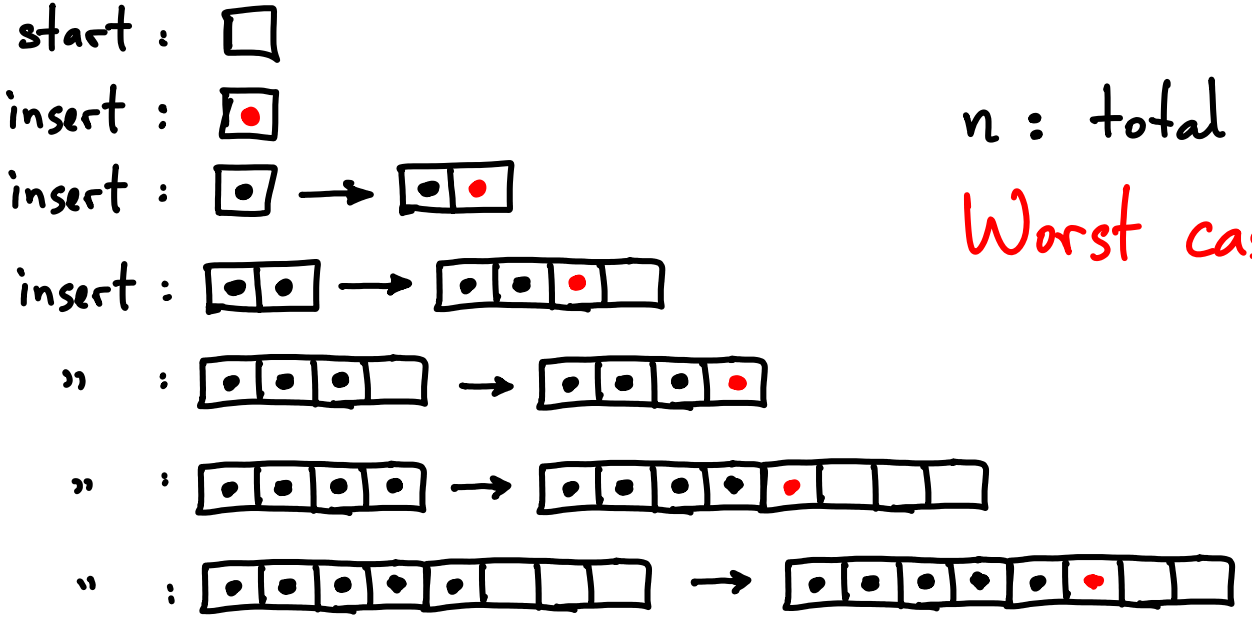
DYNAMIC TABLES & AMORTIZED ANALYSIS

Suppose you want to use an array:

You must support insertion but you don't know max #elements

array doubling: start w/ array of size 1; every time it fills up, double the size

↳ (make a new larger array & copy)



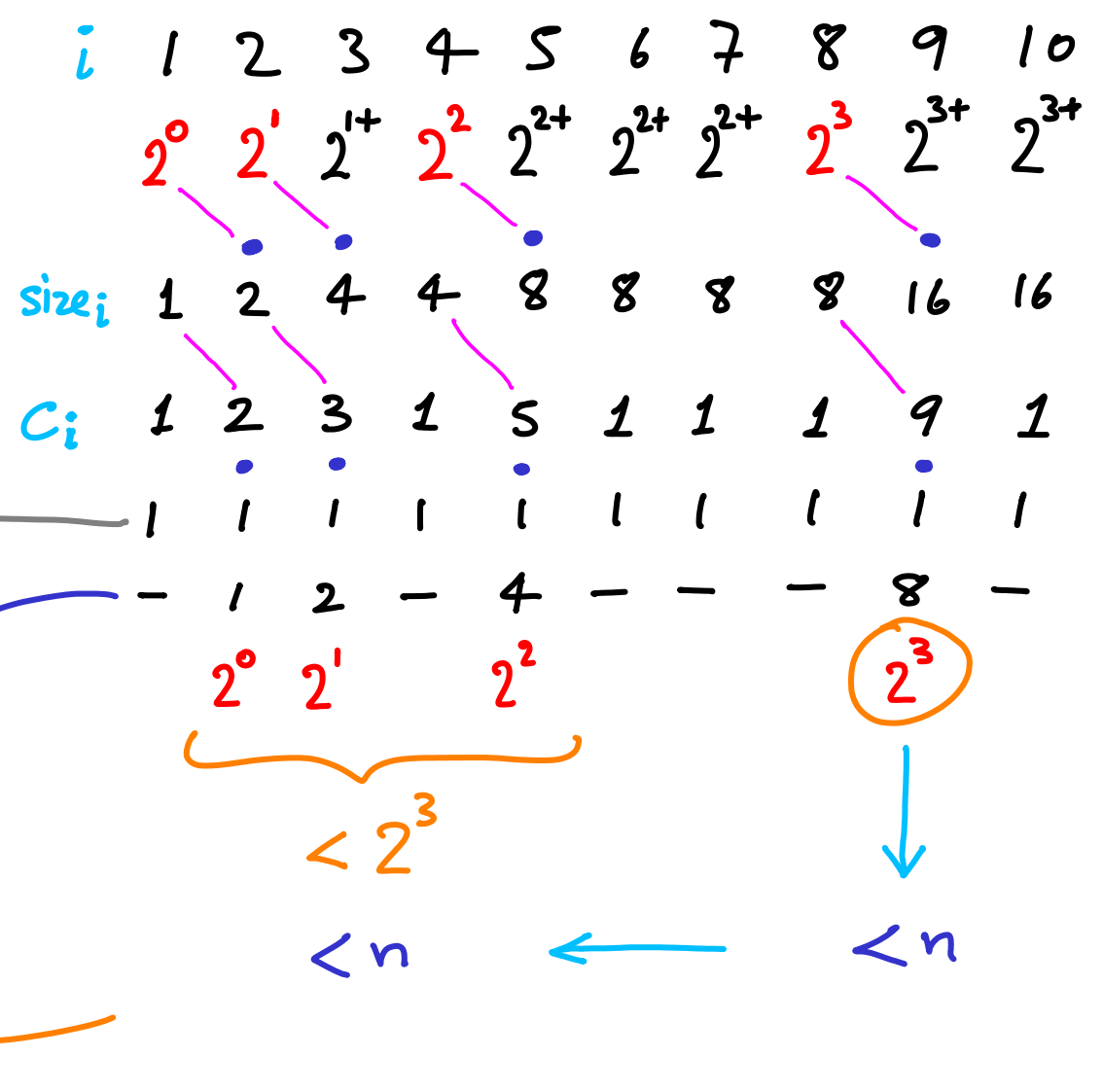
n : total number of inserts

Worst case time of an insert : $O(n)$

↳ for n inserts : $O(n^2)$

Claim : for n inserts : also $O(n)$

cost c_i $\begin{cases} i & \text{if } i-1 \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$



$$\text{cost}(n) = \sum_{i=1}^{n} c_i \leq n + 2n$$

AGGREGATE ANALYSIS

AMORTIZATION (analyzing cost)

Applies to some problems that involve many operations.

If worst case time of operation k is $O(f(k))$,
try to show that n operations cost $o(n \cdot f(n))$

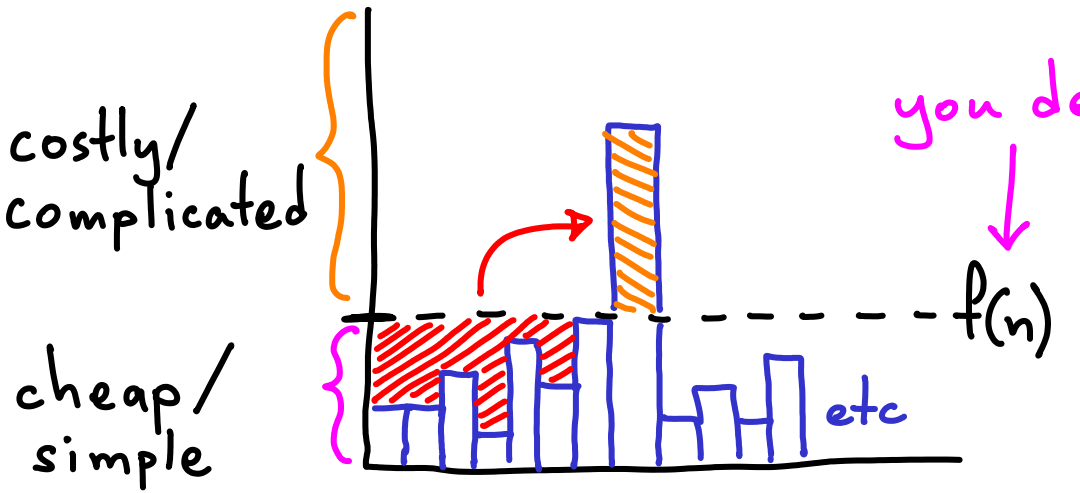
3 main ways for amortizing: aggregate, accounting, & potential method.

Accounting: saving for a rainy day

Pretend "simple" operations cost more than they do. Ideally Θ(real cost)

↳ "save" the difference → "spend" what you saved up.

"Complicated/costly" operations: pretend they cost less; pay excess via savings



you decide

Rule: Never spend more than what you saved.

Amortized cost: $n \cdot f(n) \geq$ true cost

Goal: exaggerate / save as little as possible. } minimize $f(n)$

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

let $\hat{c}_i = 3$ ⟶ 1 to cover insert cost
implies $\sum c_i \leq 3n$ 2 for eventual doubling

When table doubles, use 1 to copy each item.

BANK (savings per iteration)

1

Pretend cost is ~~3~~ 2

pay 1 to insert, save/bank ~~2~~ 1

We can even give $ to charity

i	(1)	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	(2)	3	3	3	3	3	3	3	3	3

Just this one time

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

let $\hat{c}_i = 3$ → 1 to cover insert cost
2 for eventual doubling

implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)

Spent 1 to copy 1 item into new array. → left ∅ in bank.

Added 2nd item & banked $2

i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2								

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

let $\hat{c}_i = 3$ → 1 to cover insert cost

2 for eventual doubling

implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)

Spent $2 from bank to copy 2 items.

Item #3 inserted: bank $2

i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2							

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

let $\hat{c}_i = 3$ ⟶ 1 to cover insert cost
2 for eventual doubling

implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)

Finally, bank account is growing

i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4						

Amortized cost of operation i: $\hat{c}_i$

Back to dynamic tables:

let $\hat{c}_i = 3$ ⟶ 1 to cover insert cost
implies $\sum c_i \leq 3n$ 2 for eventual doubling

When table doubles, use 1 to copy each item.

BANK (savings per iteration)

Spent 4 to copy 4.

Banked 2.

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2					

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

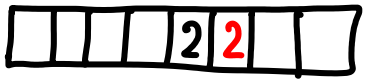
Back to dynamic tables:

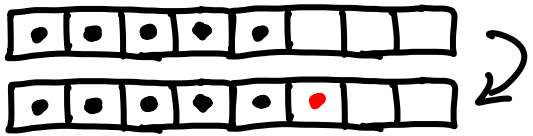
let $\hat{c}_i = 3$ → 1 to cover insert cost
2 for eventual doubling

implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)





i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4				

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

let $\hat{c}_i = 3$ ⟶ 1 to cover insert cost
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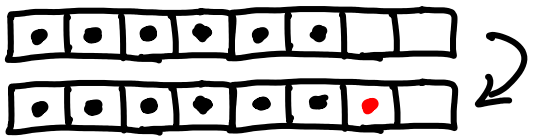
implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)



i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4	6			

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

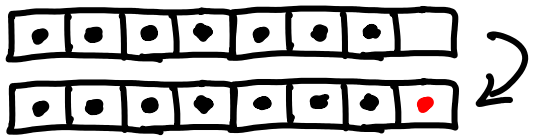
Back to dynamic tables:

let $\hat{c}_i = 3$ → 1 to cover insert cost
2 for eventual doubling

implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)



i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4	6	8		

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

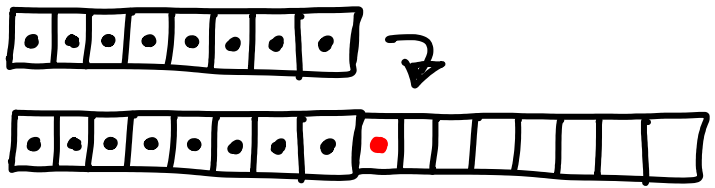
let $\hat{c}_i = 3$ → 1 to cover insert cost
2 for eventual doubling

implies $\sum c_i \leq 3n$

When table doubles, use 1 to copy each item.

BANK (savings per iteration)

use all savings to copy 8 items



insert item 9

i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4	6	8		

Amortized cost of operation i: $\hat{c}_i$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \quad \text{for all } n$$

our banking game should always exagerrate true costs.

Back to dynamic tables:

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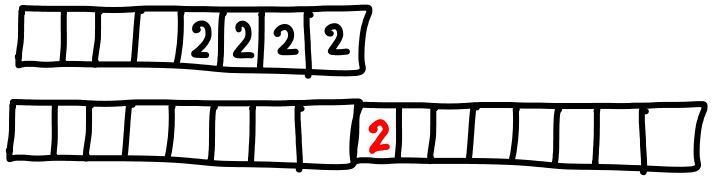
When table doubles, use 1 to copy each item.

BANK (savings per iteration)



restore the condition:

after doubling we have $2 in bank

i	1	2	3	4	5	6	7	8	9	10
c_i	1	2	3	1	5	1	1	1	9	1
	1	1	1	1	1	1	1	1	1	1
	–	1	2	–	4	–	–	–	8	–
$size_i$	1	2	4	4	8	8	8	8	16	16
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4	6	8	2	

Summary of accounting method

Estimate a cost: $\hat{c}_i$ ... higher than what you think average real cost $\frac{1}{n}\sum c_i$ will be

Prove that $\hat{c}_i$ is an overestimate of average c_i

↳ get bounds on how much you "save" & "spend"

Really does involve already having intuition.

POTENTIAL METHOD aka Physicists' method

Start with data structure D_0

Operation i : $D_{i-1} \to D_i$ cost : c_i

Potential function Φ_i maps $D_i \to \mathbb{R}$: potential value.

$\Phi_0 = 0$ $\Phi_i \geq 0$ $\Rightarrow$ 2 conditions that help.

Let $\hat{c}_i = c_i + \Phi_i - \Phi_{i-1}$

$= c_i + \Delta\Phi_i$

If $\Delta\Phi_i > 0$, $\hat{c}_i > c_i$: storing potential ... "work" in D_i

$\Delta\Phi_i < 0$, $\hat{c}_i < c_i$: release work.

$\hat{c}_i = c_i + \Delta\Phi_i \Rightarrow \sum \hat{c}_i = \sum_{i=1}^{n} (c_i + \Delta\Phi_i)$ telescoping series

$= \Phi_n - \Phi_0 + \sum_1^n c_i \geq \sum_1^n c_i$

($\Phi_n \geq 0$, $\Phi_0 = 0$)

so we know that the amortized cost will not underestimate real cost.

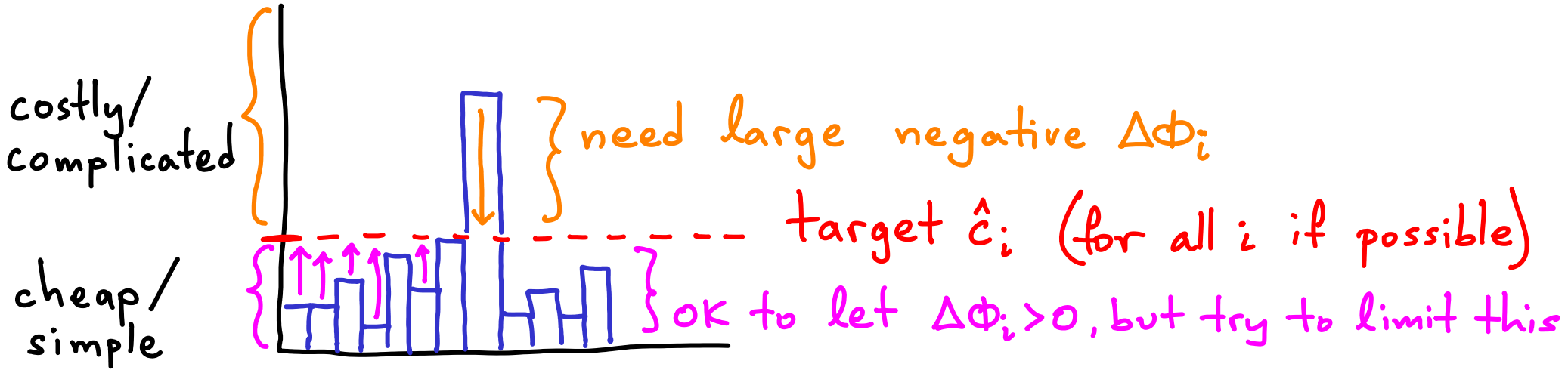
now figure out worst case for any individual $\hat{c}_i$

↳ Ideally this will give a good (and easy) bound for total cost

$$\sum c_i \leq \sum \hat{c}_i \leq n \cdot \max \hat{c}_i$$

How this works

- (Subjectively) define what a complicated / costly operation (c_i) is.
- Find something that changes a lot in the data structure in such cases.
  - ↳ Quantify this change as $\Delta\Phi_i$: let it "kill" c_i : $\hat{c}_i = c_i + \Delta\Phi_i$
  - ↳ Invent your Φ accordingly ↳ obtain low $\hat{c}_i$
  - ↳ Make sure $\Delta\Phi_i$ doesn't add much to c_i, when c_i is cheap.



Back to dynamic tables:

$$\hat{c}_i = c_i + \Delta\Phi_i$$

- (Subjectively) define what a complicated / costly operation (c_i) is.
  - ↪ whenever we insert on a full array: $c_i = \#\text{items}$
- Find something that changes a lot in the data structure in such cases.
  - ↪ size of array → try $\Phi = -\text{size}$? → $\Delta\Phi_i \sim -\#\text{items}$ ☺
    - "kills" costly c_i
    - doesn't hurt cheap c_i

⋮

but

⋮

$\Phi_i \leq 0$ ☹

$\hat{c}_i = c_i + \Delta\Phi_i \Rightarrow \sum \hat{c}_i = \sum_{i=1}^{n} (c_i + \Delta\Phi_i)$ telescoping series

$= \Phi_n - \Phi_0 + \sum_{1}^{n} c_i \geq \sum_{1}^{n} c_i$

dynamic tables: $\Phi_i = 2 \cdot (\#\text{items in table}) - (\text{size of table})$ $\left\{ \begin{array}{l} \Phi_0 = 0 \\ \Phi_i \geq 0 \end{array} \right.$

(and doesn't change rapidly) ← always $\geq \frac{1}{2}$ table

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type 1 : $c_i = 1$ (when element i doesn't trigger a doubling)

$\hat{c}_i = 1 + [2i - \text{Size}_i] - [2(i-1) - \text{Size}_{i-1}] = 3$

type 2 : $c_i = i$ (when element i does trigger a doubling)

$\hat{c}_i = i + [2i - \text{Size}_i] - [2(i-1) - \text{Size}_{i-1}]$

$= i + [2i - 2(i-1)] - [2(i-1) - (i-1)] = 3i - 3(i-1) = 3$

$\sum \hat{c}_i = 3n$

Never needed to know how many of each type, or order of operations

Summary of potential method

Amortized cost is calculated for each operation.
↳ or at least each "type" of operation

} Conclude n · worst type OR analyze further

Requires an inspired choice of function Φ
↳ typically something that changes a lot when you have those rare expensive operations.

Once you have Φ, the rest can be easy.

Find $\Delta\Phi$ once for each "type"