Applications of amortized analysis

Binary search trees of bounded balance α $\rightarrow$ BB(α)

(weight-balanced trees)

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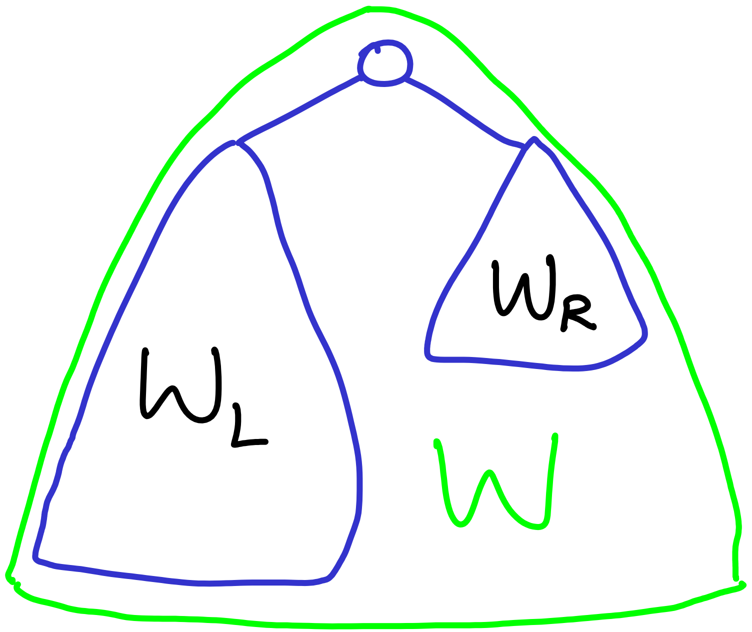
node with subtree weight $\underline{W}$ and — left subtree weight W_L
— right subtree weight W_R



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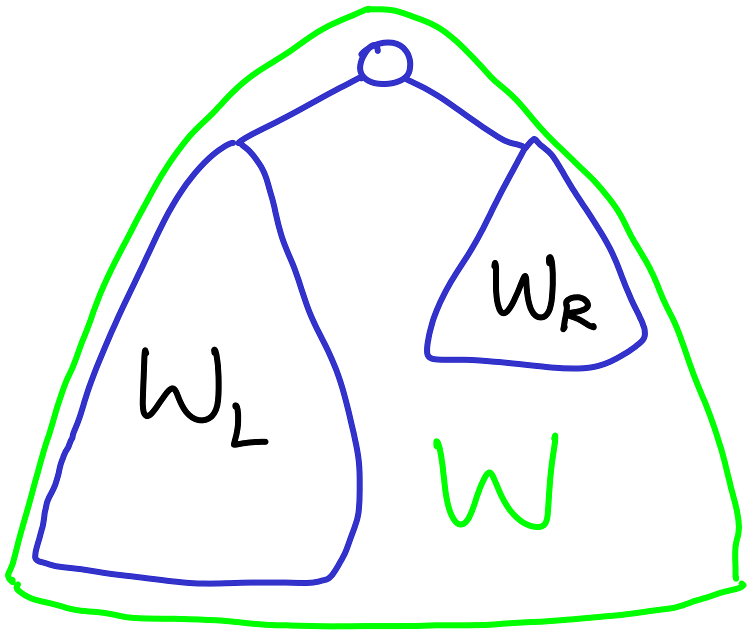

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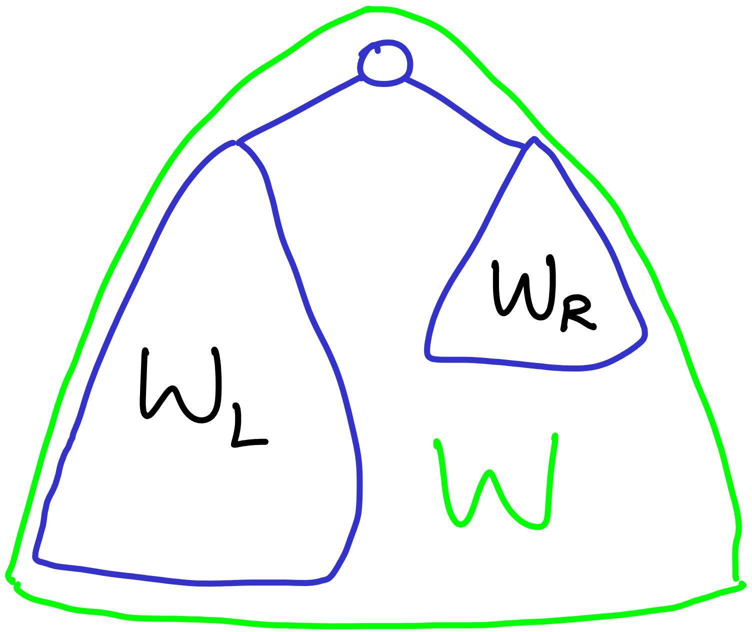
Height: ?



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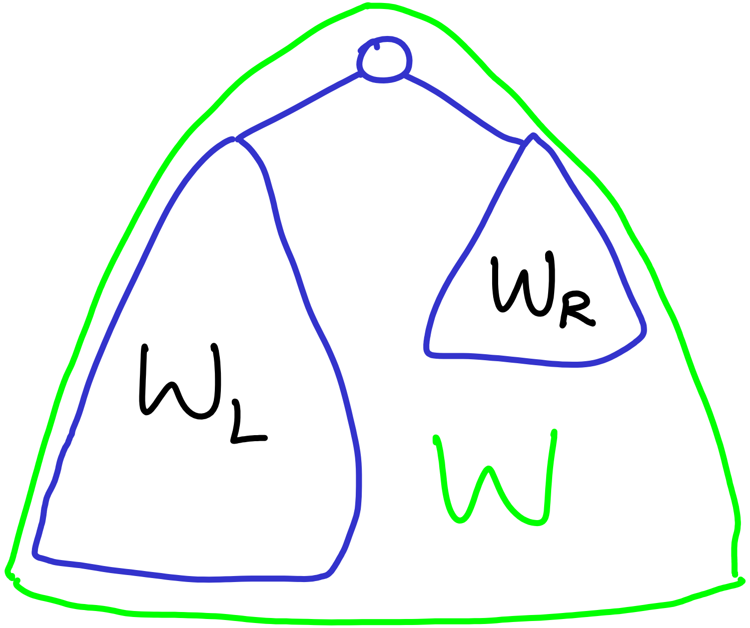
Height: $H(n) \leq 1 + H\big(\underbrace{(1-\alpha)}_{\geq \frac{1}{2}} n\big)$



Binary search trees of bounded balance α $\rightarrow$ $BB(\alpha)$

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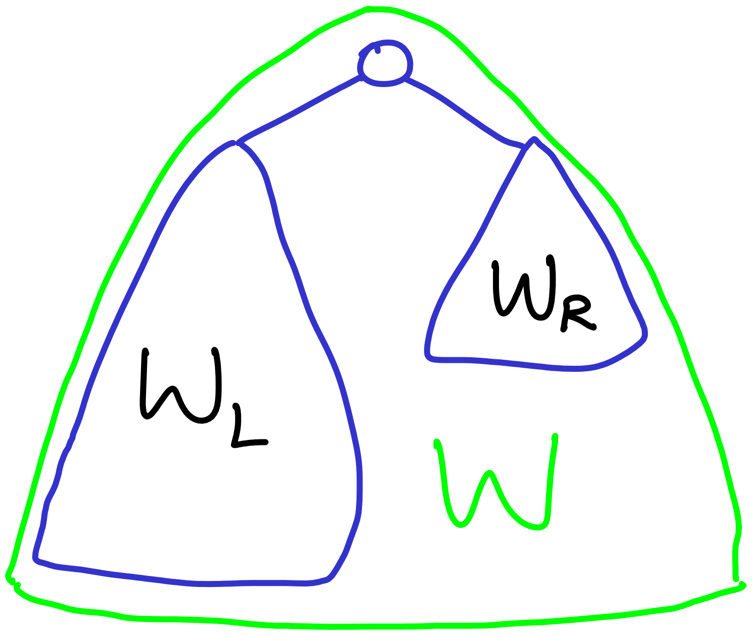

Height: $H(n) \leq 1 + H(\underbrace{(1-\alpha)}_{\geq \frac{1}{2}} n)$

standard geometric series:

$H \sim \log_{1/(1-\alpha)} n = \Theta(\log n)$

Insertion / Deletion in BB(α) weight-balanced tree

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Any ancestor could now have:

$W_R < \alpha W$ (wlog)

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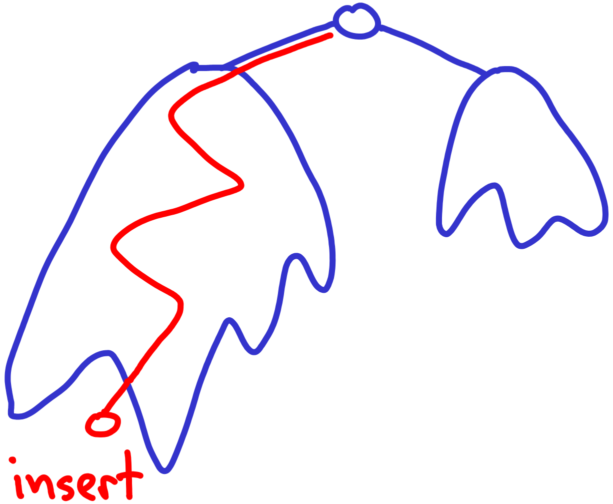

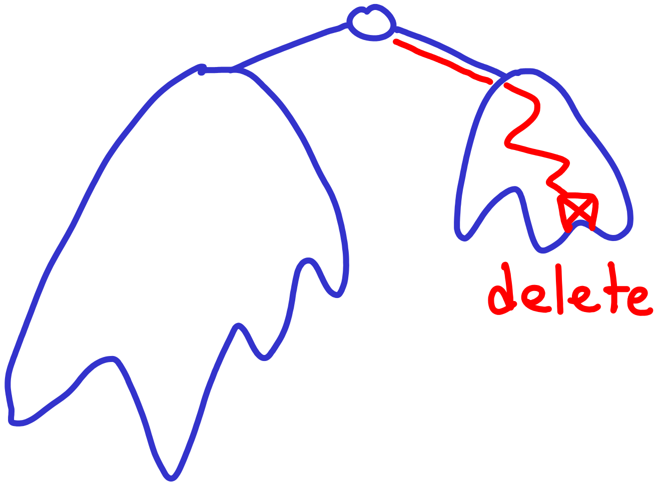

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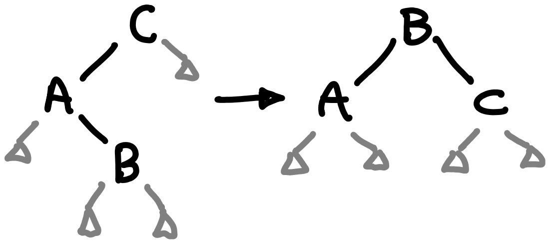
FYI

actually, 2 types of rotation
↳ standard & "split"



... depends on amount of imbalance

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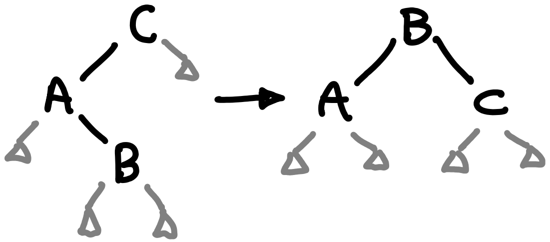


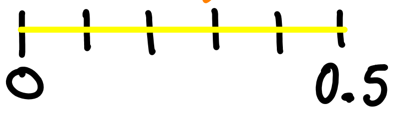
... depends on amount of imbalance

Works for $\alpha < 1 - \frac{1}{\sqrt{2}}$ ~ 0.3 // Proof involves some annoying counting

Can walk up & rebalance ancestors : $O(\log n)$

Insertion / Deletion in BB(α) weight-balanced tree

$$\alpha < 1 - \frac{1}{\sqrt{2}}$$



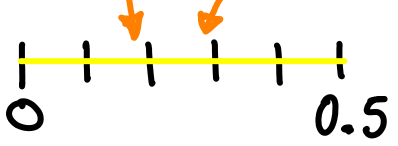
FYI

Insertion / Deletion in BB(α) weight-balanced tree

lower bound:

$\frac{2}{11} < \alpha$

$\alpha < 1 - \frac{1}{\sqrt{2}}$



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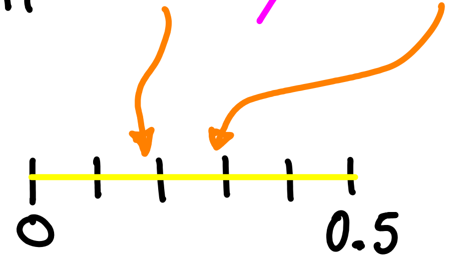

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Claim:

can rebalance for α outside this range with more complicated rotations

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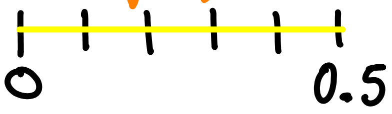

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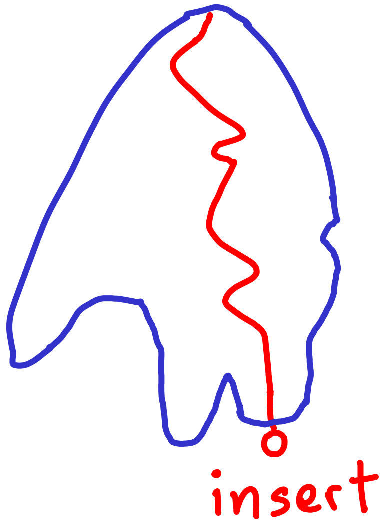
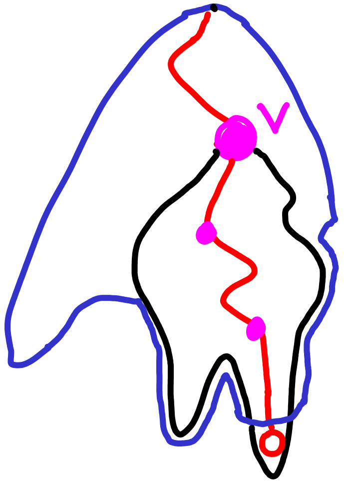
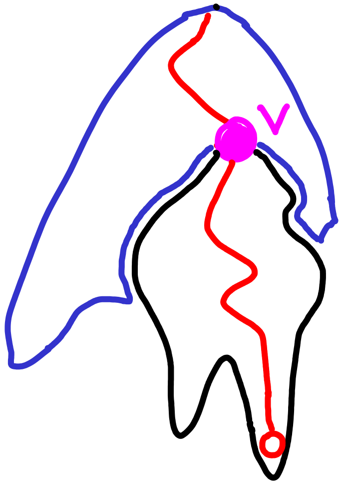
Instead, we will avoid rotations


insert



α-violation may occur for any ancestor

Let v be highest violation



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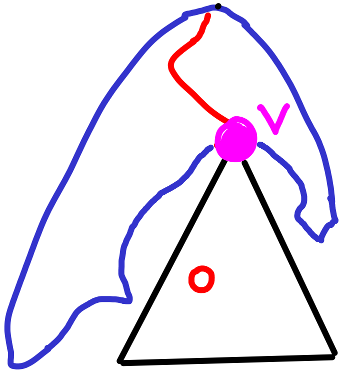

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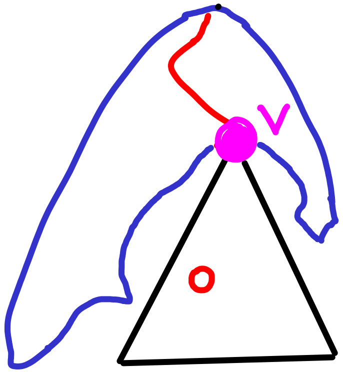
Rebuild subtree(v)   cost ?



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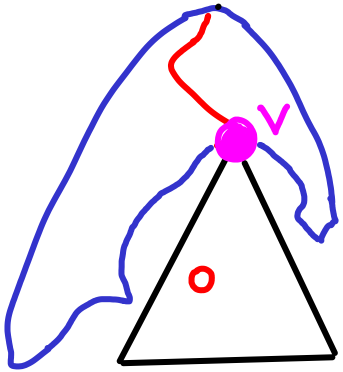
Rebuild subtree(v) $O(\log n) + \Theta(\text{size}(v)) = O(n)$

AMORTIZE ... ?



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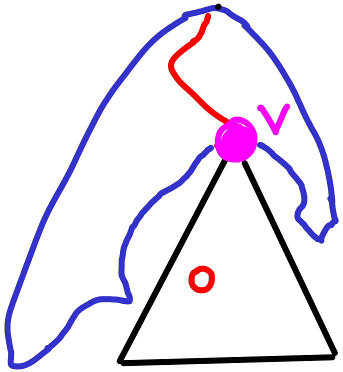
$$\Phi = \sum_{\substack{\text{all} \\ \text{nodes}}} \frac{1}{1-2\alpha} |W_L - W_R|$$ $\rightarrow$ technically, only if diff ≥ 2

(diff = 1 unavoidable)

e.g. $\alpha = \frac{1}{3}$: $\Phi = 3 \cdot \sum |W_L - W_R|$



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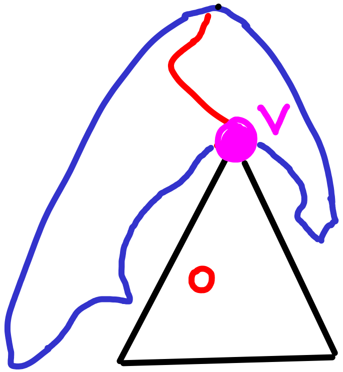

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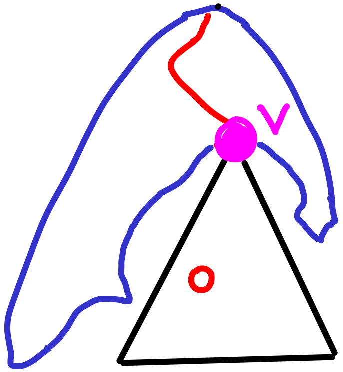

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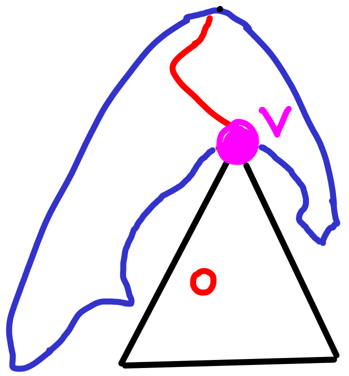

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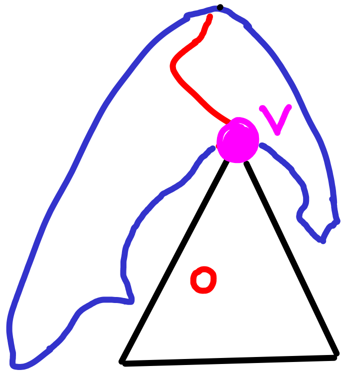

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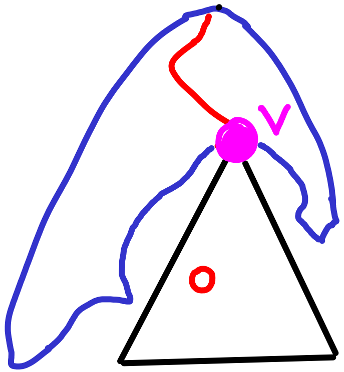
every $\Delta\Phi$ term ≤ 0



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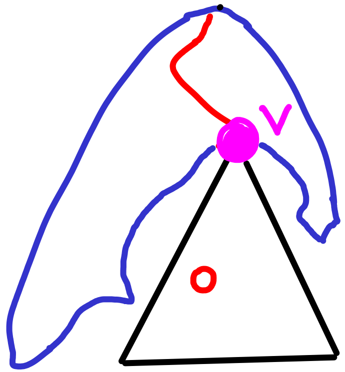

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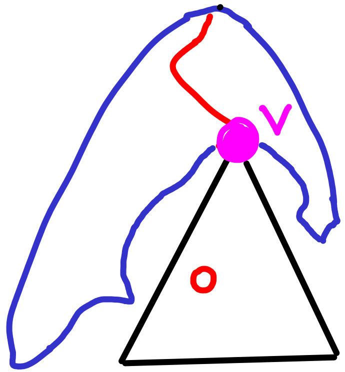

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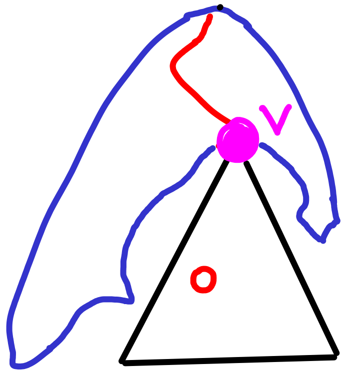

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$$\underbrace{\tfrac{1}{3}W}_{\Phi_{i-1}}$$

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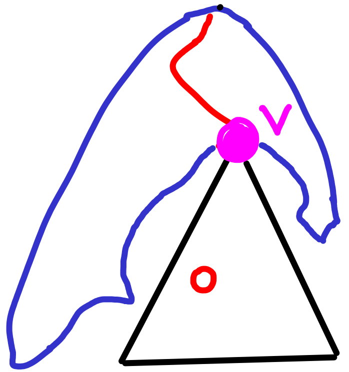

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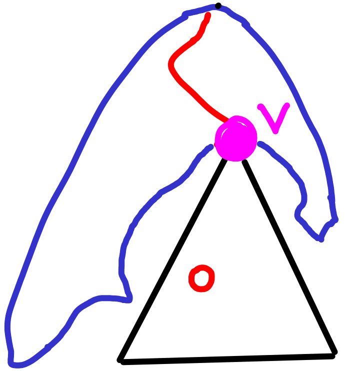
$$\leq 3 \cdot [\underset{\downarrow}{0} - \underbrace{\tfrac{1}{3} W}] \sim -\text{size}(v)$$

$\Phi_i - \Phi_{i-1}$

violation: $\boxed{\text{wlog } W_R < \frac{1}{3} W < \frac{2}{3} W < W_L}$

$\square$



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$$\leq \frac{1}{1-2\alpha} \cdot [0 - \underbrace{(1-2\alpha)}_{\Phi_{i-1}} W] \sim -\text{size}(v)$$

$\downarrow$ Φ_i

violation: $\boxed{W_R < \alpha W < (1-\alpha) W < W_L}$

wlog

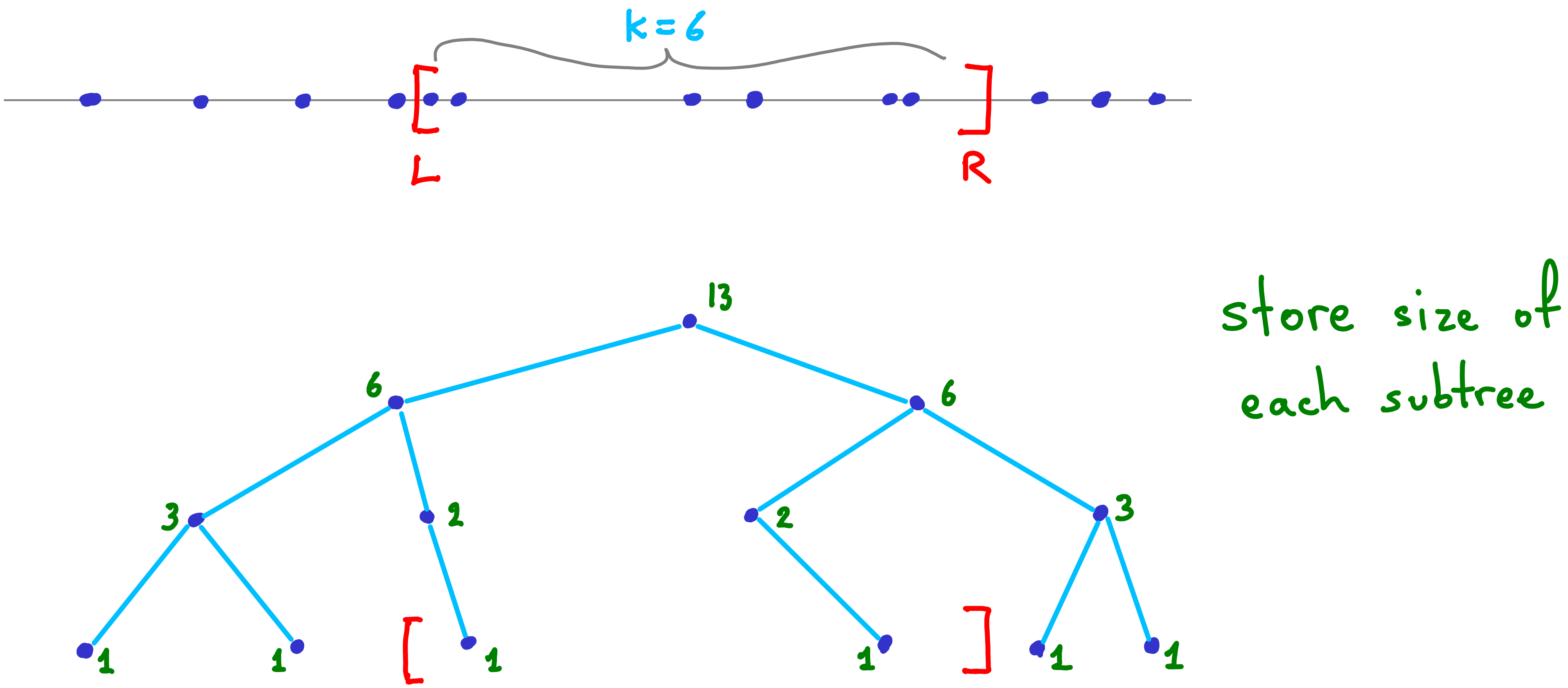
$\square$

An application of BB(α)

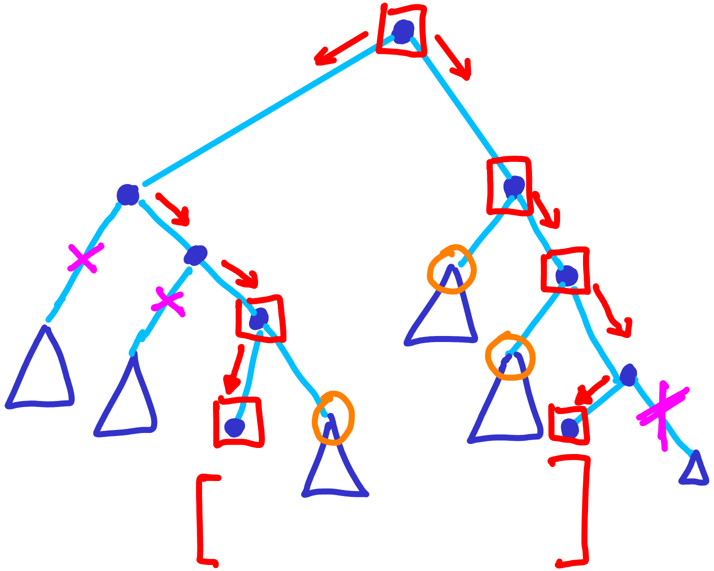
Dynamic high-dimensional multilayered range trees

↪ not easy to update efficiently with rotations

FYI

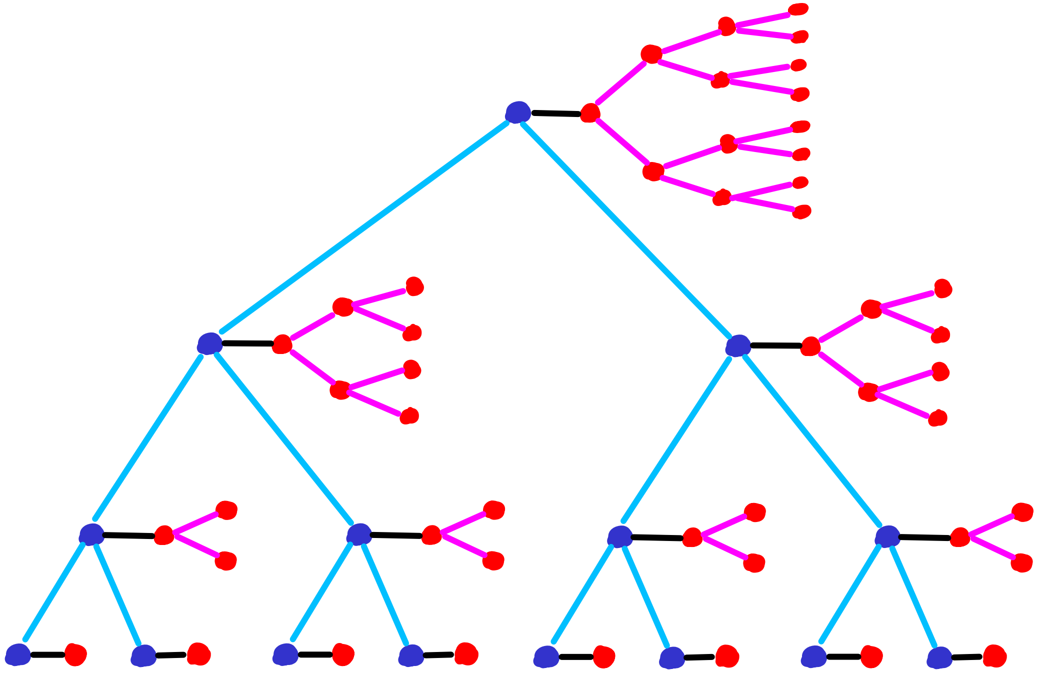


1D range counting



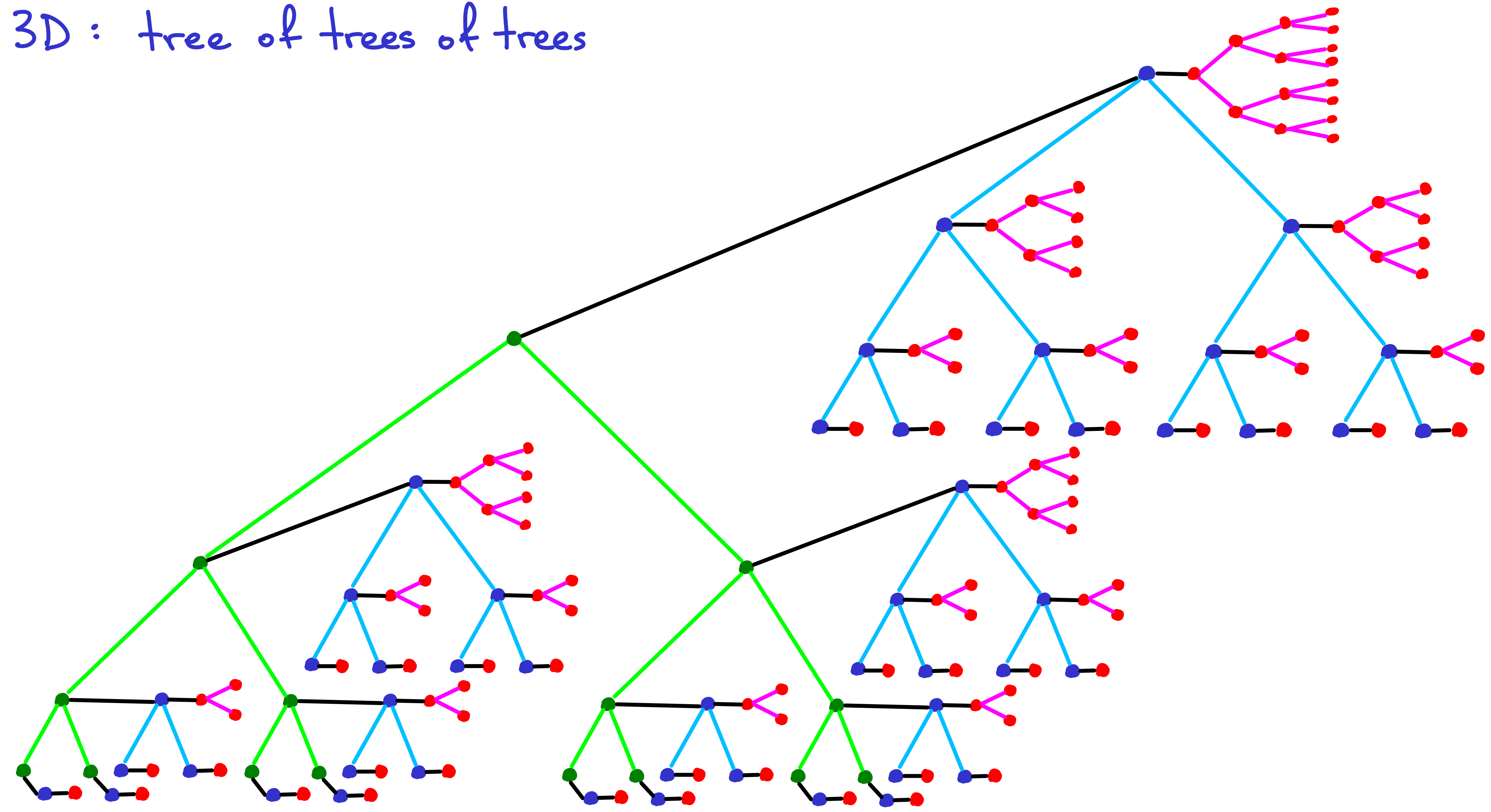
Every X-range is represented by O(logn) nodes

For each node, create a new (aux.) tree containing all nodes of subtree, sorted by Y.



2D range counting

3D : tree of trees of trees

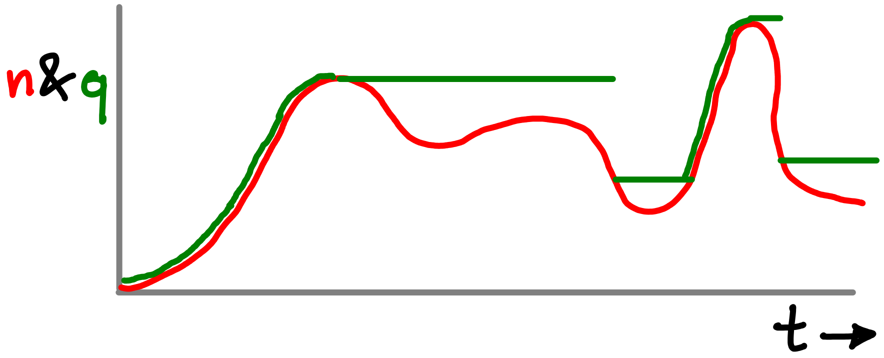


More results that use amortized analysis

SCAPEGOAT TREES : amortized balanced dynamic BST

n = # keys

q = variable s.t. $q/2 \le n \le q$ →



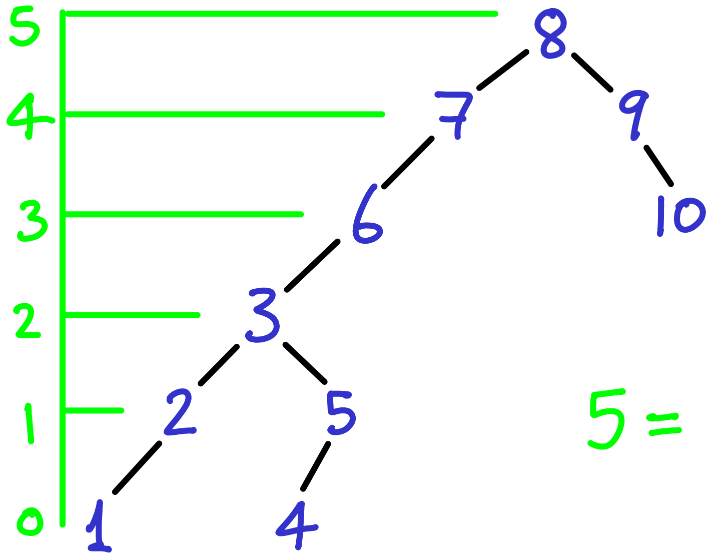
α = balance factor,

$\frac{1}{2} < \alpha < 1$

determines max allowed height.

$h \le \log_{1/\alpha} q$

$\le \log_{1/\alpha} 2n = \log_{1/\alpha} n + O(1)$



$n = 10$

$10 \le q \le 20$

$\alpha = \frac{2}{3}$

$5 = h \le \log_{1.5} q \sim 5.68$ for $q=10$

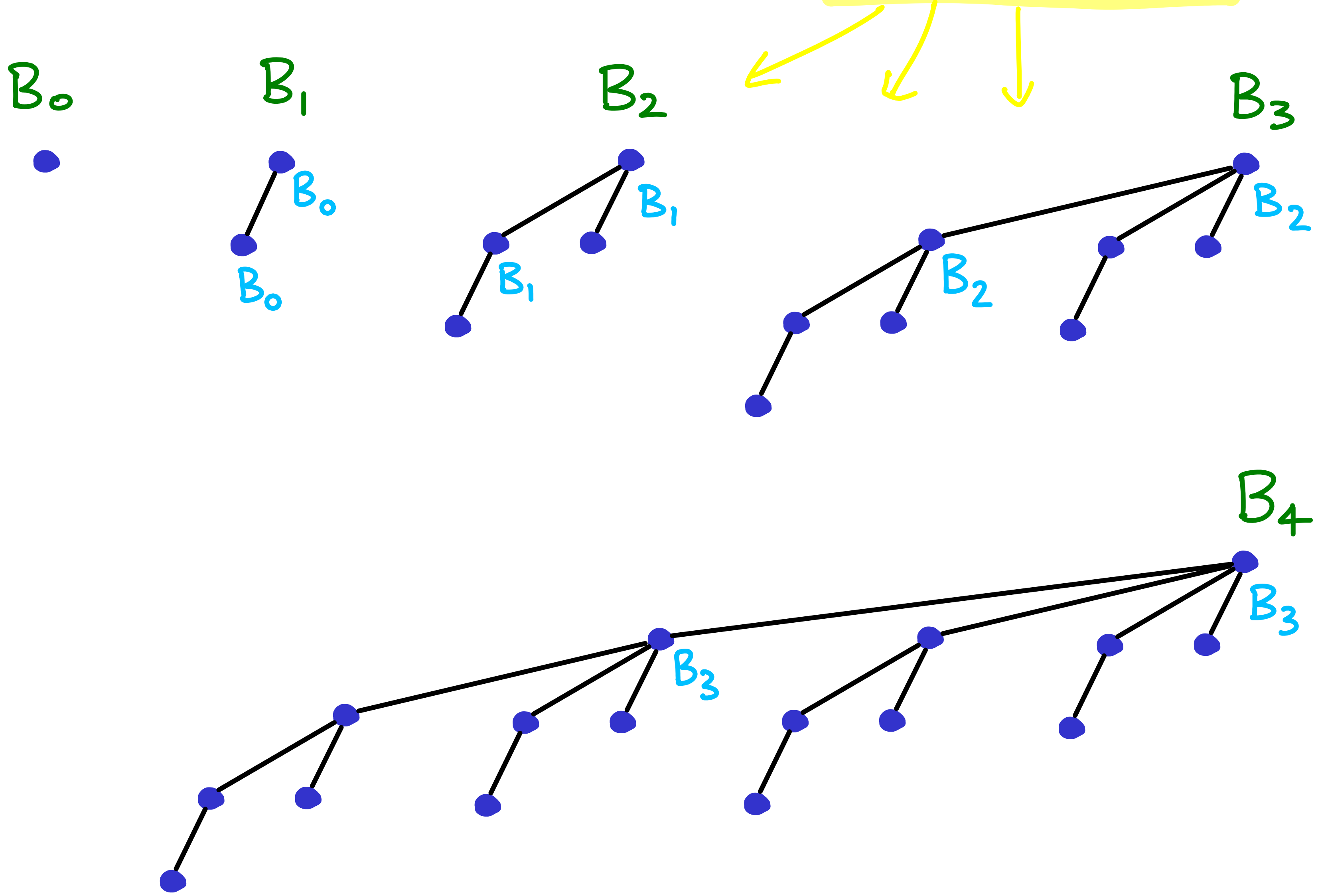
OK for $q > 10$

Heap building: we have seen $O(n)$ instead of $O(n\log n)$

Heap building: we have seen $O(n)$ instead of $O(n\log n)$

Binomial heaps: collections of binomial trees

BINOMIAL TREES



Heap building: we have seen $O(n)$ instead of $O(n\log n)$

Binomial heaps: collections of binomial trees

size = power of 2
only one of each size
trivially mergeable

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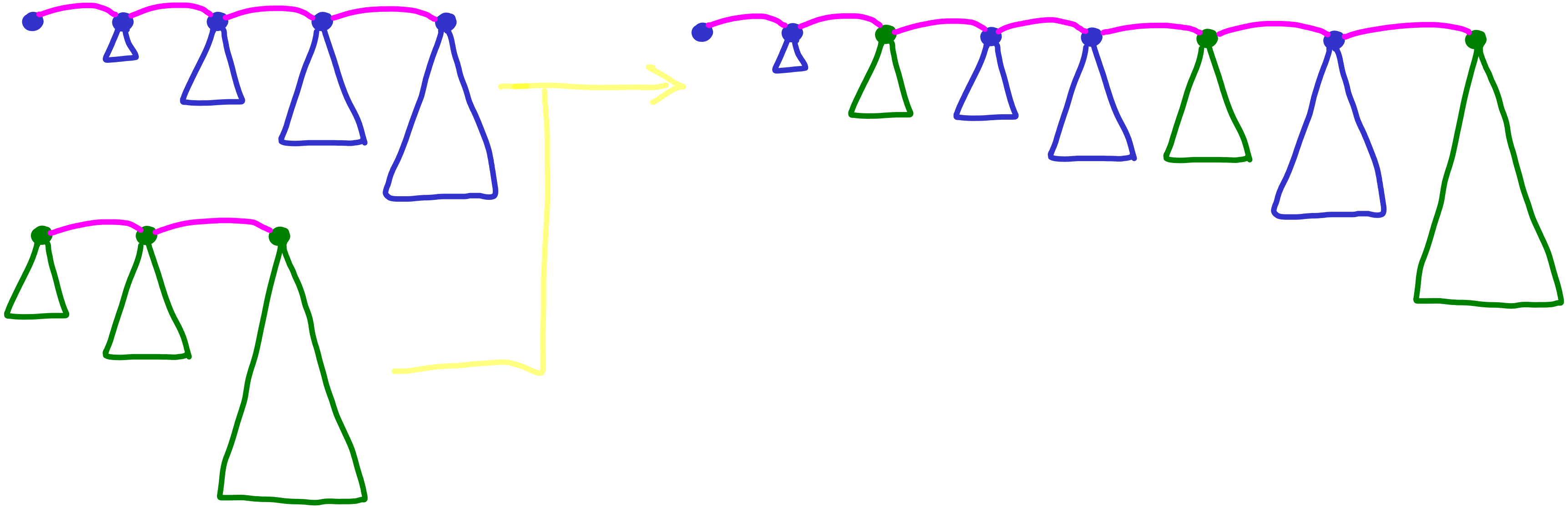

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but O(1) amortized

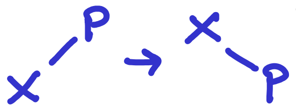
HEAPS:	REGULAR	BINOMIAL	FIBONACCI
REPORT MIN:	$O(1)$	←	←
EXTRACT MIN:	$O(\log n)$	←	$O(n)$ $O(\log n)$ amortized
INSERT:	$O(\log n)$ $O(1)$ amortized	←	$O(1)$
DECREASE KEY:	$O(\log n)$	←	$O(n)$ * $O(1)$ amortized
DELETE:	$O(\log n)$	←	$O(n)$ $O(\log n)$ amortized
MERGE/UNION:	$O(n)$	$O(\log n)$	* $O(1)$

HEAPS:

	REGULAR	BINOMIAL	QUAKE
REPORT MIN:	$O(1)$	←	←
EXTRACT MIN:	$O(\log n)$	←	$O(n)$ $O(\log n)$ amortized
INSERT:	$O(\log n)$ $O(1)$ amortized	←	$O(1)$
DECREASE KEY:	$O(\log n)$	←	$O(1)$
DELETE:	$O(\log n)$	←	$O(n)$ $O(\log n)$ amortized
MERGE/UNION:	$O(n)$	$O(\log n)$	$O(1)$

Splay trees: whenever you access a node, bring it to the top

Amortized cost O(logn)

(if no G then )