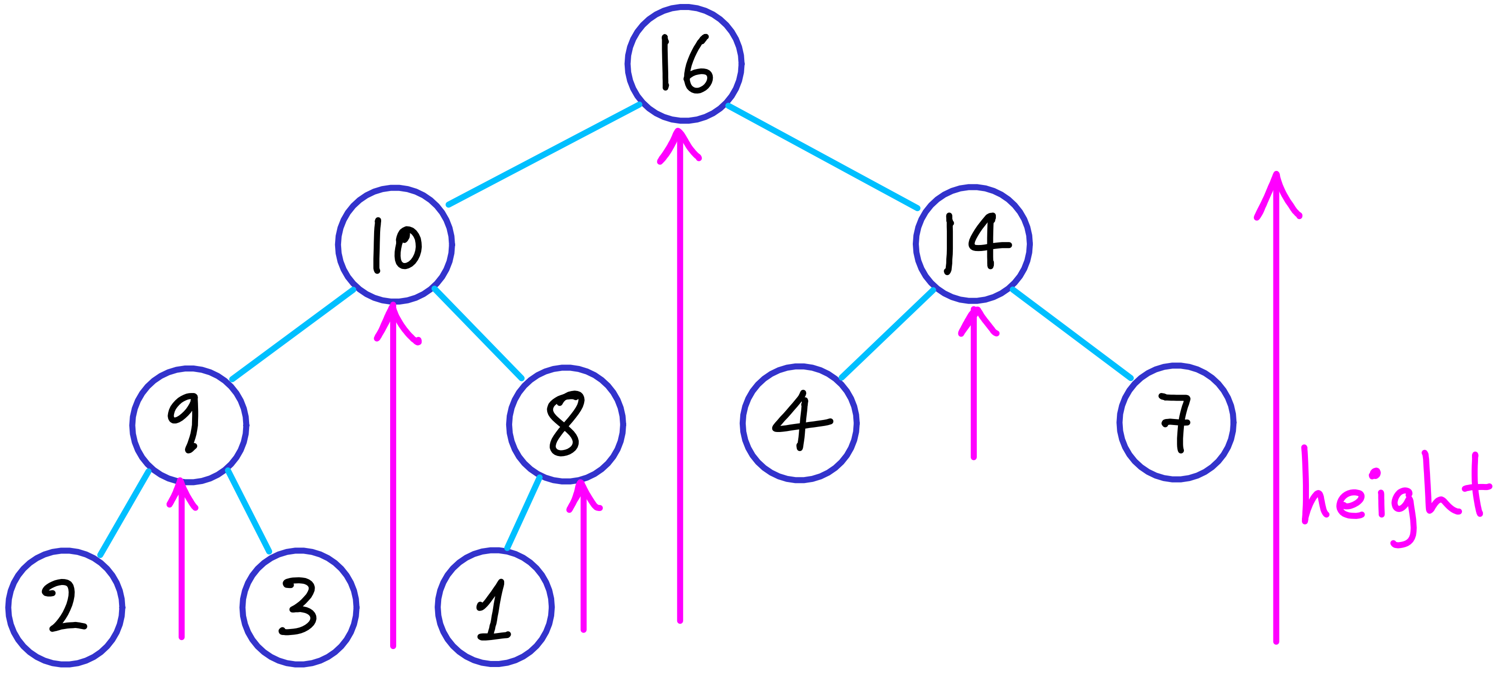
1	2	3	4	5	6	7	8	9	10
16	10	14	9	8	4	7	2	3	1



Time ?

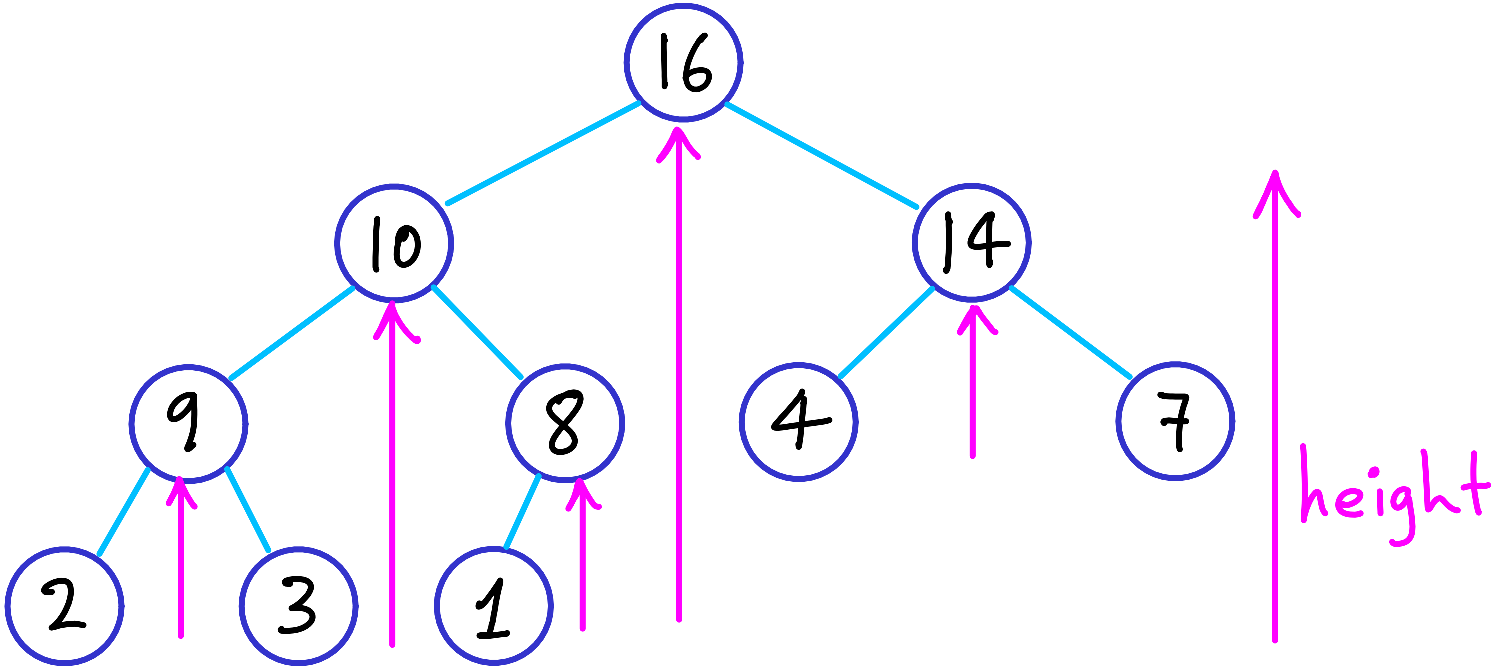
$\text{heapify}(x) = O(\text{height}(x))$

$\sum_{\text{all } x} \text{height}(x) = O(n\log n)$



better calculation

$$\sum_{\text{all } x} \text{height}(x)$$



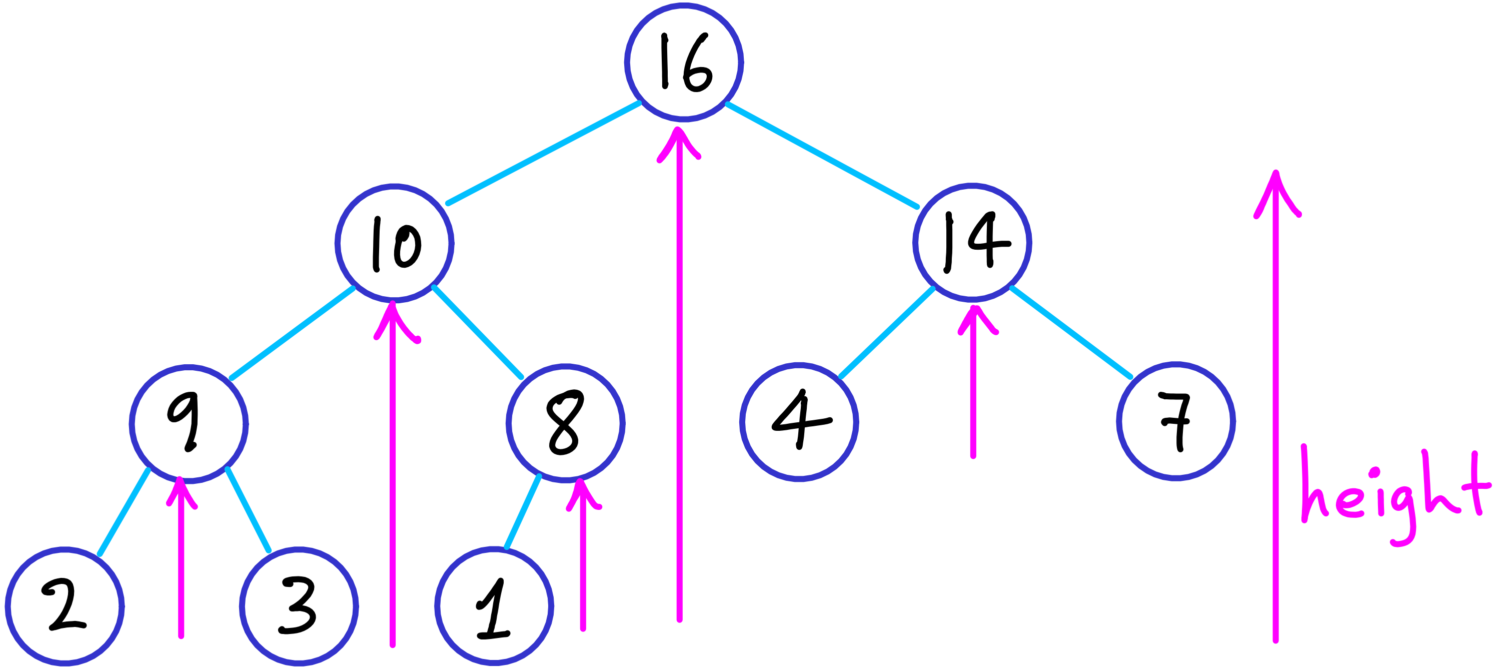
better calculation

$$\sum_{\text{all } x} \text{height}(x)$$

$$\sum \le \underbrace{\frac{n}{2}}_{\text{\#nodes}}\cdot\underbrace{1}_{\text{height}} + \underbrace{\frac{n}{4}}\cdot\underbrace{2} + \underbrace{\frac{n}{8}}\cdot\underbrace{3} + \cdots + \underbrace{2}\cdot\underbrace{((\log n)-1)} + \underbrace{1}_{\text{\#nodes}}\cdot\underbrace{\log n}_{\text{height}}$$

lowest level

root level

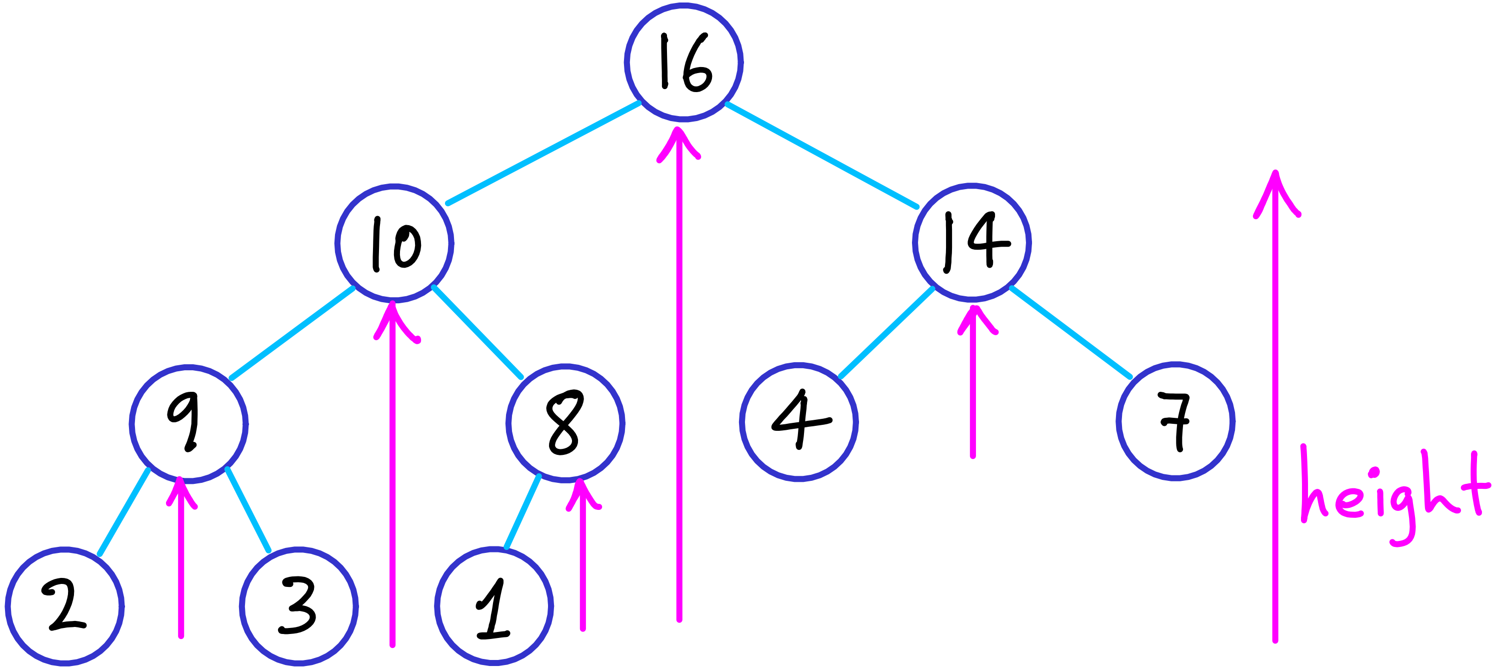


better calculation

$$\sum_{\text{all } x} \text{height}(x)$$

$$\sum \le \frac{n}{\underbrace{2}}\cdot\underset{\smile}{1} + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \cdots + 2\cdot((\log n)-1) + 1\cdot\log n$$

$$= \sum_{h=1}^{\log n} \frac{n}{2^h}\cdot h$$

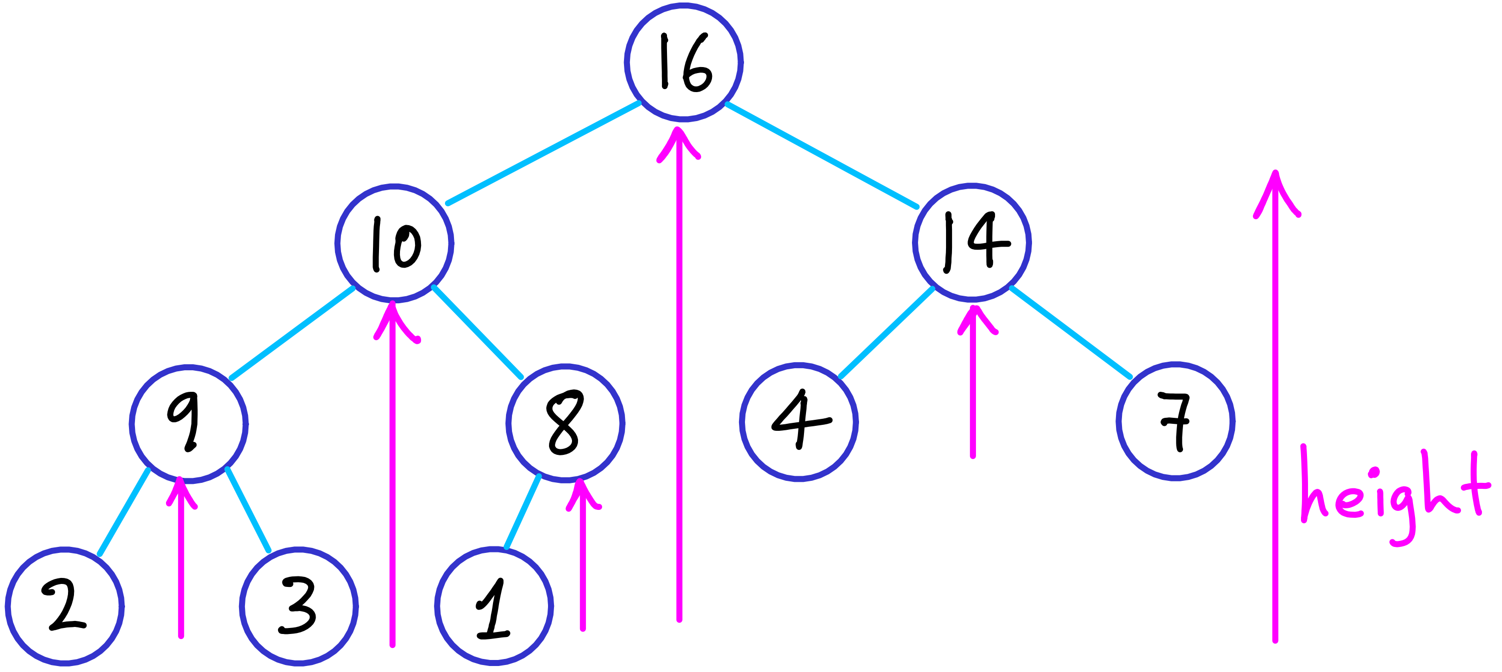


better calculation

$$\sum_{\text{all } x} \text{height}(x)$$

$$\sum \le \frac{n}{2}\cdot 1 + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \cdots + 2\cdot((\log n)-1) + 1\cdot \log n$$

$$= \sum_{h=1}^{\log n} \frac{n}{2^h}\cdot h = n\cdot \sum \frac{h}{2^h}$$



better calculation

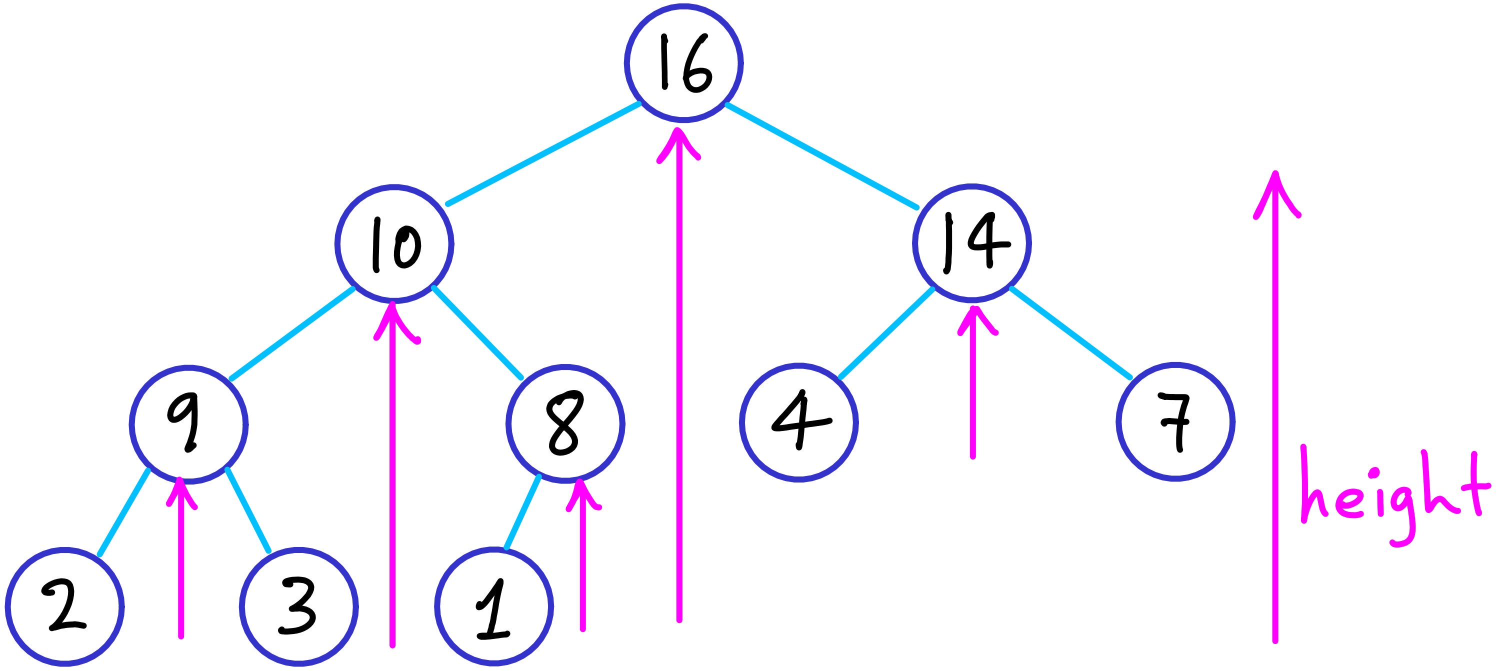
$$\sum_{\text{all } x} \text{height}(x)$$

$$\sum \leq \frac{n}{2}\cdot 1 + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \cdots + 2\cdot((\log n)-1) + 1\cdot \log n$$

$$= \sum_{h=1}^{\log n} \frac{n}{2^h}\cdot h = n \cdot \sum \frac{h}{2^h} \leq n \frac{1/2}{(1-1/2)^2}$$

CLRS 1148

$$\left[\text{use } \sum_0^{\infty} k x^k \right]$$



better calculation

$$\sum_{\text{all } x} \text{height}(x)$$

$$\sum \leq \frac{n}{2}\cdot 1 + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \cdots + 2\cdot((\log n)-1) + 1\cdot \log n$$

$$= \sum_{h=1}^{\log n} \frac{n}{2^h}\cdot h = n\cdot \sum \frac{h}{2^h} \leq n\,\frac{1/2}{(1-1/2)^2} = O(n)$$

CLRS 1148

$$\left[\text{use } \sum_0^\infty k x^k\right]$$