AUGMENTING DATA STRUCTURES (BSTs)

FINDING THE RANK OF AN ELEMENT IN A SET

Use array:

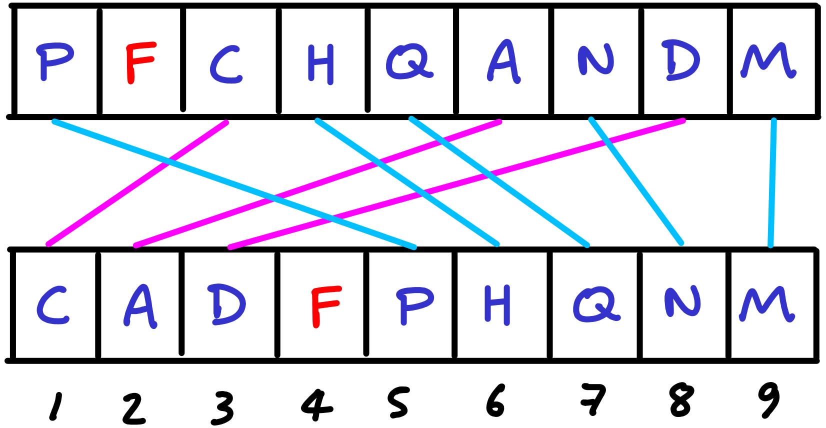
P	F	C	H	Q	A	N	D	M
1	2	3	4	5	6	7	8	9

rank(F) = ?

FINDING THE RANK OF AN ELEMENT IN A SET

Use array:

partition



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$\Theta(?)$

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1	2	3	4	5	6	7	8	9

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partition

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1	2	3	4	5	6	7	8	9

$\Theta(n)$

OK if done once.

Not for multiple queries

suggestions?

FINDING THE RANK OF AN ELEMENT IN A SET

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$\Theta(n)$

ok if done once.

Not for multiple queries

Preprocess (sort)

A	C	D	F	H	M	N	P	Q
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$O(n \log n)$

Now all queries: $O(1)$

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Now all queries: $O(1)$

What if we want to insert/delete?

FINDING THE RANK OF AN ELEMENT IN A SET

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→ partition

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$\Theta(n)$ {
OK if done once.
Not for multiple queries

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A	C	D	F	H	M	N	P	Q
1	2	3	4	5	6	7	8	9

$O(n \log n)$

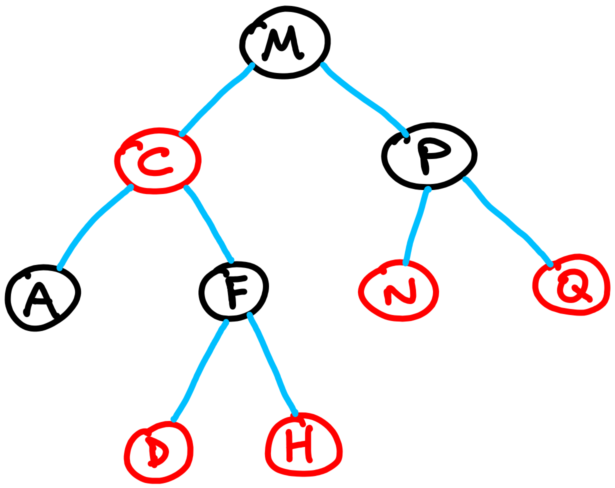
Now all queries: $O(1)$

What if we want to insert/delete? → bad $O(n)$

FINDING THE RANK OF AN ELEMENT in a DYNAMIC SET with PREPROCESSING

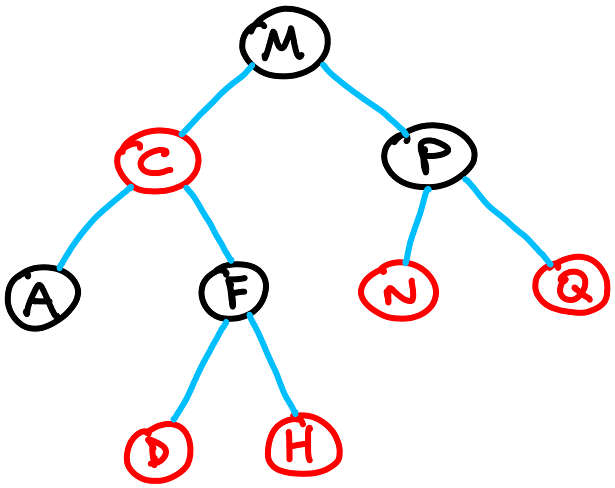
Allow insertions & deletions "quickly"

FINDING THE RANK OF AN ELEMENT in a DYNAMIC SET with PREPROCESSING



RB-tree contains sorted letters

FINDING THE RANK OF AN ELEMENT in a DYNAMIC SET with PREPROCESSING

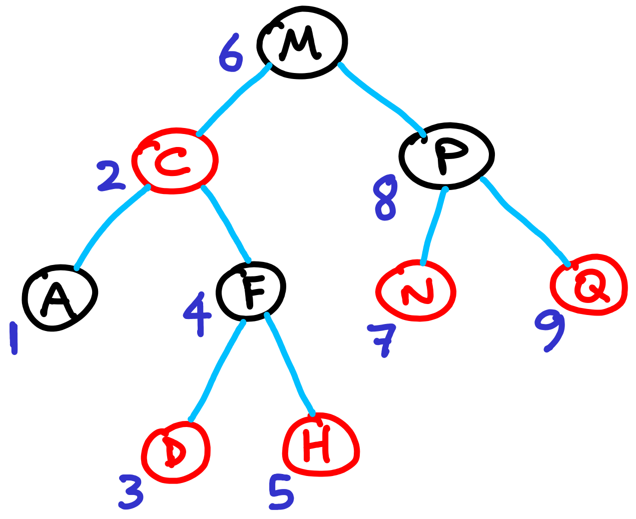


RB-tree contains sorted letters

Now we can quickly restore sorted order

(still need to get ranks)

FINDING THE RANK OF AN ELEMENT in a DYNAMIC SET with PREPROCESSING

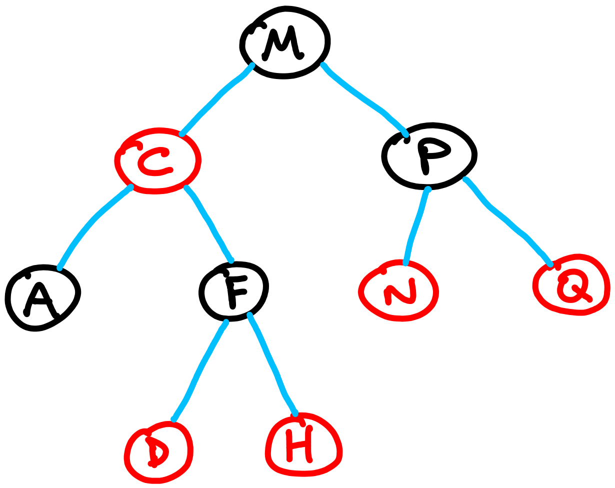


RB-tree contains sorted letters

Now we can quickly restore sorted order

Store ranks

FINDING THE RANK OF AN ELEMENT in a DYNAMIC SET with PREPROCESSING



RB-tree contains sorted letters

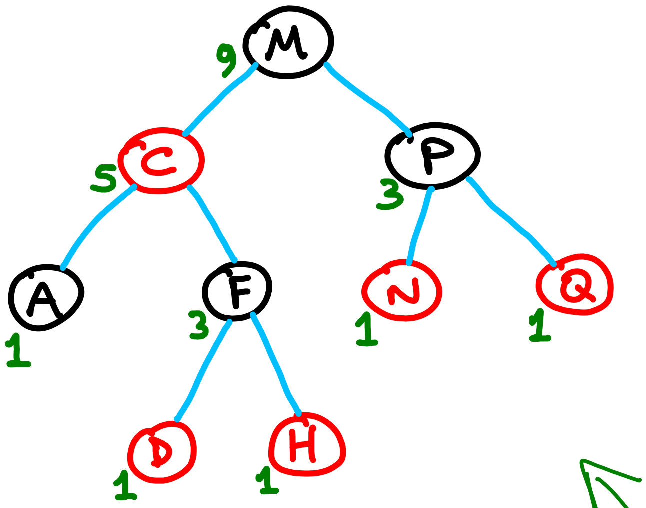
Now we can quickly restore sorted order

Store ranks... → bad

(too many ranks change w/ insert)

Dynamic X

FINDING THE RANK OF AN ELEMENT in a DYNAMIC SET with PREPROCESSING



RB-tree contains sorted letters

Now we can quickly restore sorted order

Store ranks... → bad

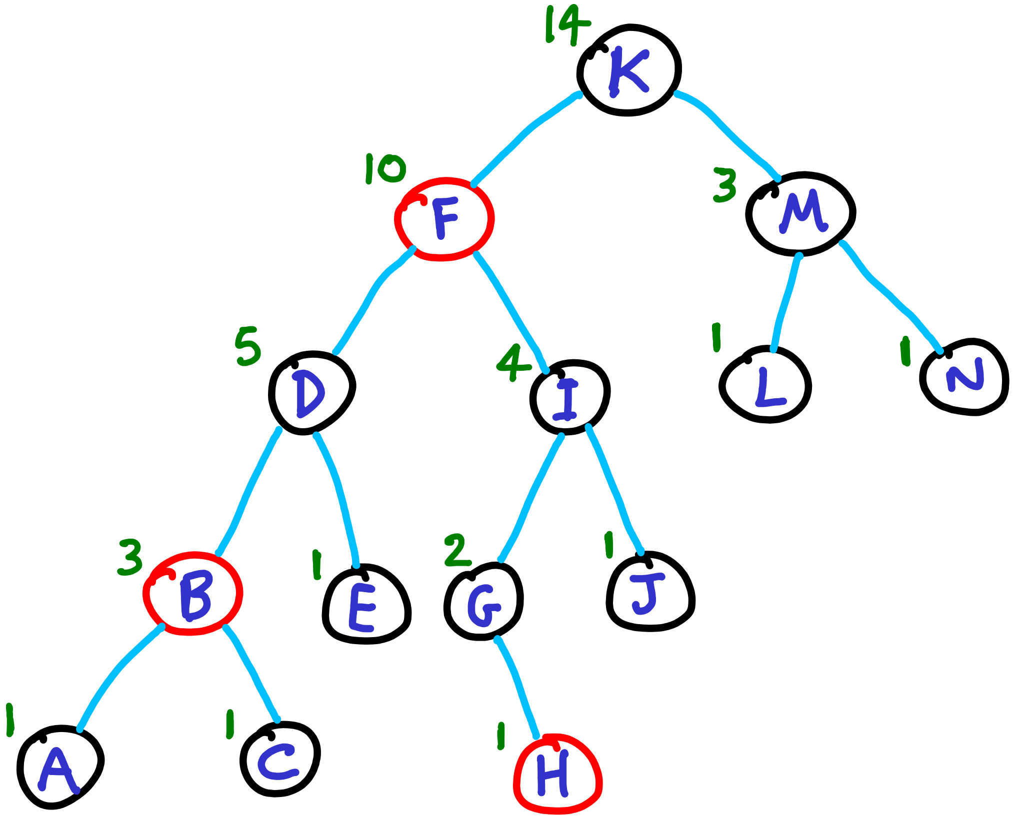
(too many ranks change w/ insert)

Dynamic ?

Store subtree sizes

USING AN AUGMENTED R-B TREE TO FIND RANKS

(with subtree sizes)



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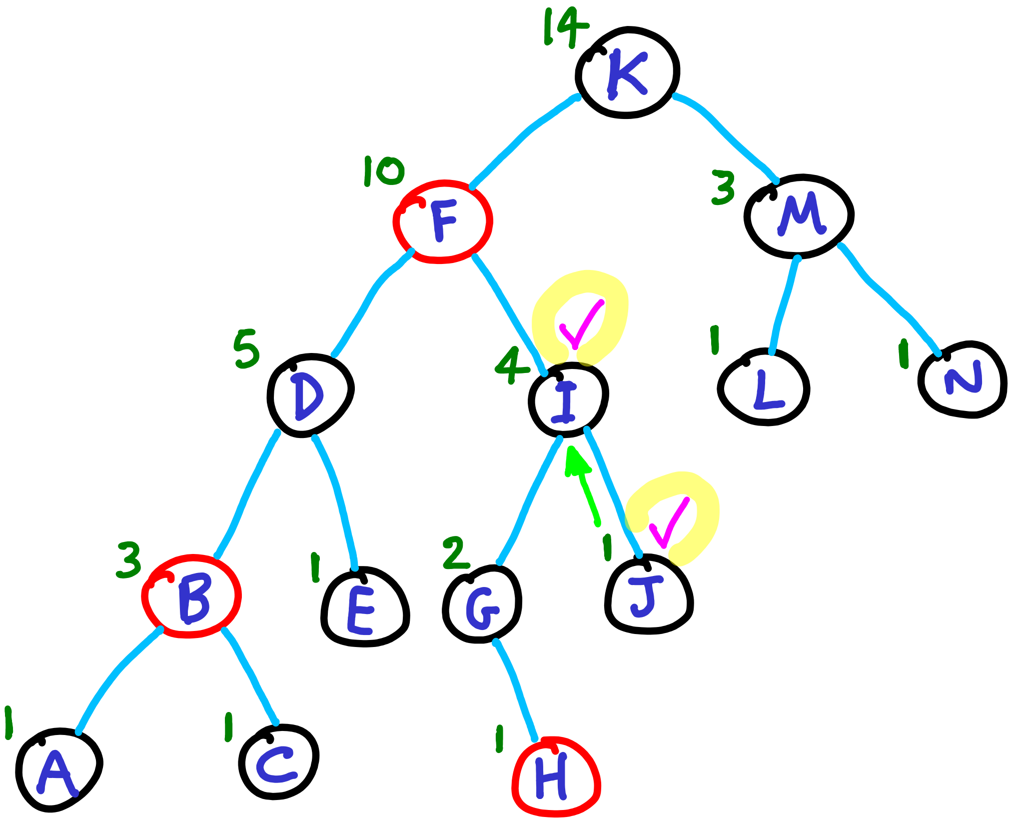


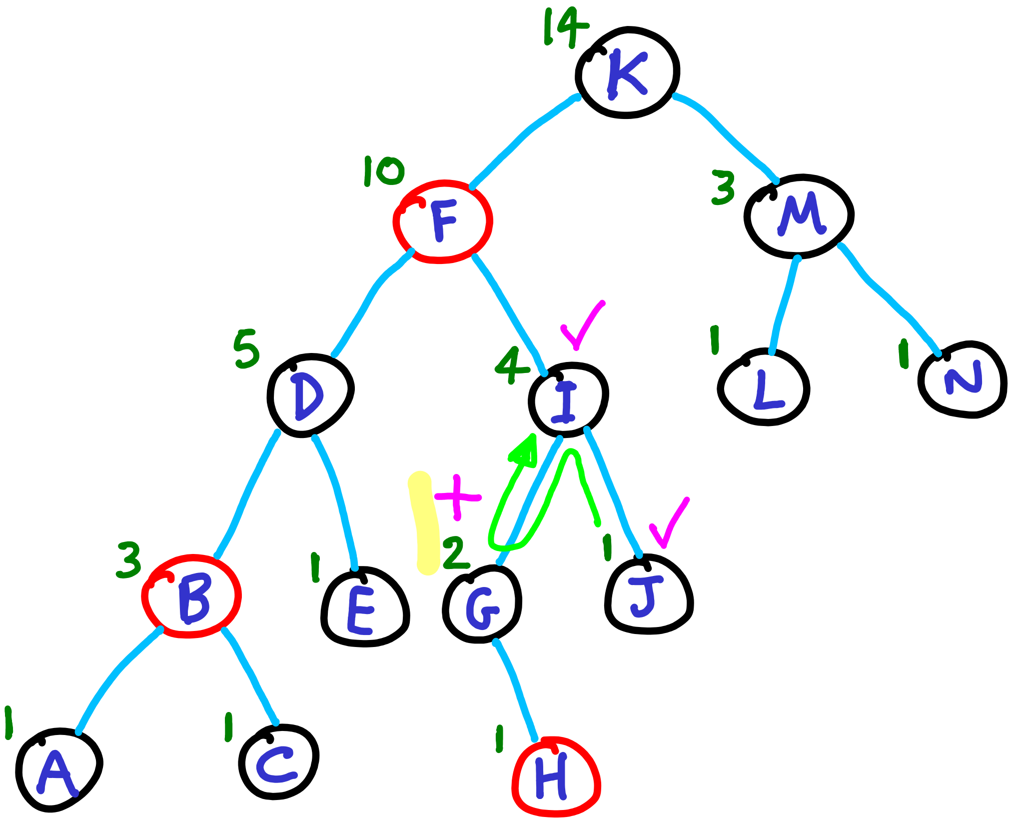
Rank(J) : Walk up,

✓ count smaller ancestors

J I = 2

USING AN AUGMENTED R-B TREE TO FIND RANKS

(with subtree sizes)



Rank(J) : Walk up,

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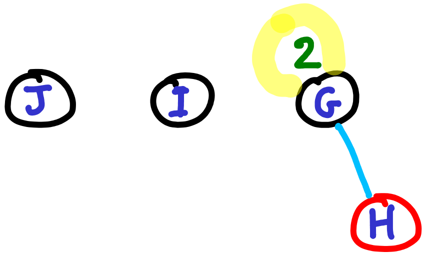
but also

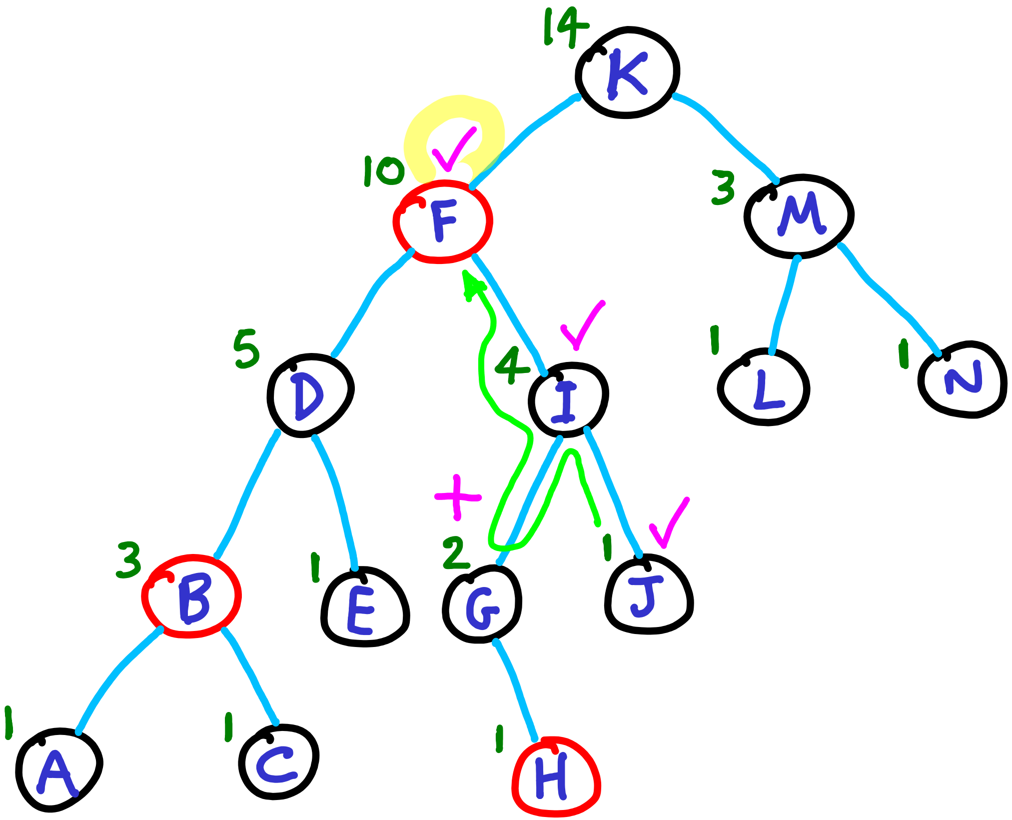
+ add sizes of subtrees containing smaller numbers.



= 4

USING AN AUGMENTED R-B TREE TO FIND RANKS

(with subtree sizes)



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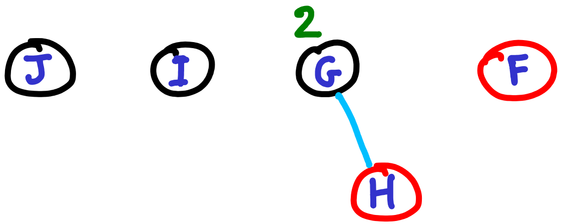
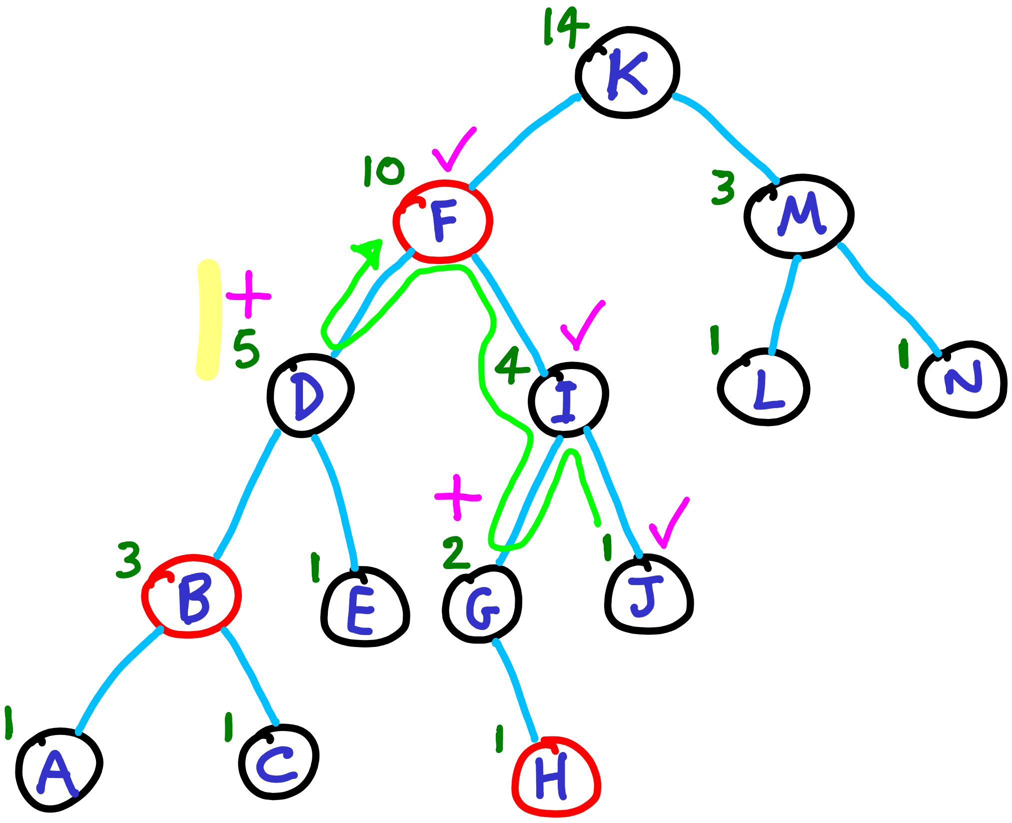


= 5

USING AN AUGMENTED R-B TREE TO FIND RANKS

(with subtree sizes)



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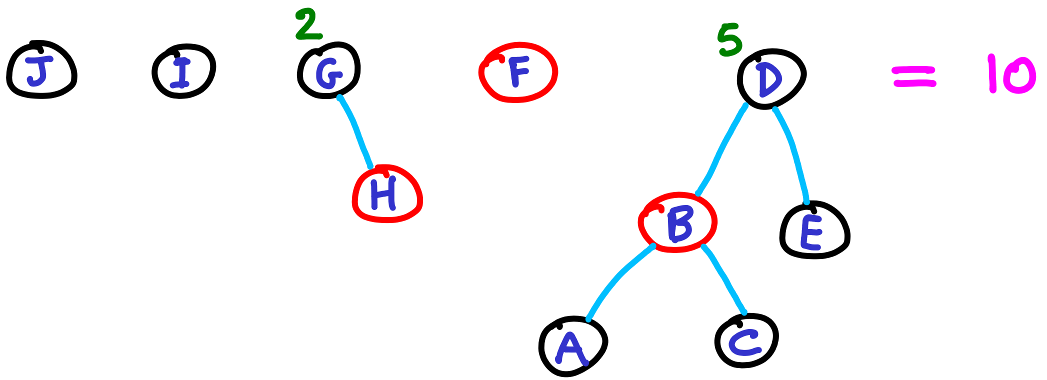
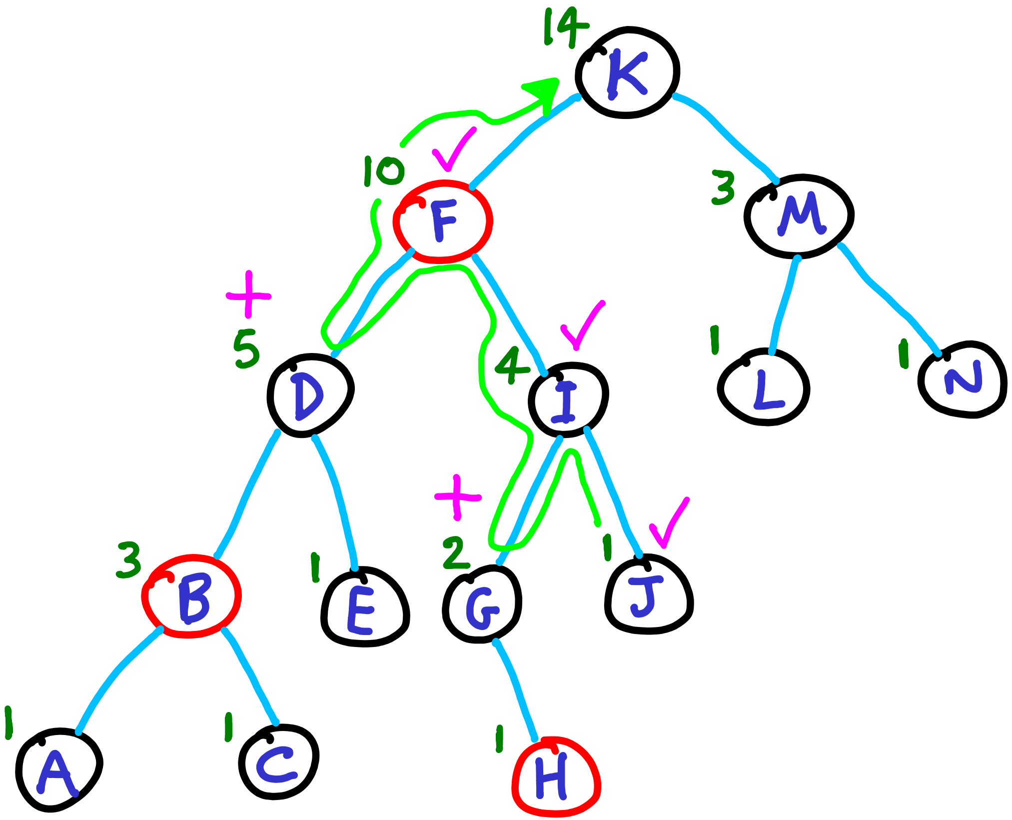


USING AN AUGMENTED R-B TREE TO FIND RANKS
(with subtree sizes)



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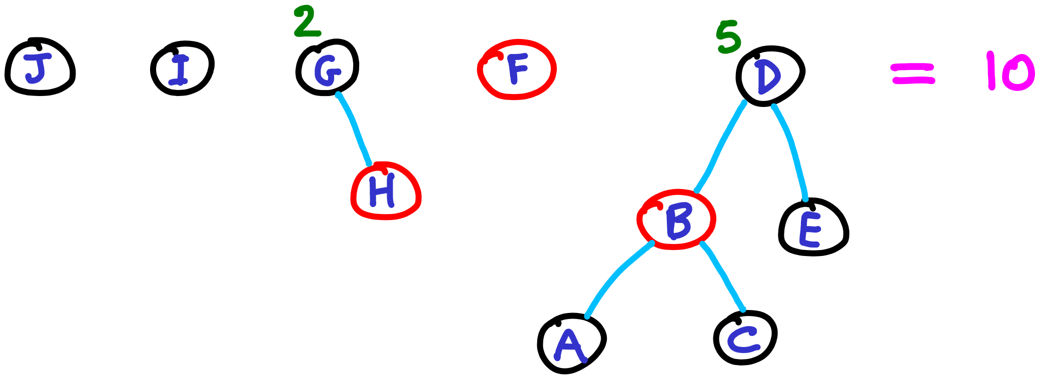
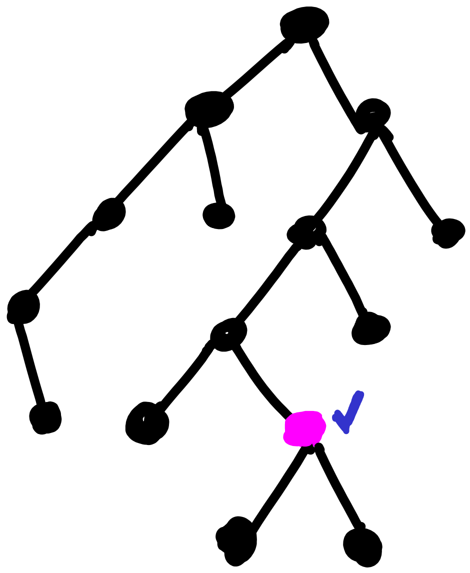
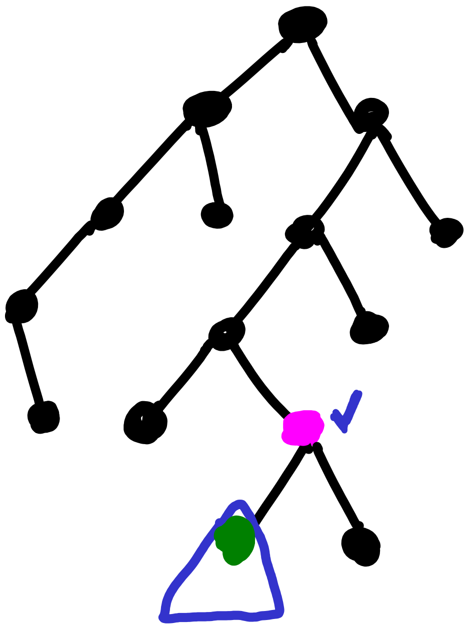
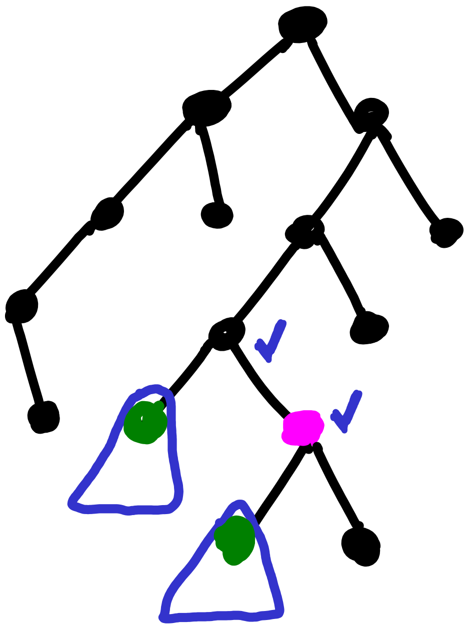
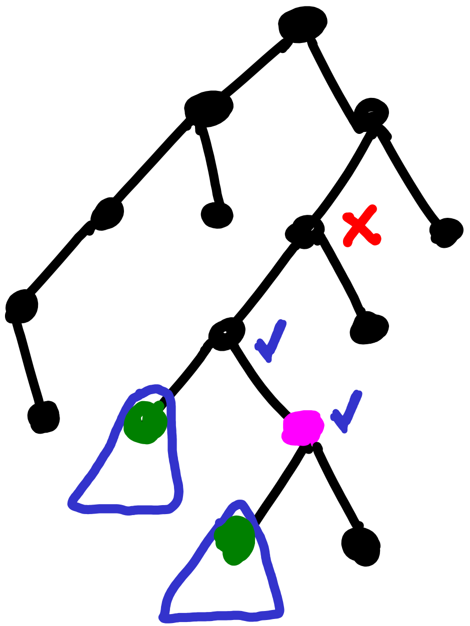
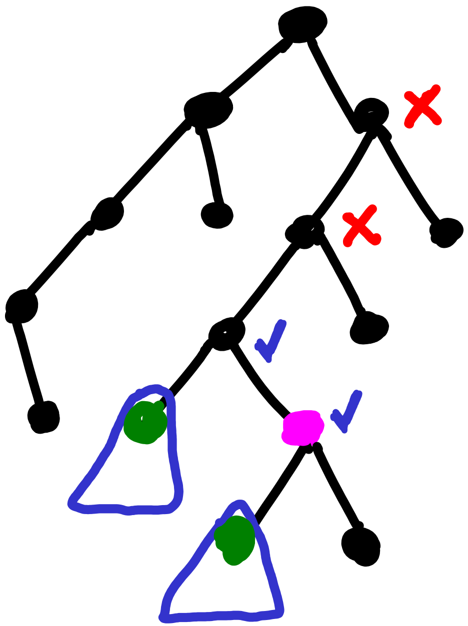




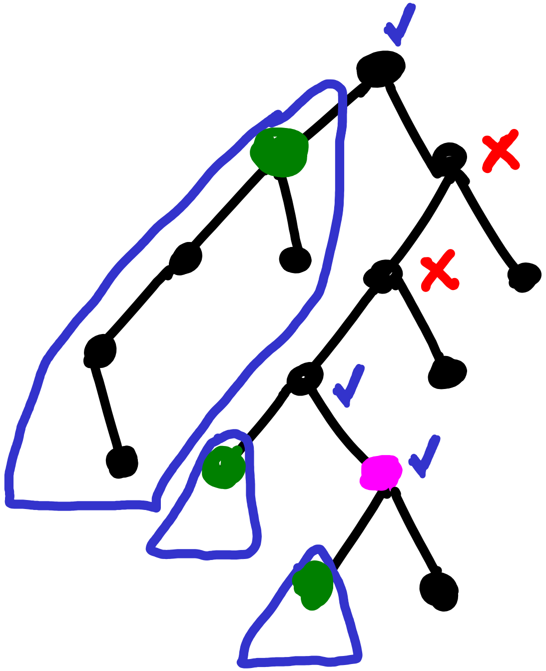


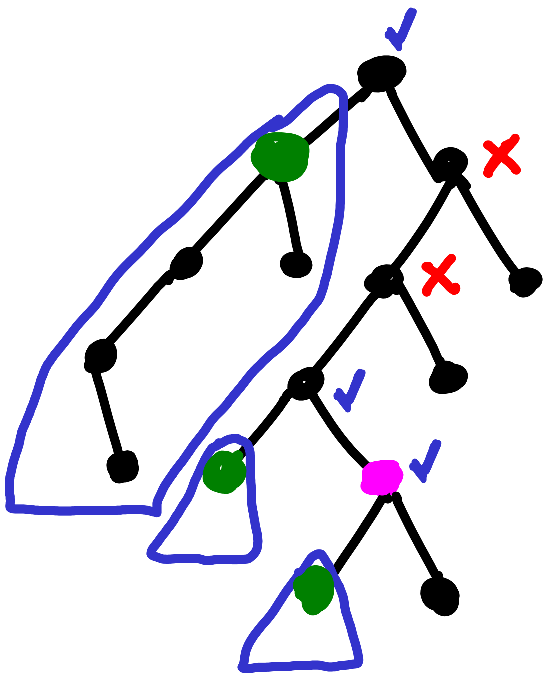


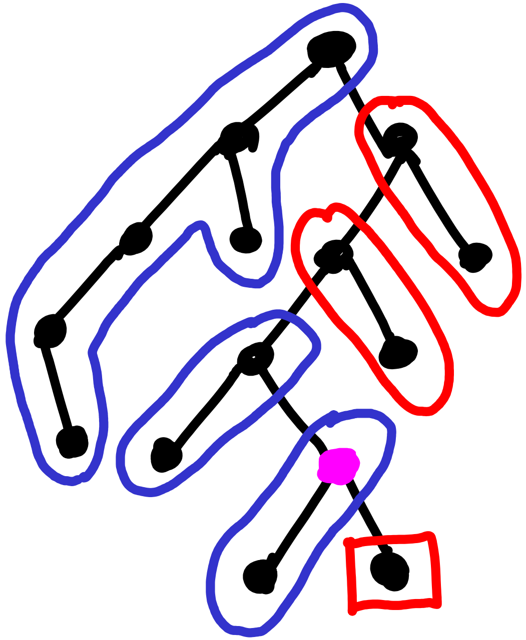




Don't forget to walk all the way up.

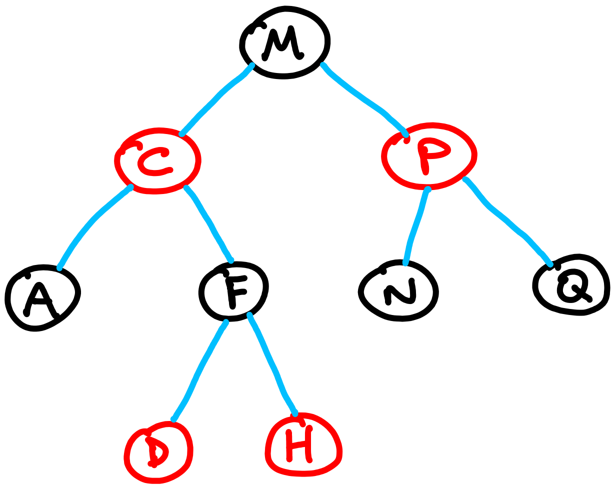






$O(\log n)$ time

The balanced BST can be built in $\Theta(n \log n)$ time



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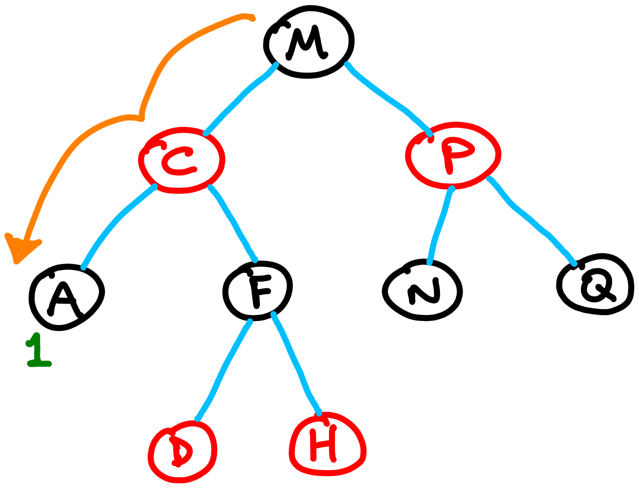
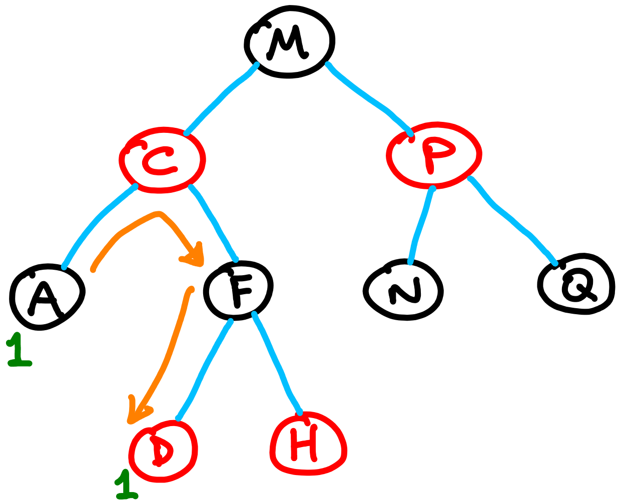


Compute subtree sizes after building by postorder walk

The balanced BST can be built in $\Theta(n \log n)$ time



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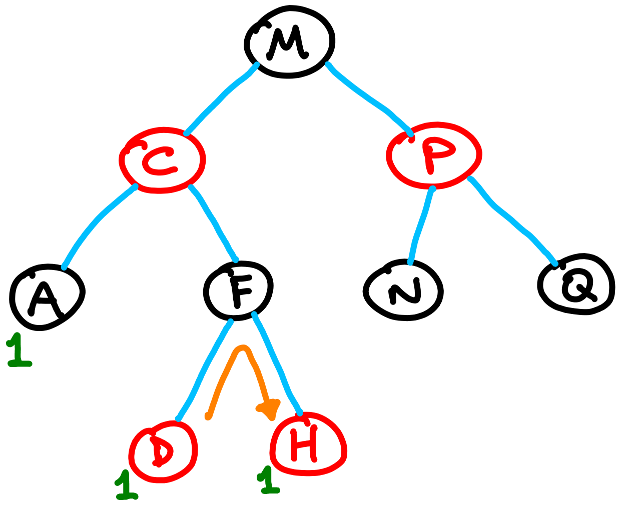

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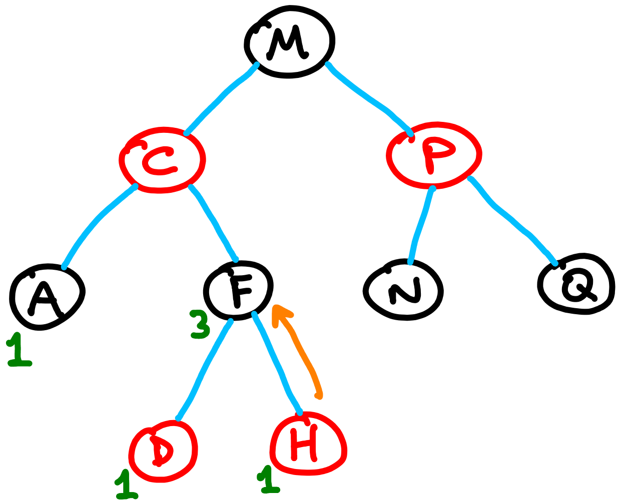

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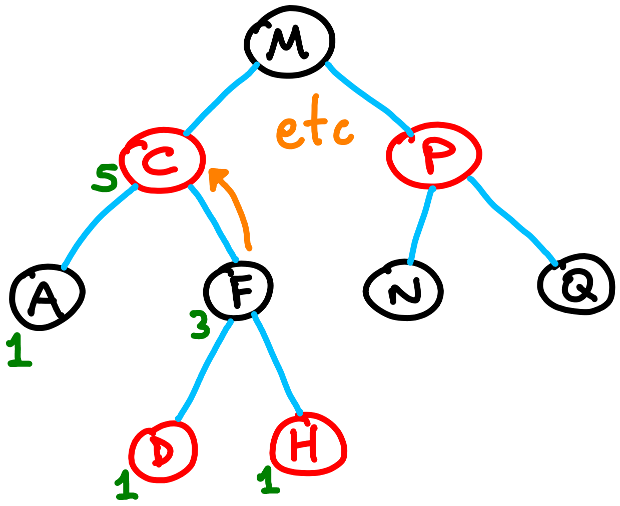

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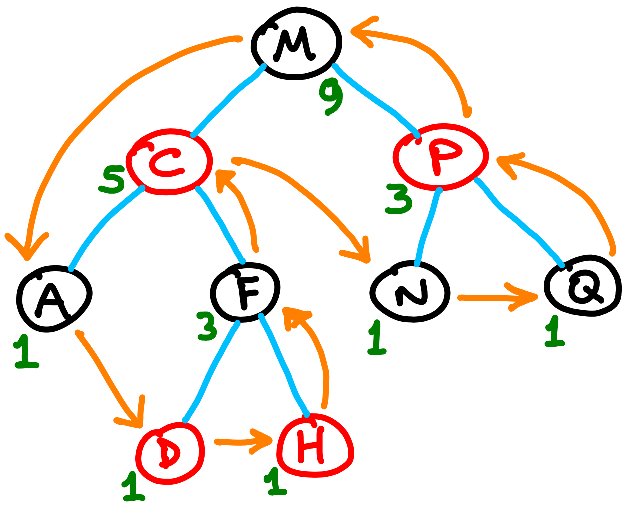
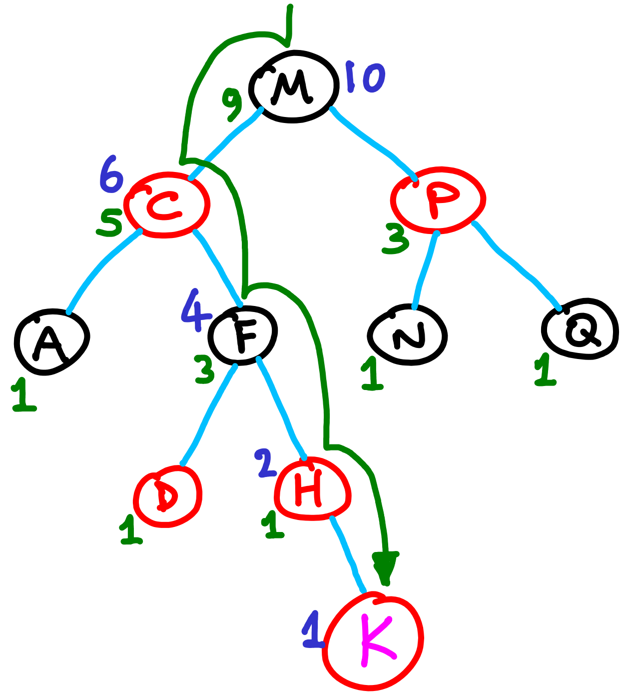


Compute subtree sizes after building by postorder walk...

... or update path when inserting }



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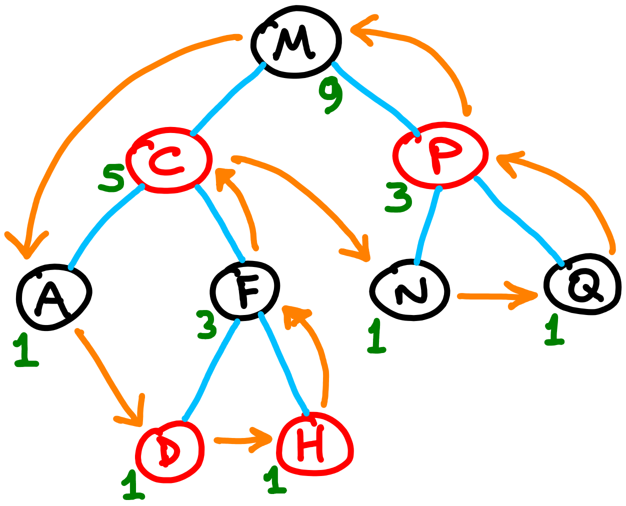
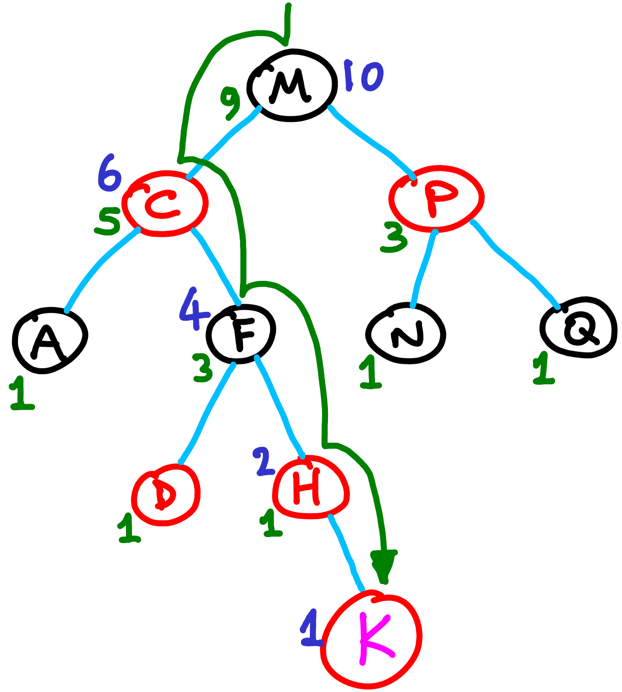


Compute subtree sizes after building

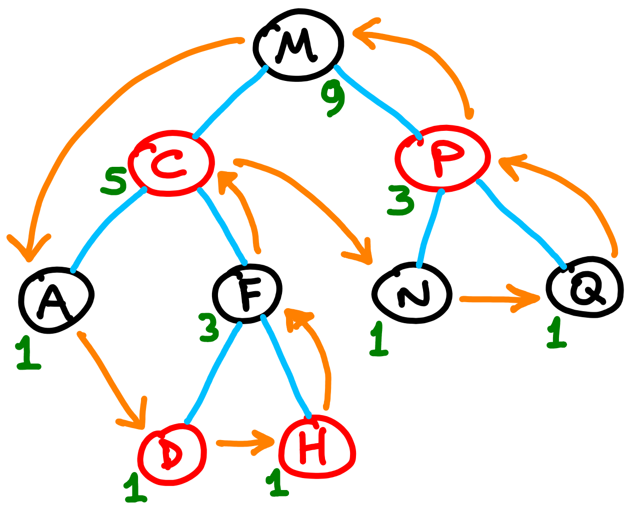
by postorder walk...

... or update path
when inserting

BUT...



The balanced BST can be built in $\Theta(n \log n)$ time

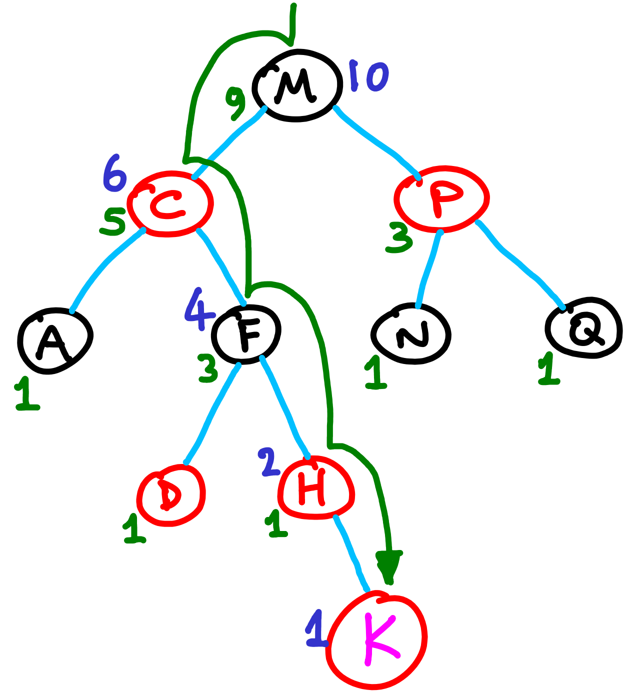


Compute subtree sizes after building by postorder walk...

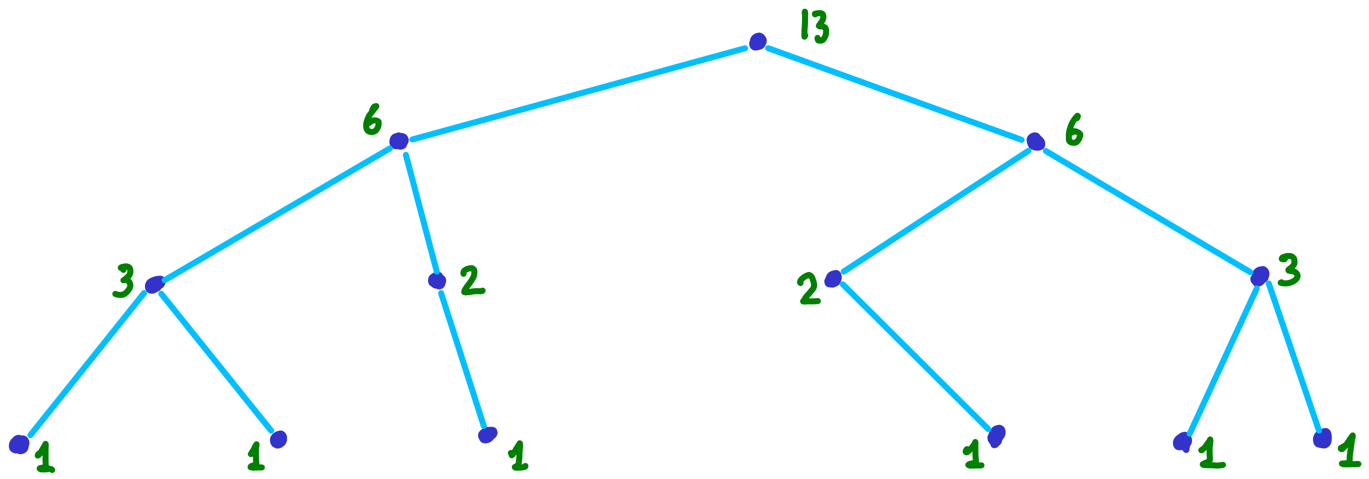
... or update path when inserting }

BUT...

we will need to rebalance



Can we update subtree sizes when inserting/deleting data?



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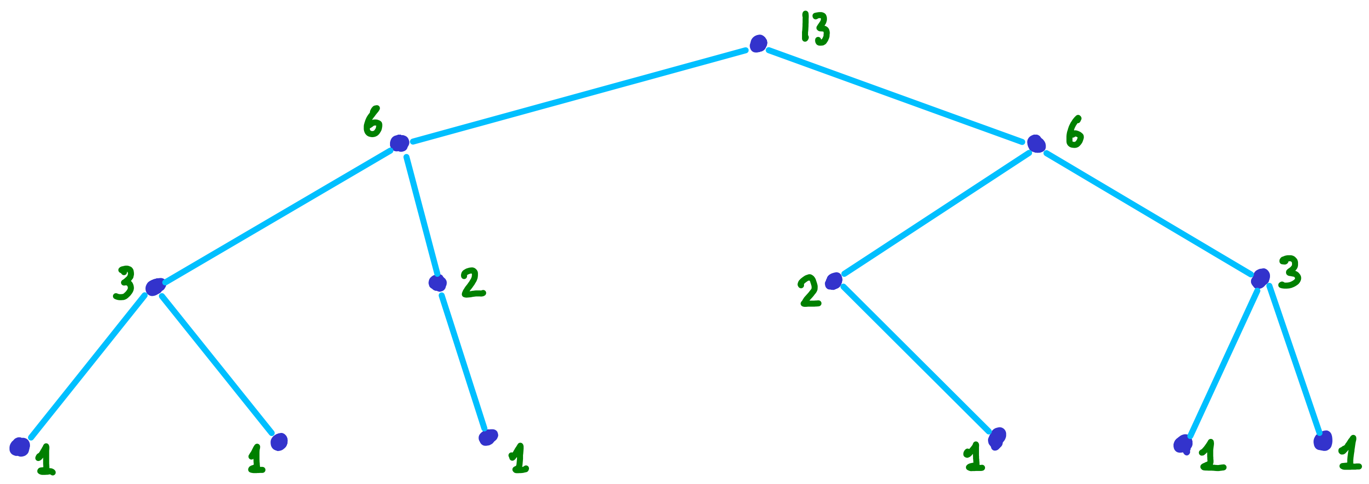
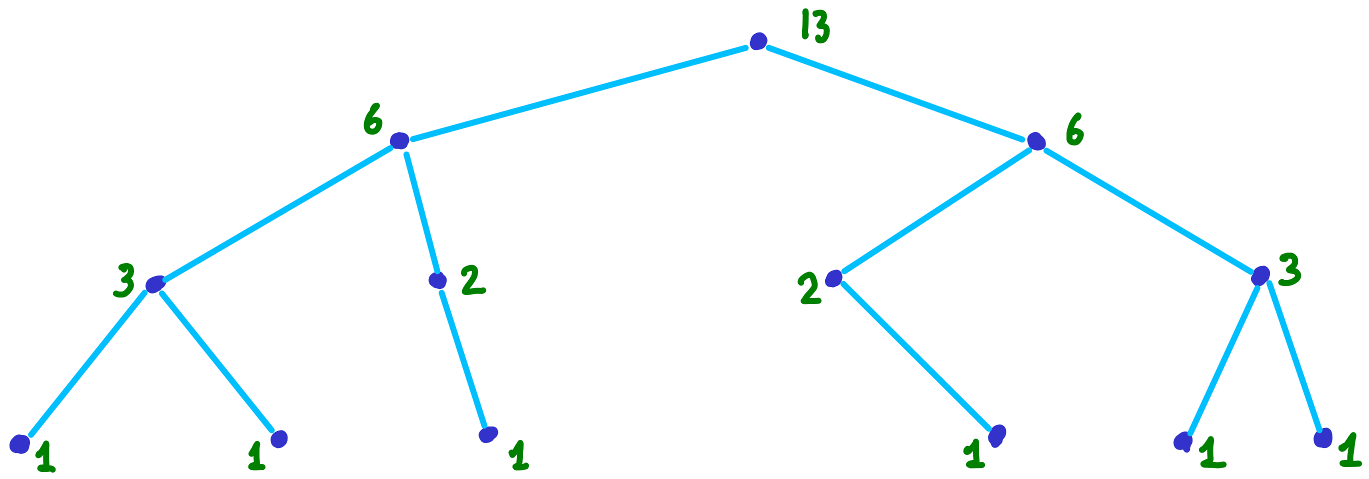


Use a RB tree

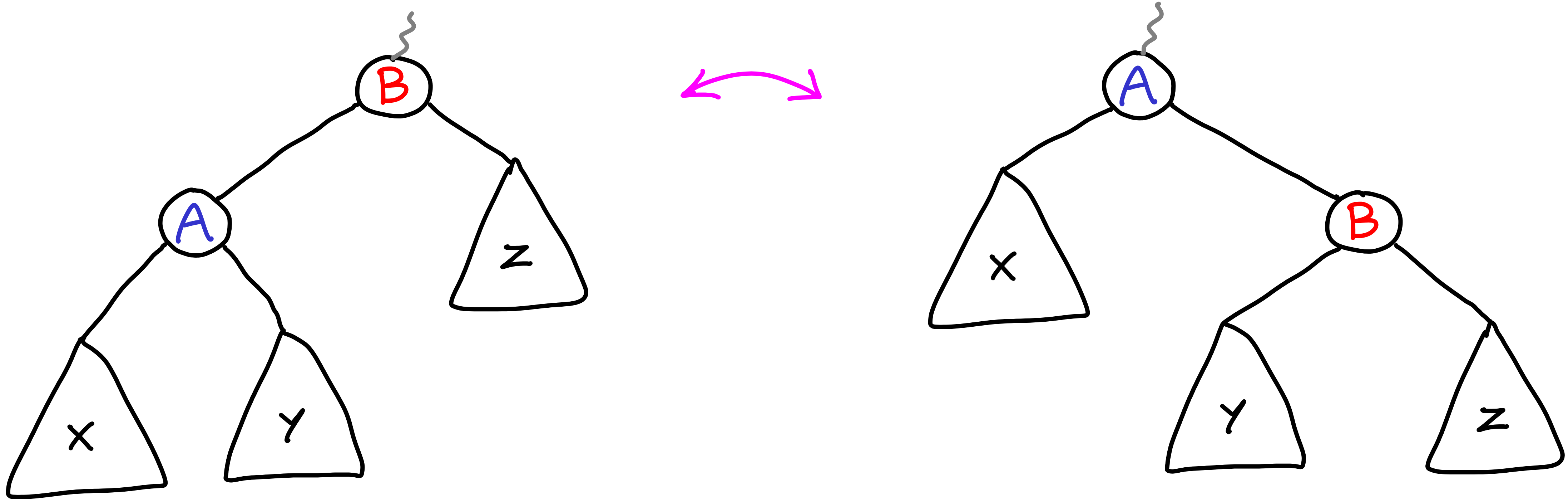
↳ when are subtree sizes affected?

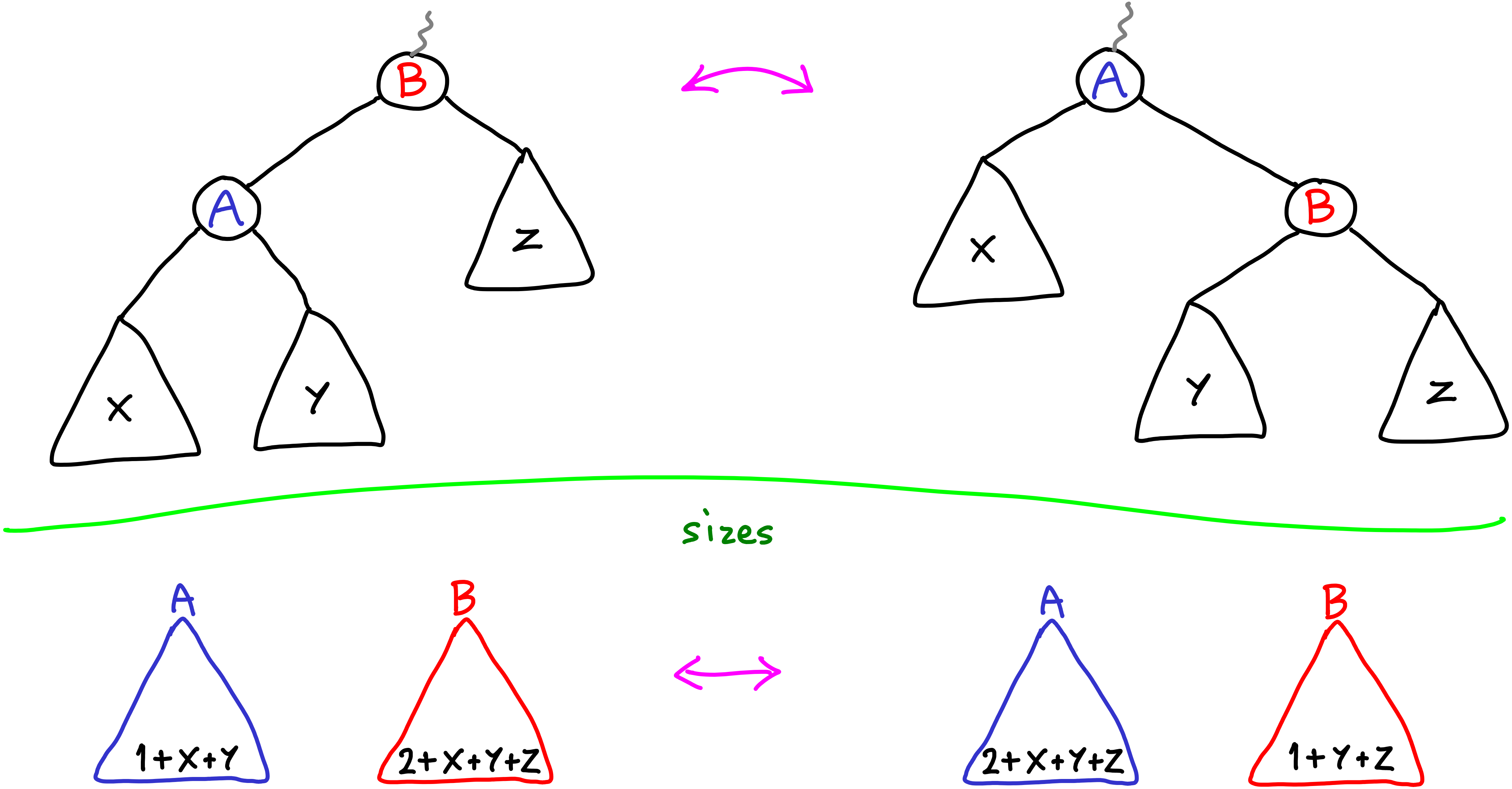
Can we update subtree sizes when inserting/deleting data?



Use a RB tree

↳ when are subtree sizes affected? Rotations


B
A
z
x
y
A
B
x
y
z


B
A
X
Y
Z
A
B
X
Y
Z
sizes
A
1+X+Y
B
2+X+Y+Z
A
2+X+Y+Z
B
1+Y+Z

AUGMENTED TREE TO FIND RANKS

- easy to find rank:
  - look at ancestor path & some adjacent subtree sizes
- subtree sizes can be updated when inserting and rebalancing

$O(\log n)$ per search/insertion/deletion

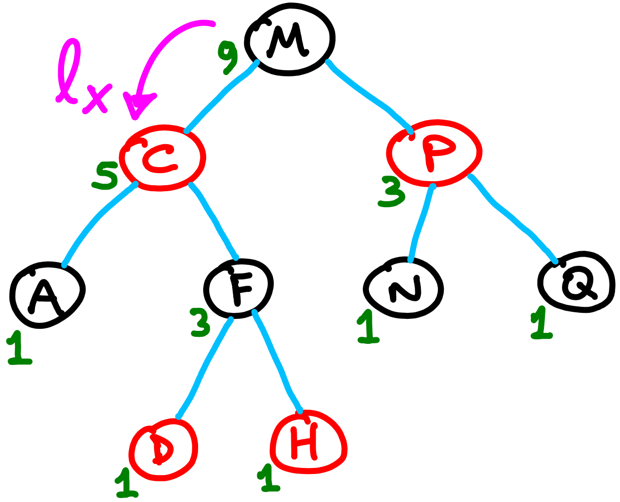
DYNAMIC SELECTION

find the i-th smallest element in a set

Static: $\Theta(n)$

Dynamic: $O(n \log n)$ preprocessing $\rightarrow$ balanced BST w/ subtree sizes

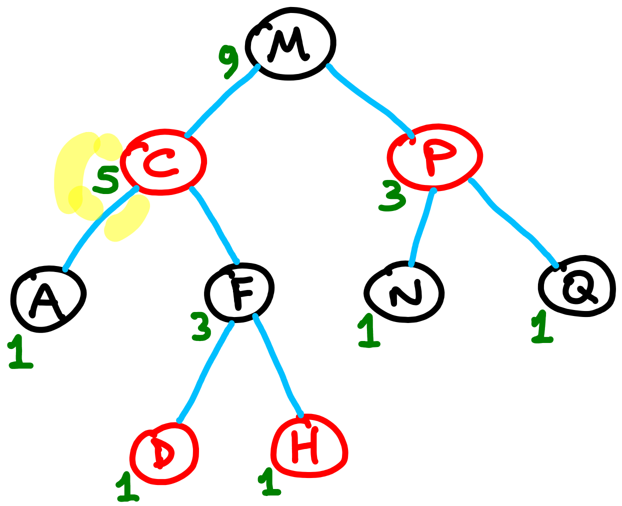
$O(\log n)$ after that



Select(x, i) \\ get i-th element in subtree rooted at x.

$k \leftarrow 1 + \text{size}(l_x)$ \\ l_x : left child of x

if $i = k$, return x.



Select(x, i) \\ get i-th element in subtree rooted at x.

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if $i = k$, return x.

example: $i=5$

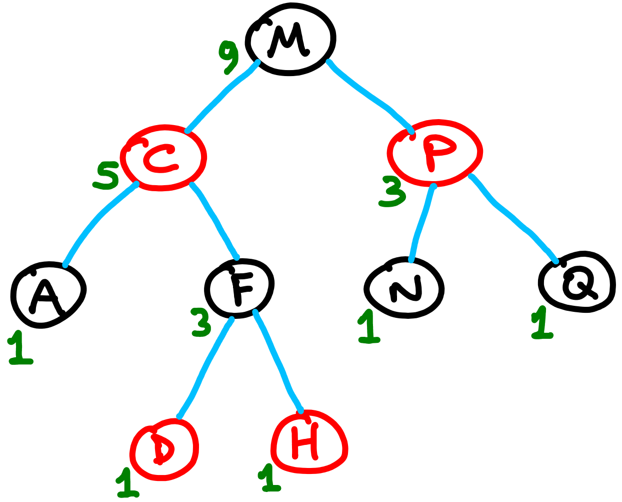
$k=6$

Select(root, 5)

$k \leftarrow 1 + 5$

$i < k$

Now what?



Select(x, i) \\ get i-th element in subtree rooted at x.

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if $i = k$, return x.

else if $i < k$, return Select(ℓ_x, i)

example: $i=5$

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Select(root, 5)

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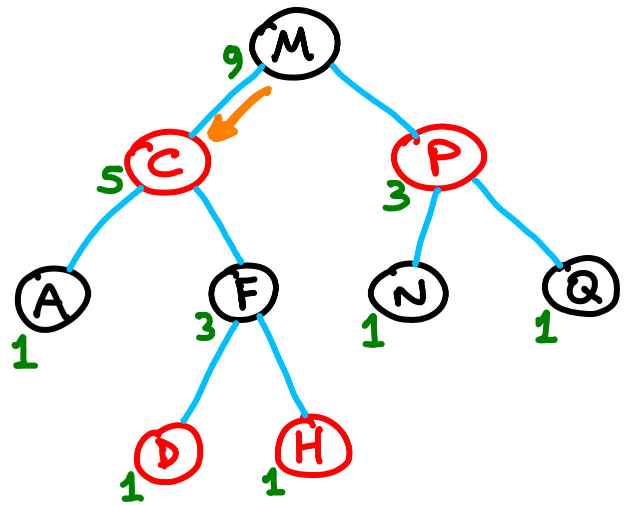


$\text{Select}(x, i)$ \\ get i-th element in subtree rooted at x.

$k \leftarrow 1 + \text{size}(l_x)$ \\ l_x: left child of x

if $i = k$, return x.

else if $i < k$, return $\text{Select}(l_x, i)$

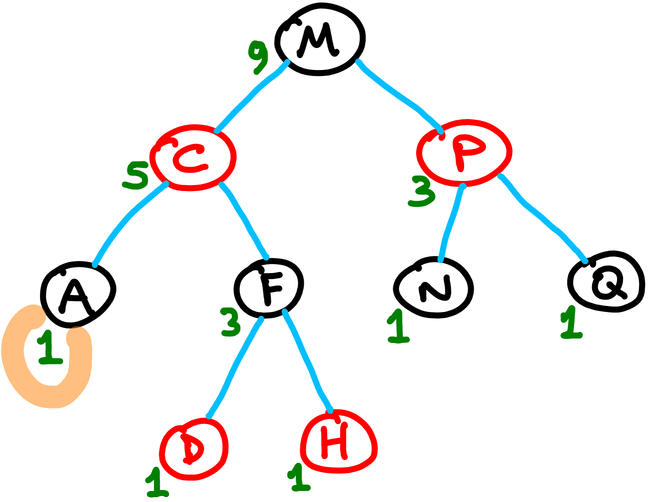
example: $i=5$

$k=6$

$\text{Select}(\text{root}, 5)$

$k \leftarrow 1 + 5$

$i < k \Rightarrow \text{Select}(c, 5)$



$Select(x, i)$ \\ get i-th element in subtree rooted at x.

$k \leftarrow 1 + size(l_x)$ \\ l_x : left child of x

if $i = k$, return x.

else if $i < k$, return $Select(l_x, i)$

example: $i=5$

$k=6$

$i=5$, $k=2$

$Select(root, 5)$

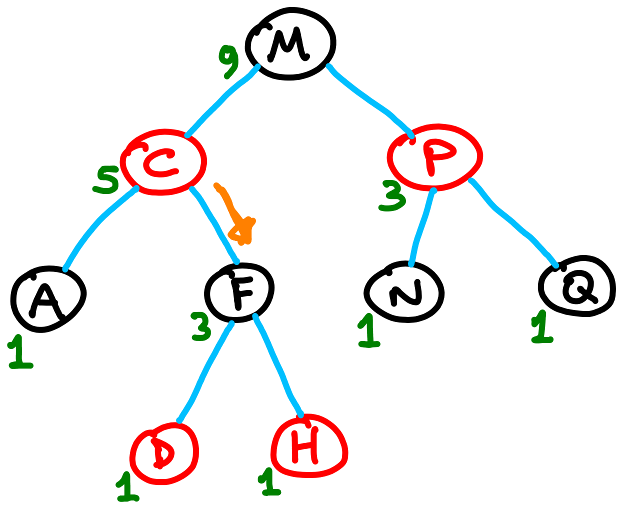
$k \leftarrow 1 + 5$

$i<k \Rightarrow Select(c, 5)$

$k = 1 + 1$

$i > k$

... next ?



Select(x, i) \\ get i-th element in subtree rooted at x.

$k \leftarrow 1 + \text{size}(l_x)$ \\ l_x : left child of x

if $i = k$, return x.

else if $i < k$, return Select(l_x, i)

else $(i > k)$ return Select$(r_x, i-k)$

example: $i=5$

$k=6$

$i=5$, $k=2$

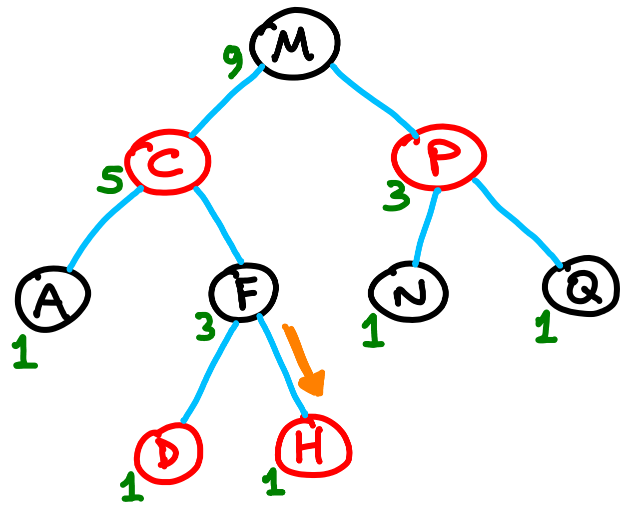
Select(root, 5)

$k \leftarrow 1 + 5$

$i<k \Rightarrow$ Select(C, 5)

$k = 1 + 1$

$i>k \Rightarrow$ Select(F, 3)



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$k=6$

$i=5$, $k=2$

$i=3$, $k=2$

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$k \leftarrow 1 + 5$

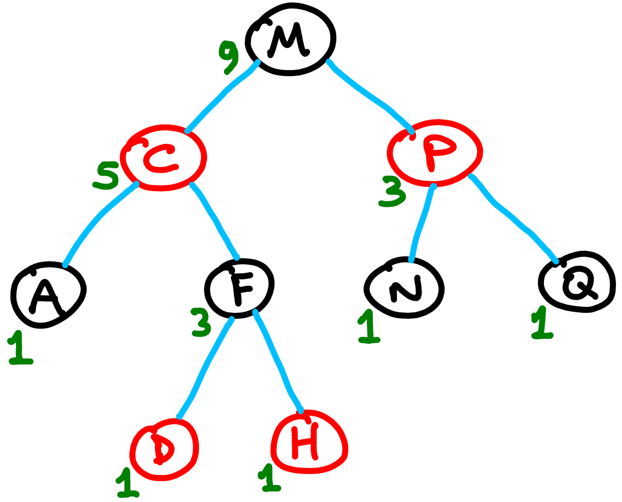
$i<k \Rightarrow$ Select(C, 5)

$k = 1 + 1$

$i>k \Rightarrow$ Select(F, 3)

$k = 1 + 1$

$i>k \Rightarrow$ Select(H, 1)



Select(x, i) \\ get i-th element in subtree rooted at x.

$k \leftarrow 1 + \text{size}(\ell_x)$ \\ ℓ_x: left child of x

if $i = k$, return x.

else if $i < k$, return Select(ℓ_x, i)

else $(i > k)$ return Select$(r_x, i-k)$

example: $i=5$

$k=6$

$i=5$, $k=2$

$i=3$, $k=2$

$i=1$, $k=1$

Select(root, 5)

$k \leftarrow 1 + 5$

$i<k \Rightarrow$ Select(C, 5)

$k = 1+1$

$i>k \Rightarrow$ Select(F, 3)

$k = 1+1$

$i>k \Rightarrow$ Select(H, 1)

$k = 1+0$

$i=k \Rightarrow$ return H

DYNAMIC SELECTION

find the i-th smallest element in a set

Static: $\Theta(n)$

Dynamic: $O(n \log n)$ preprocessing → balanced BST w/ subtree sizes

$O(\log n)$ query / insert / delete