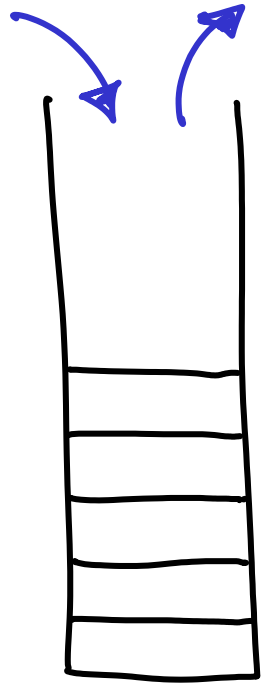
Amortization

Stack operations: push pop

Cost = 1 Cost = 1



Stack operations:

push	pop	multipop(k)
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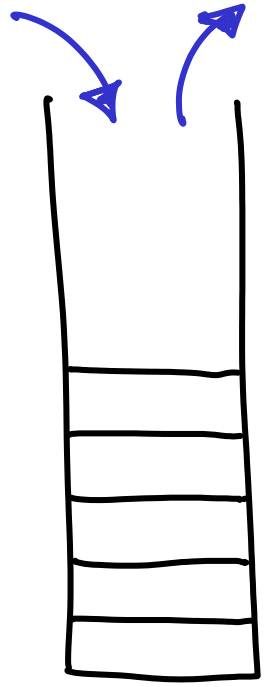

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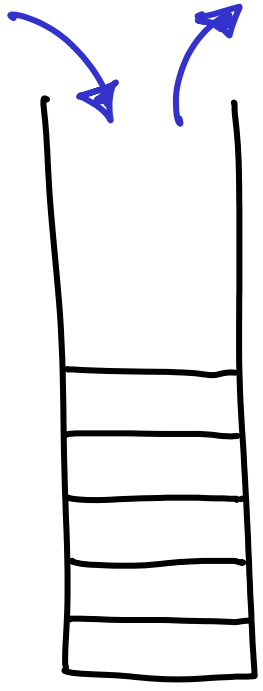

What will n operations cost ?

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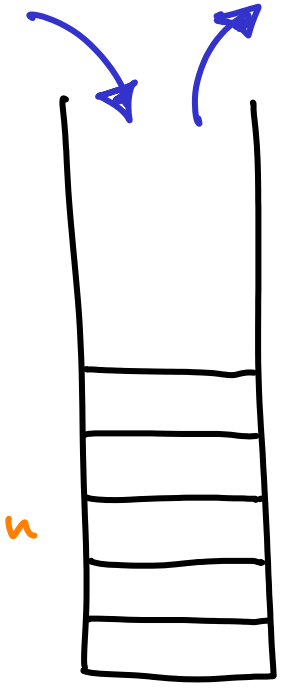

What will n operations cost ?

Worst case: n. (max individual op) = n.k = k per operation

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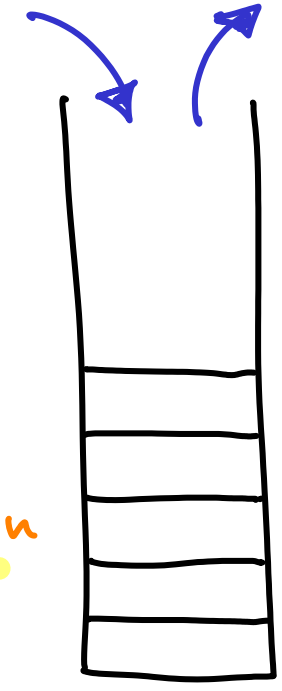

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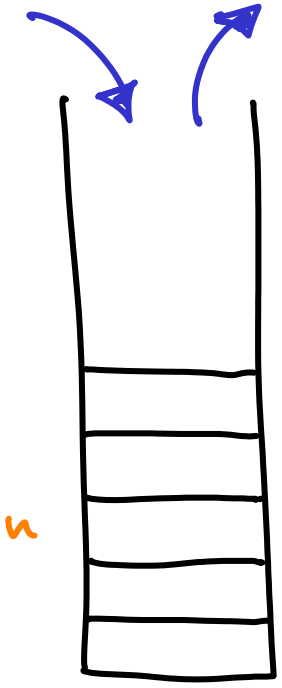

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Intuitive: each element can be pushed precisely once

but also only popped or multipopped once

Contribution per element ≤ 2

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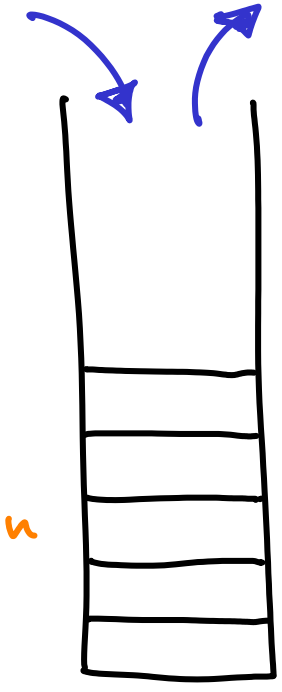

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Amortized = O(1) per operation

Intuitive: each element can be pushed precisely once

but also only popped or multipopped once

Contribution per element ≤ 2 / #elements ≤ n / Total cost = O(n)

AMORTIZATION

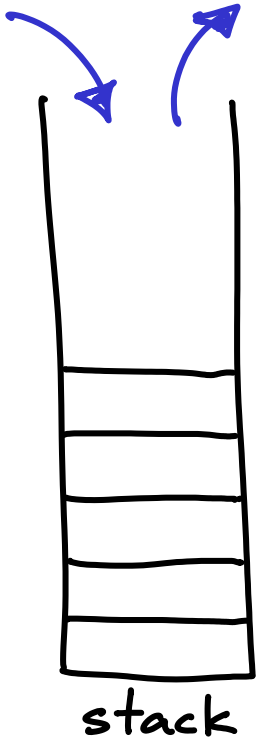
Applies to some problems that involve many operations.

If worst case time of operation k is $O(f(k))$,
try to show that n operations cost $o(n \cdot f(n))$

Stack operations: push pop multipop(k)

Cost = 1 Cost = 1 Cost = k

Pretend 2



Accounting method

- Pretend some cheap operation type costs more

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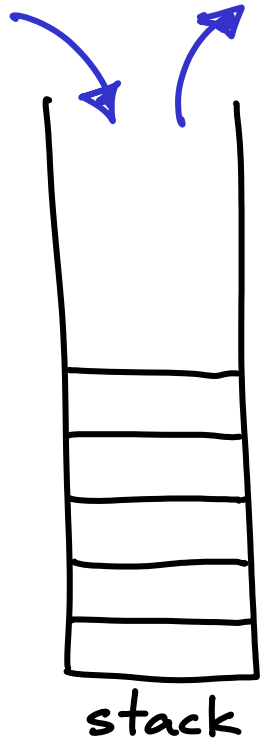
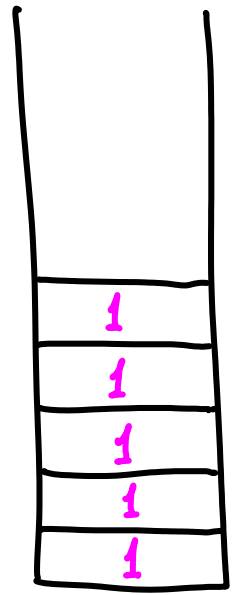
Pretend 2



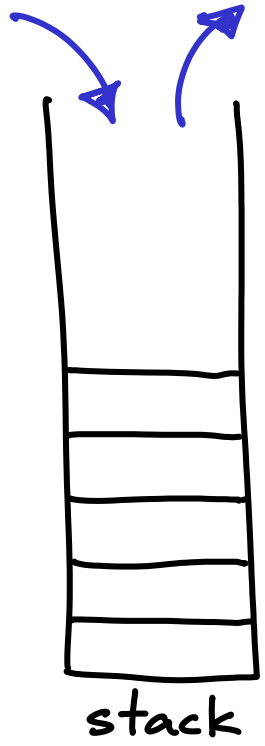
Accounting method

- Pretend some cheap operation type costs more
- Save the difference



Stack operations:

	push	pop	multipop(k)
	Cost = 1	Cost = 1	Cost = k
	Pretend 2	use savings	
Amortized cost:	2	0	0



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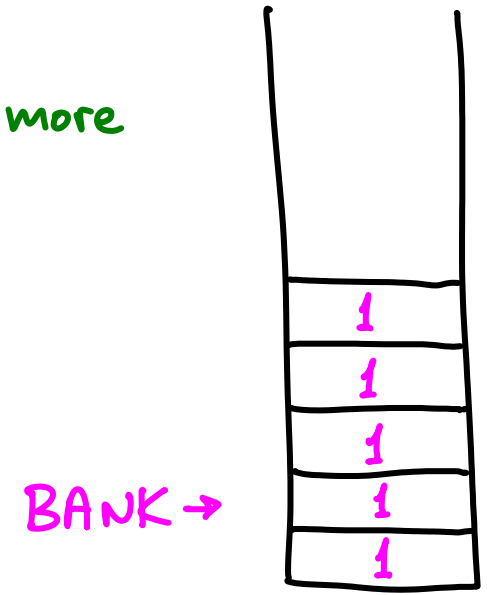

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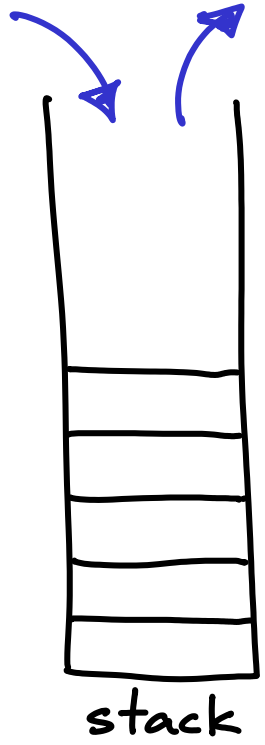

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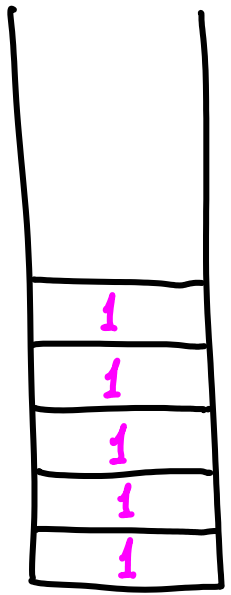

True cost $\leq n \cdot$ (worst amortized cost)

$= n \cdot 2$

ACCOUNTING : saving for a rainy day

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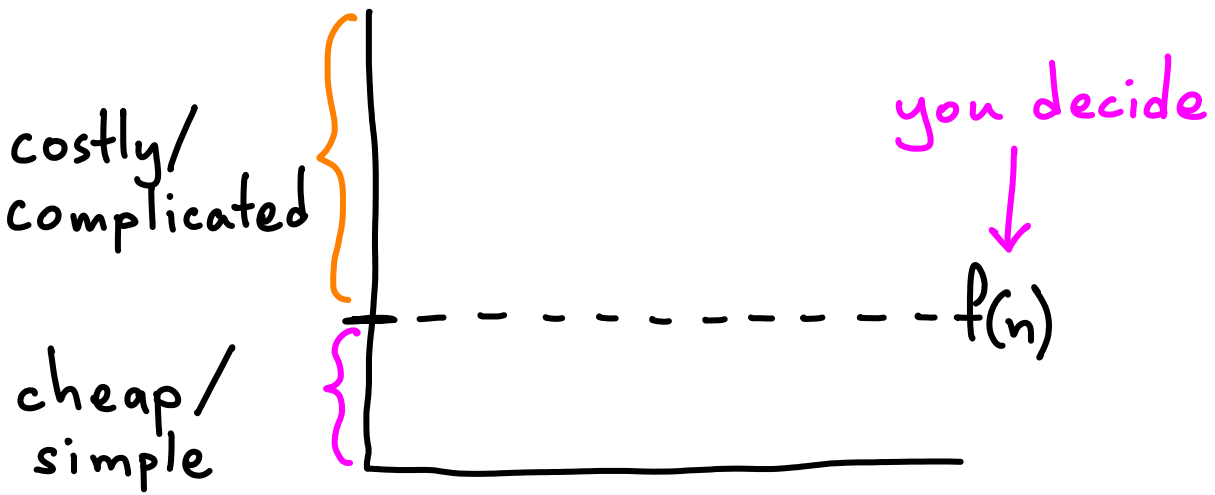
"simple" operations

"Complicated/costly" operations



ACCOUNTING : saving for a rainy day

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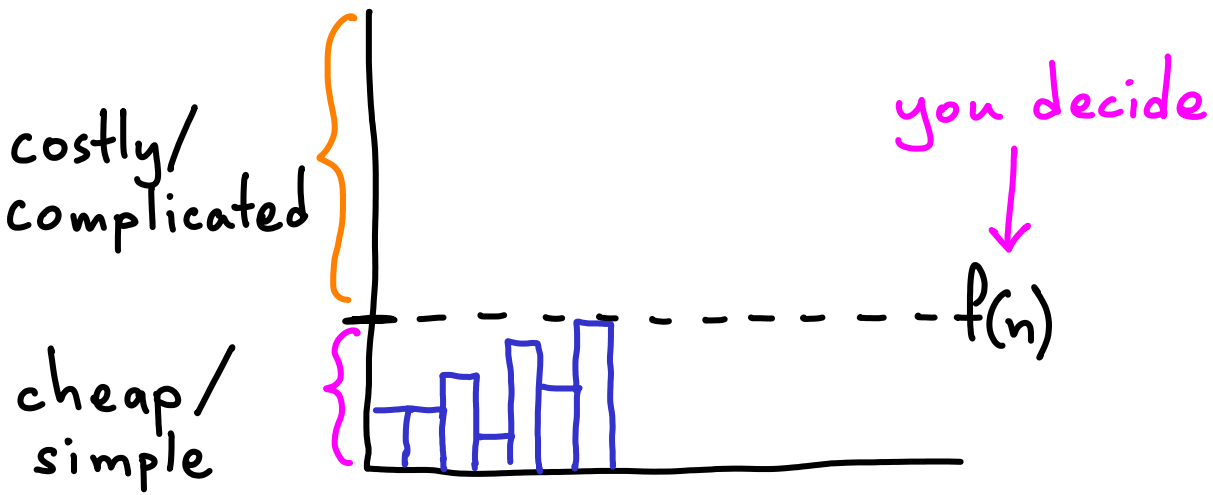

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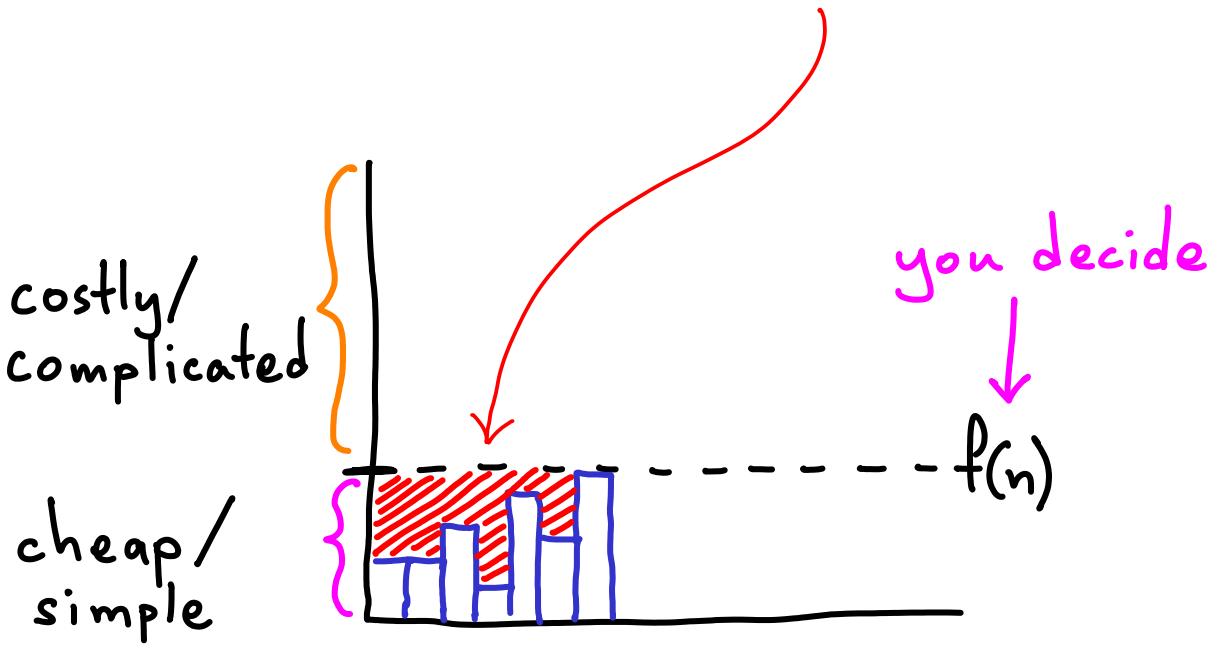
Pretend "simple" operations cost more than they do. Ideally Θ(real cost)

↳ "save" the difference



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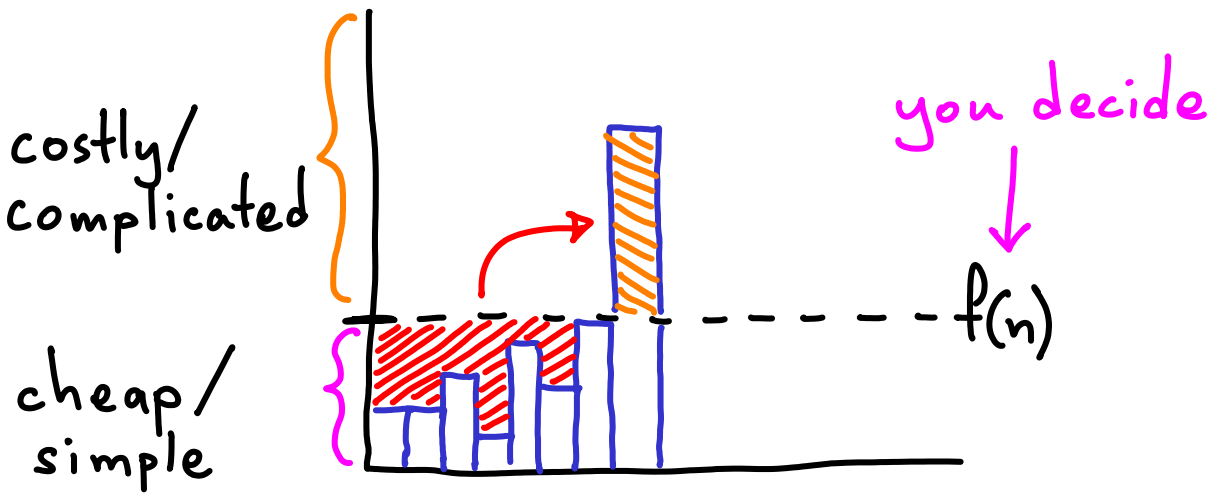
↳ "save" the difference → "spend" what you saved up.

"Complicated/costly" operations: pretend they cost less; pay excess via savings



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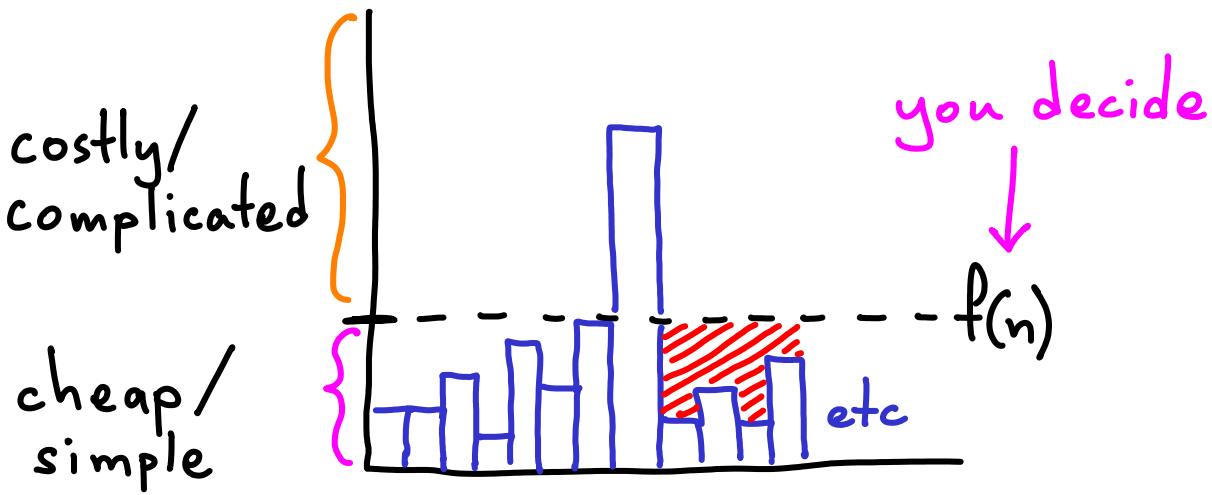

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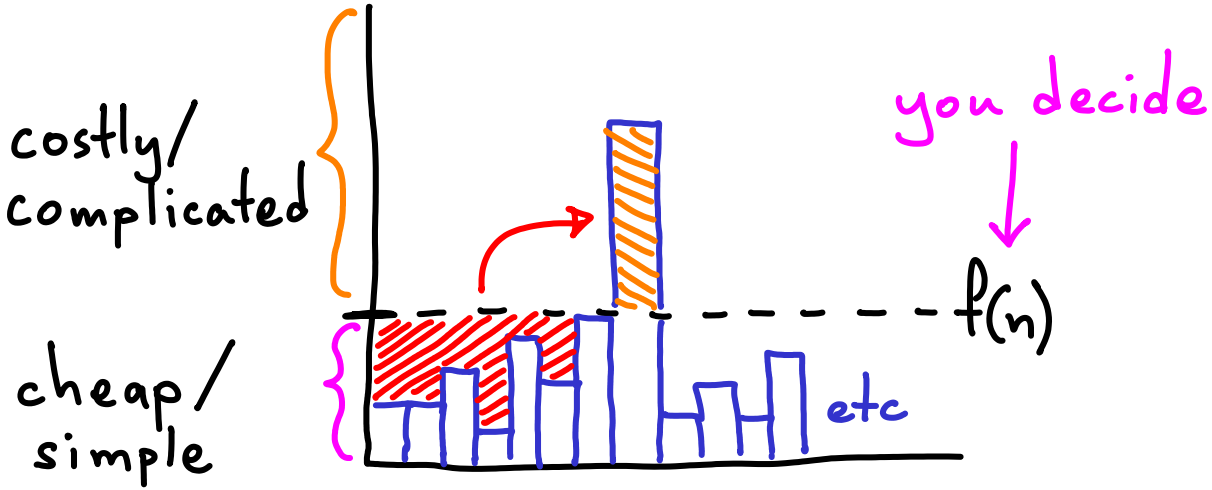

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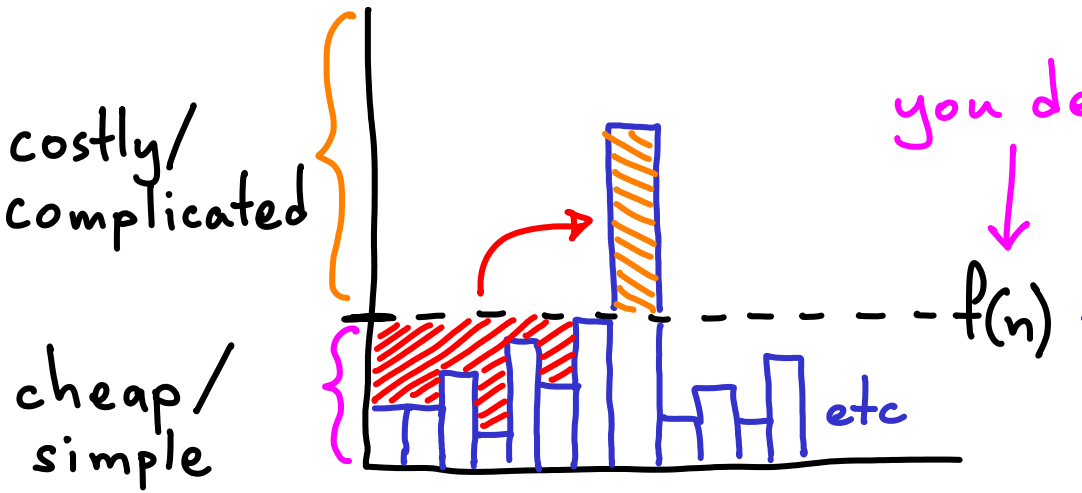

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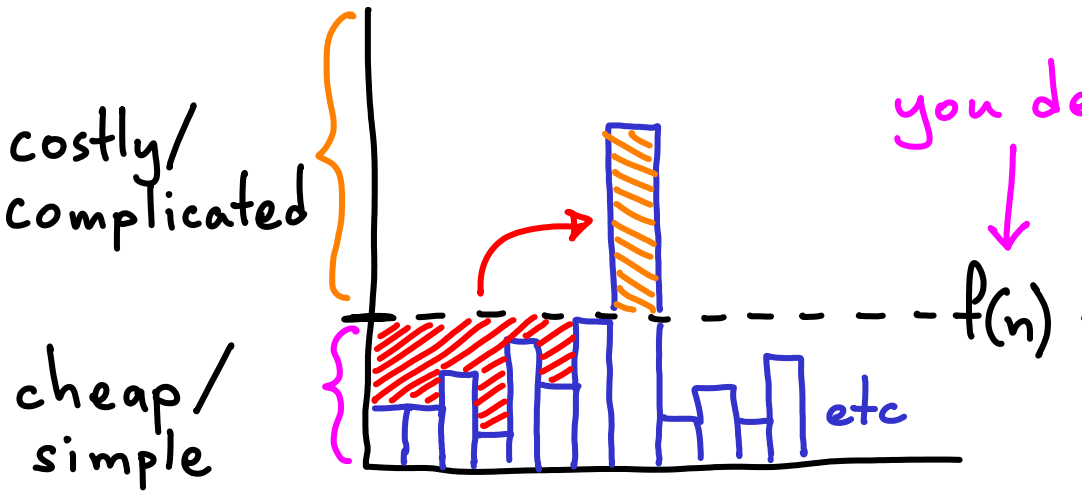

you decide

Rule: Never spend more than what you saved.

Amortized cost: $n \cdot f(n) \geq$ true cost

Goal: exaggerate / save as little as possible. } minimize $f(n)$

POTENTIAL METHOD

aka Physicist's method

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Start with data structure D_0

Operation i : $D_{i-1} \rightarrow D_i$ cost : c_i

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$\Delta\Phi_i < 0$, $\hat{c}_i < c_i$: release work.

$$\hat{c}_i = c_i + \Delta\Phi_i \quad \Rightarrow \quad \sum \hat{c}_i = \sum_{i=1}^{n} (c_i + \Delta\Phi_i) \quad = \; ?$$

$$\hat{c}_i = c_i + \Delta\Phi_i \quad \Rightarrow \quad \sum \hat{c}_i = \sum_{i=1}^{n} (c_i + \Delta\Phi_i)$$

telescoping series

$$= \Phi_n - \Phi_0 + \sum_{1}^{n} c_i$$

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($\Phi_n \geq 0$, $\Phi_0 = 0$)

so we know that the amortized cost will not underestimate real cost.

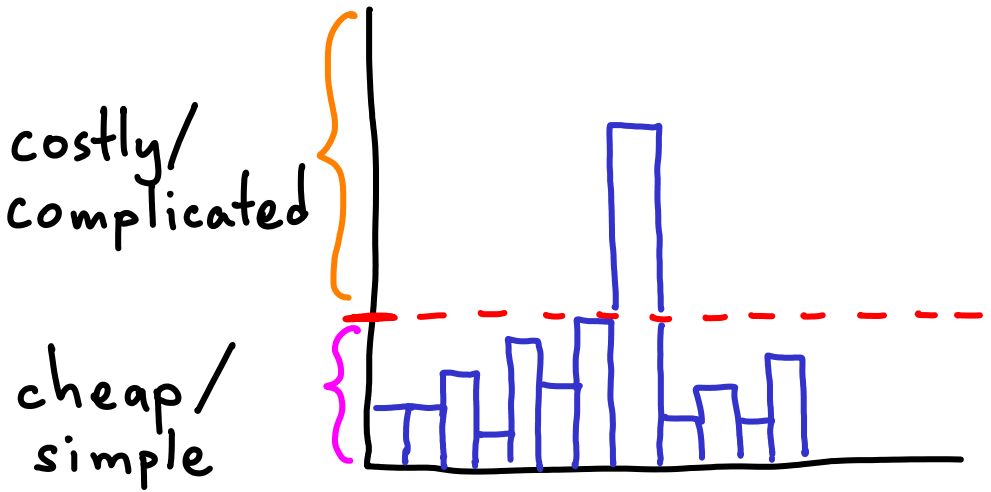
now figure out worst case for any individual $\hat{c}_i$

↳ Ideally this will give a good (and easy) bound for total cost

$$\sum c_i \leq \sum \hat{c}_i \leq n \cdot \max \hat{c}_i$$

How this works

- (Subjectively) define what a complicated / costly operation (c_i) is.



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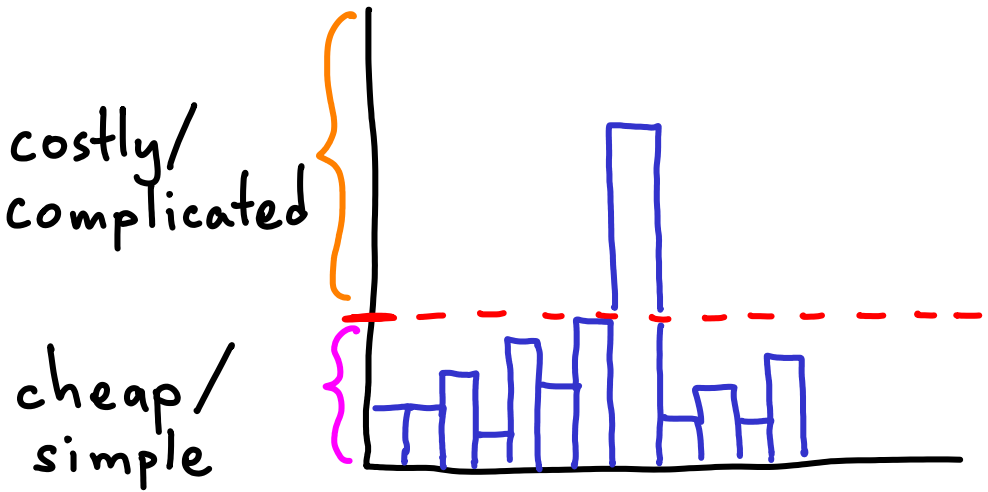

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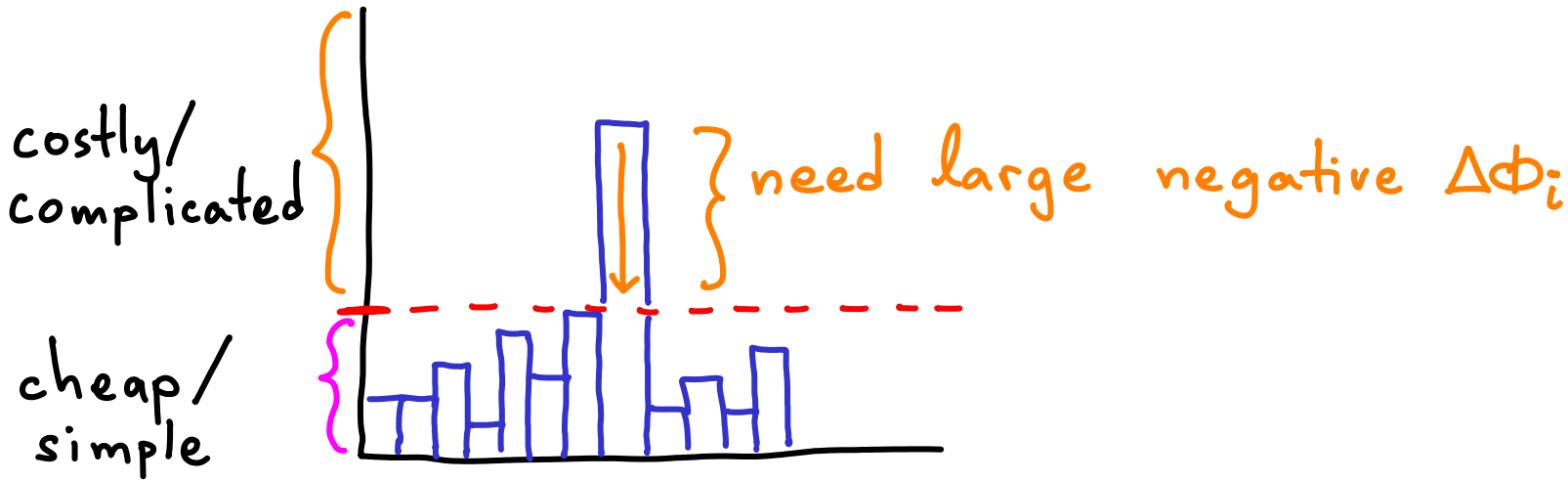

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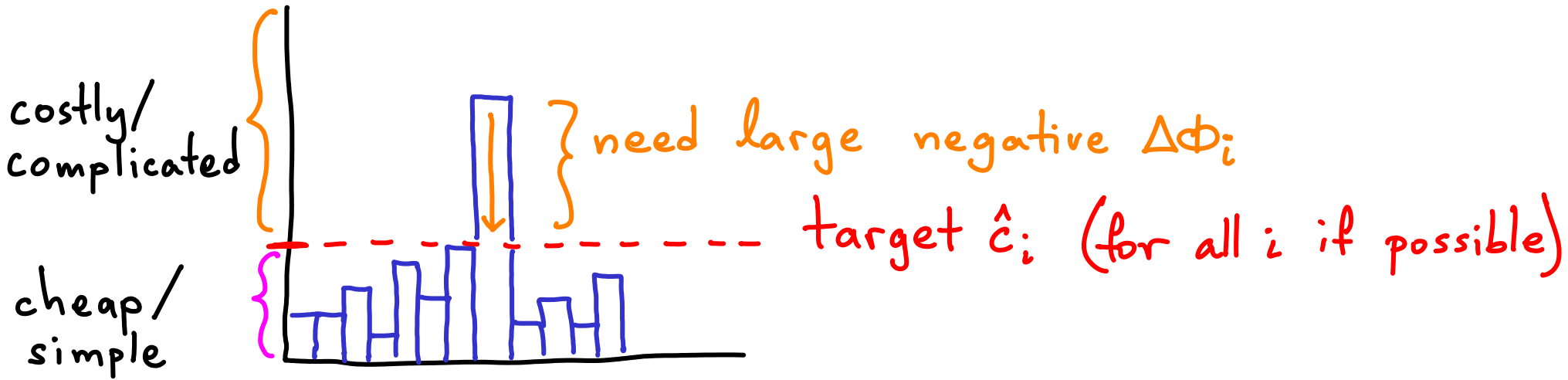

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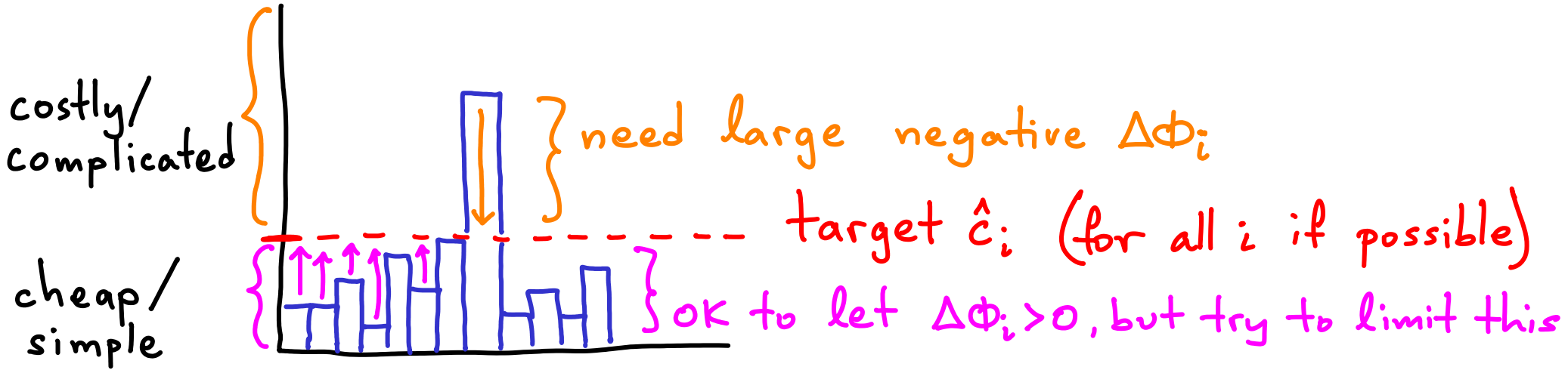
  - ↳ Quantify this change as $\Delta\Phi_i$: let it "kill" c_i : $\hat{c}_i = c_i + \Delta\Phi_i$
  - ↳ Invent your Φ accordingly ↳ obtain low $\hat{c}_i$
  - ↳ Make sure $\Delta\Phi_i$ doesn't add much to c_i, when c_i is cheap.



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Let Φ = # elements in stack

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then $\sum_{i=1}^{n} c \leq \sum_{i=1}^{n} \hat{c}_i$

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if ≥ 0 then $\sum_{i=1}^{n} c \leq \sum_{i=1}^{n} \hat{c}_i$

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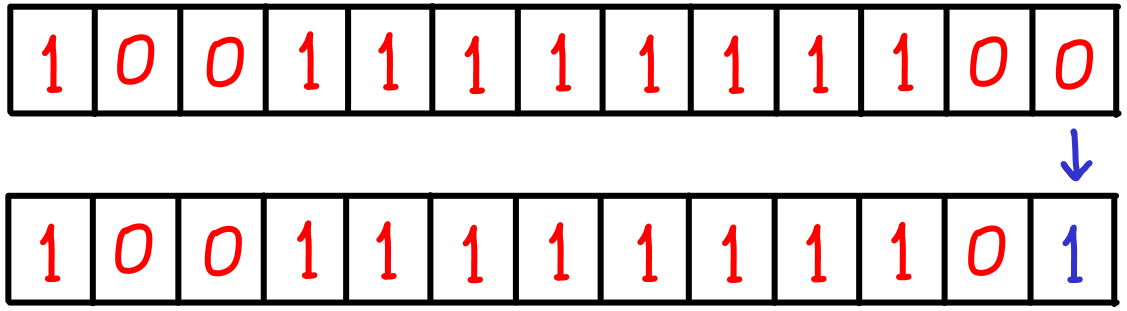
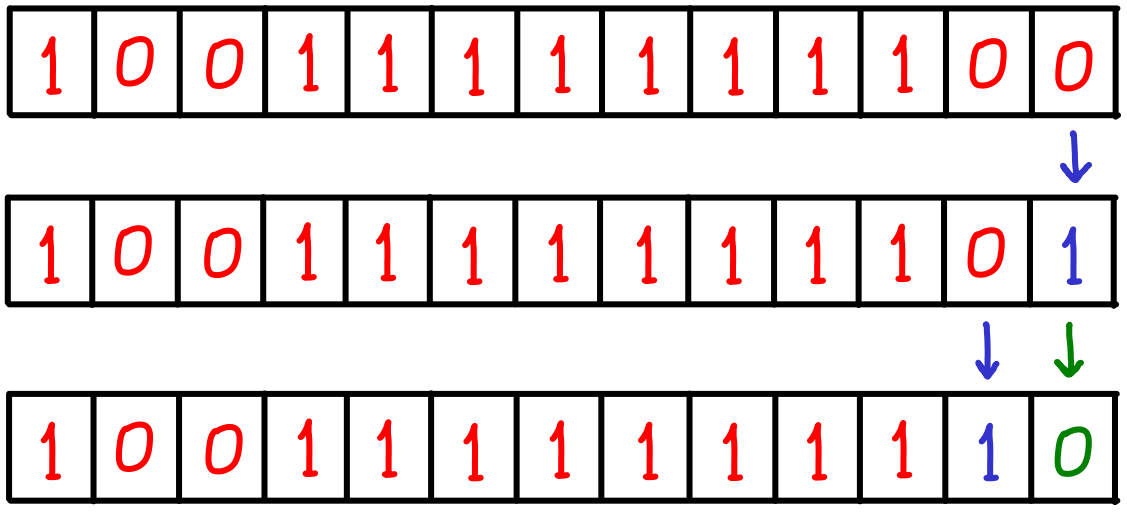
$$\sum_{i=1}^{n} \hat{c}_i \leq n \cdot \max \hat{c}_i = n \cdot 2 \rightarrow \sum_{i=1}^{n} c \leq 2n$$

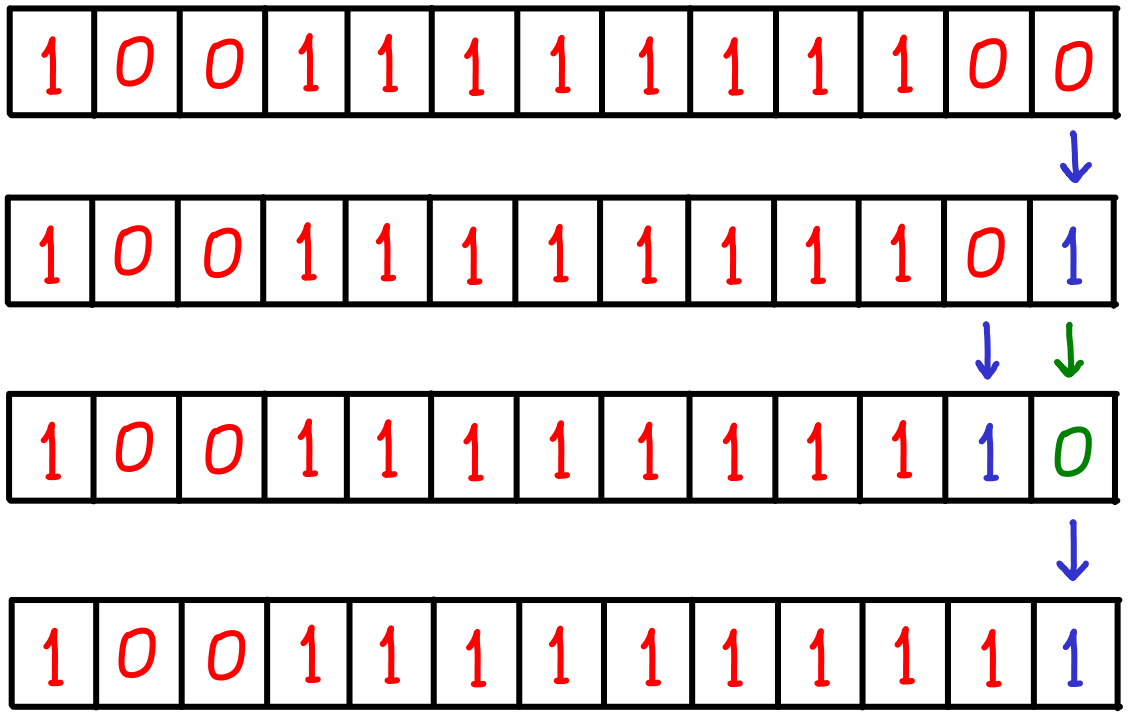
Incrementing a k-bit counter



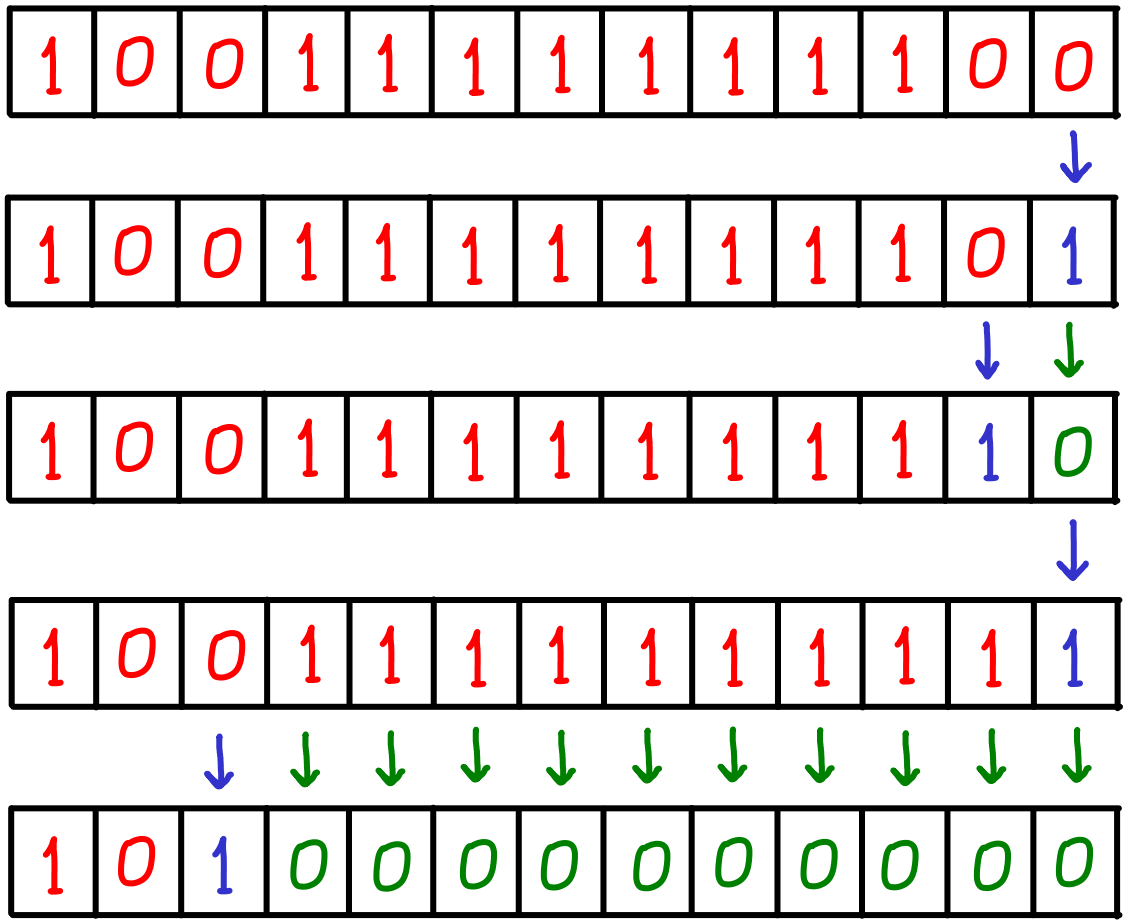
Incrementing a k-bit counter



Incrementing a k-bit counter

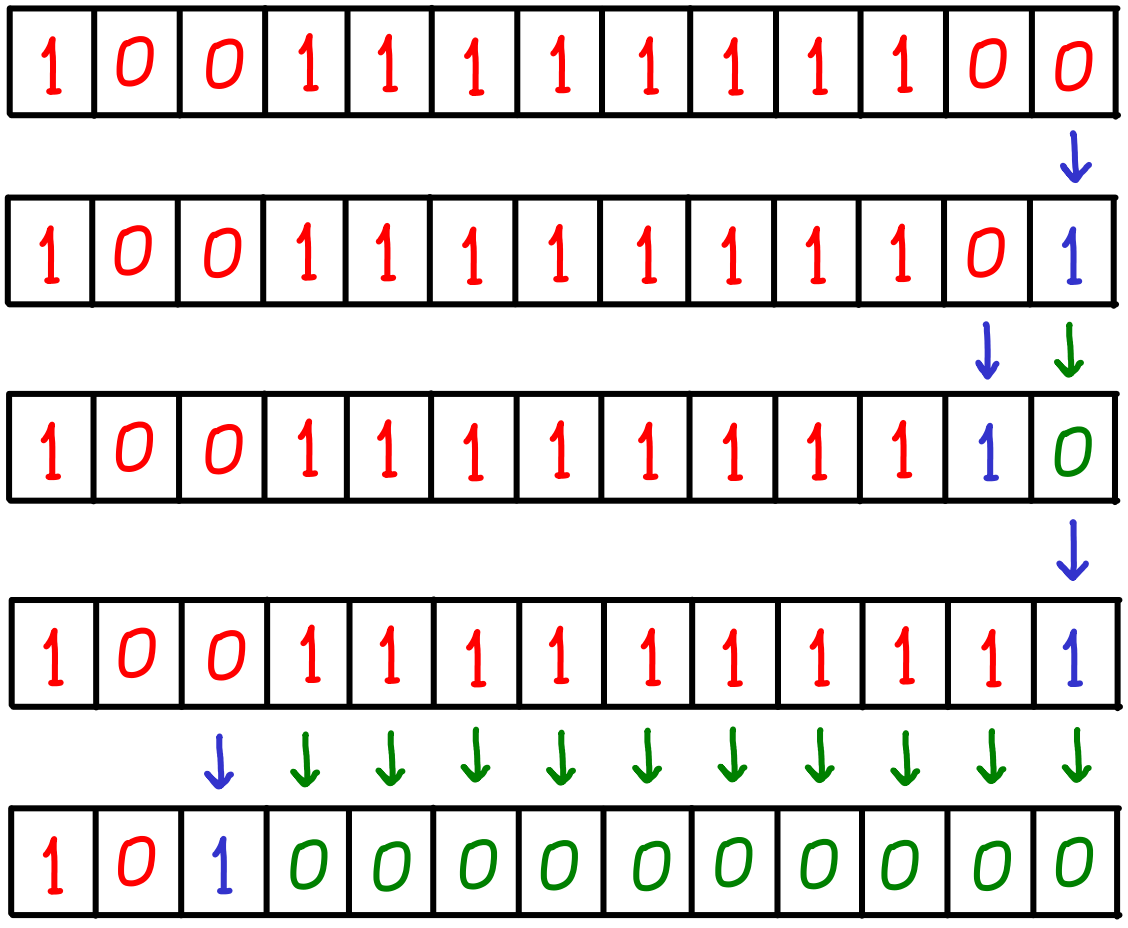


Incrementing a k-bit counter



Incrementing a k-bit counter

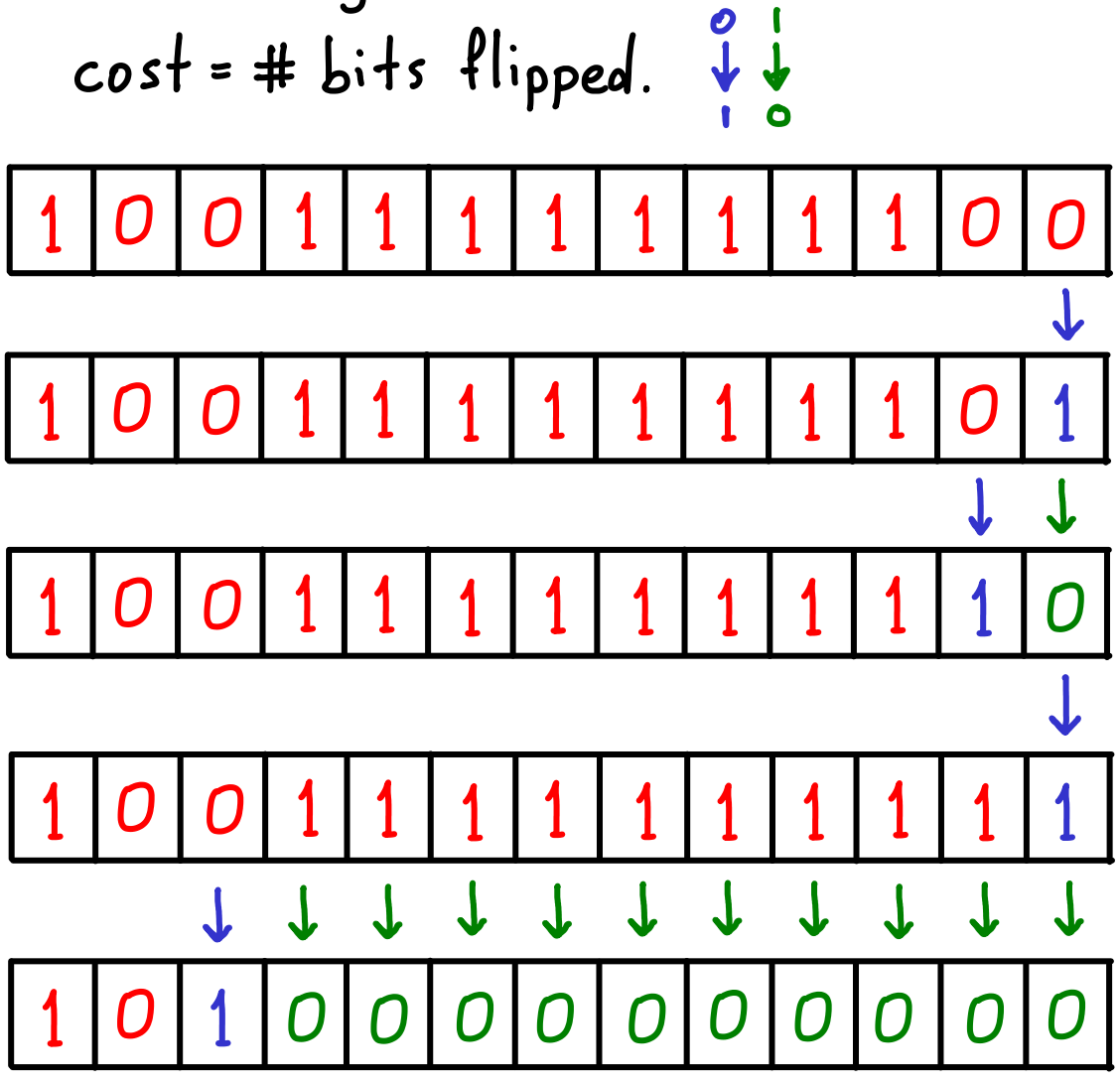
cost = # bits flipped.



Incrementing a k-bit counter: cost of n increments? (assume max = 111111...)

($< 2^k$)

cost = # bits flipped.

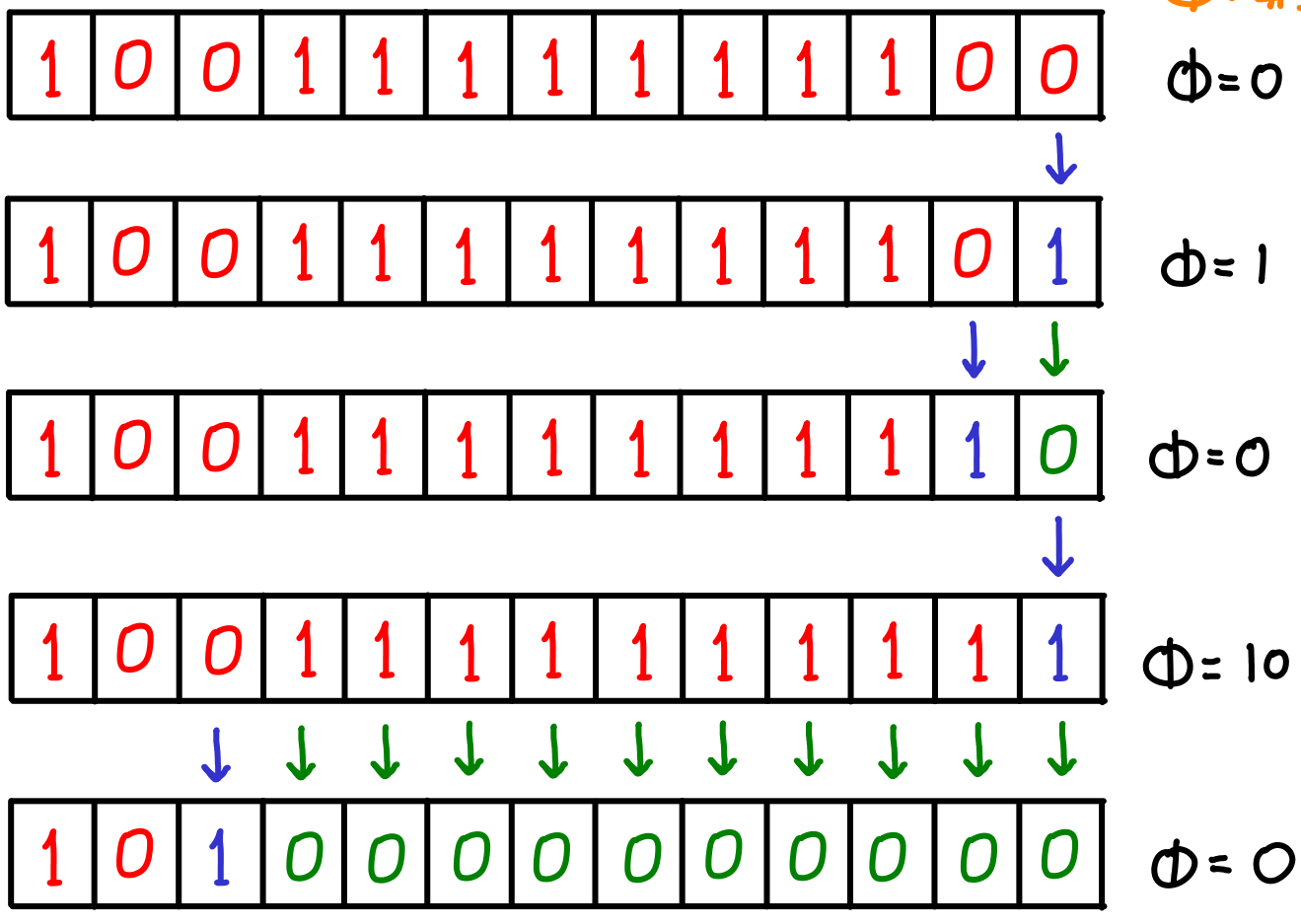


Incrementing a k-bit counter: cost of n increments? (assume max = 111111...)

cost = # bits flipped.

($< 2^k$)

Φ: #leading 1's ?



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cost = # bits flipped.

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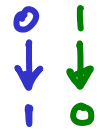
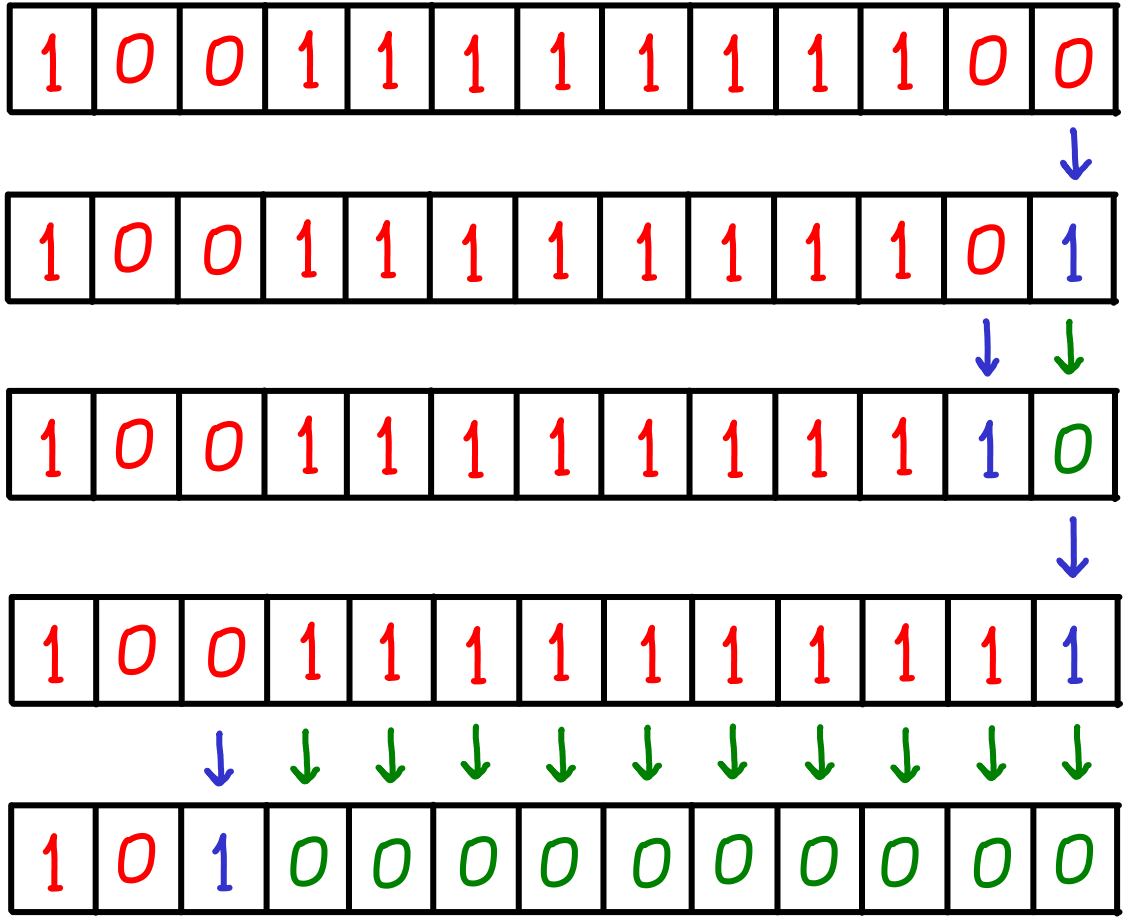


Φ: #leading 1's



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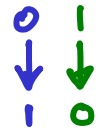

($< 2^k$)

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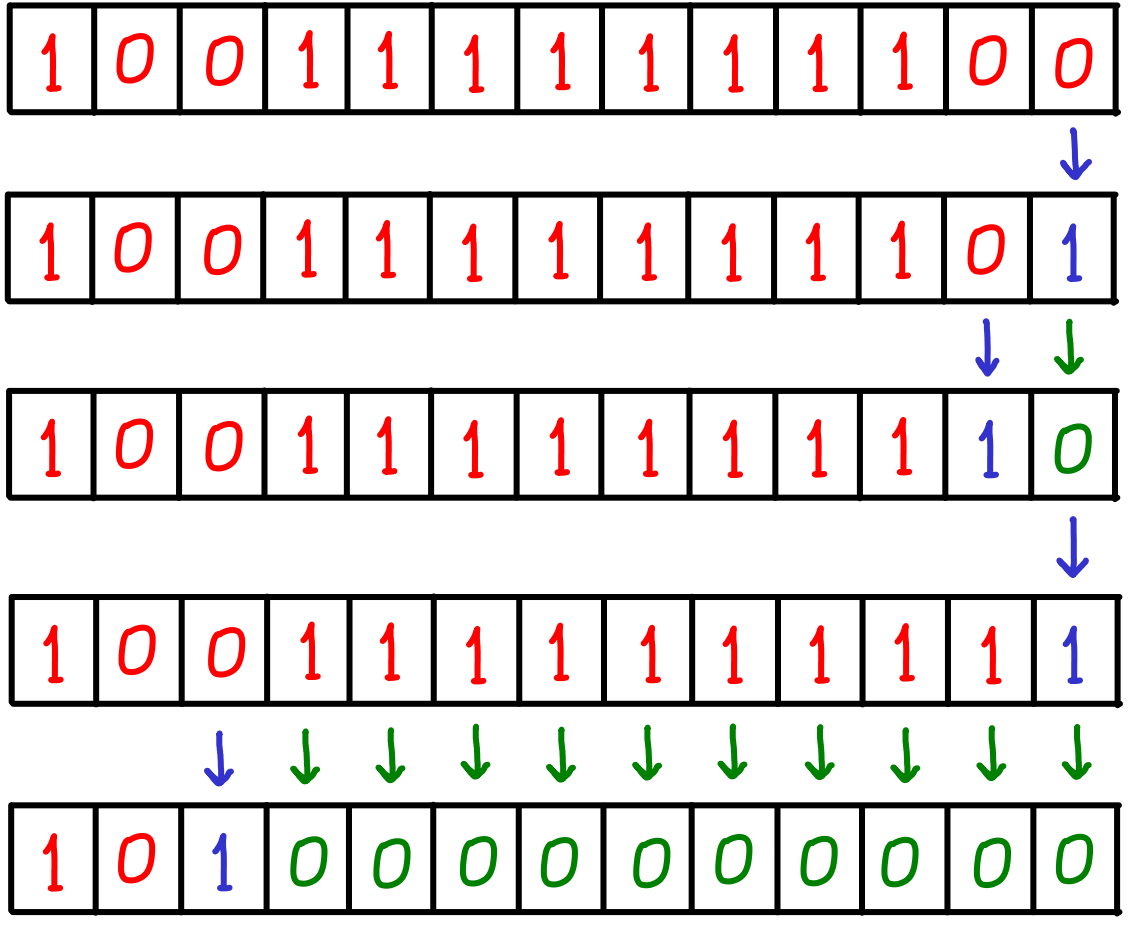

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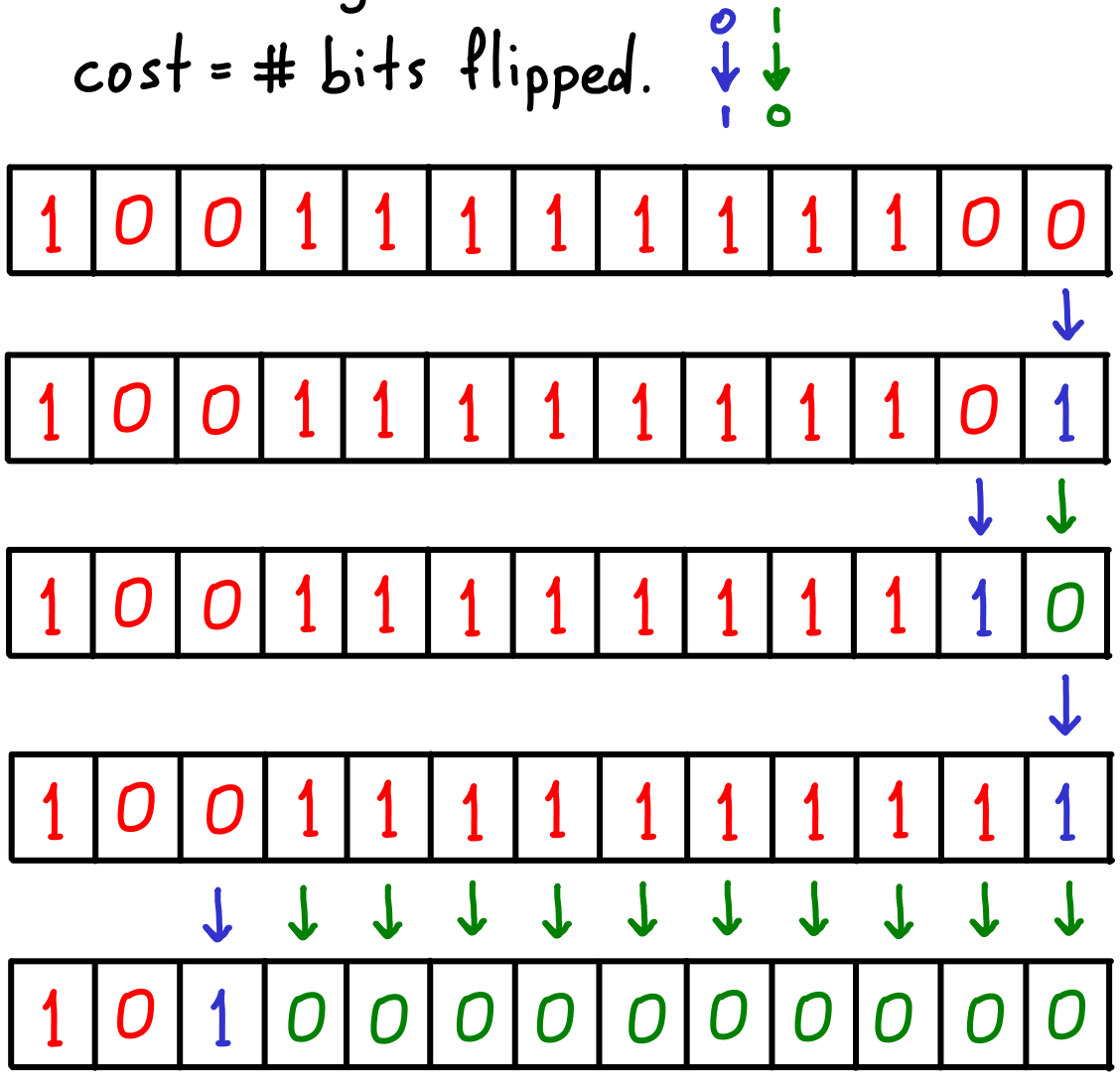
($< 2^k$)



Φ: #leading 1's

$\Phi = 0$

$\hat{c} = 1 + (1 - 0) = 2$

$\Phi = 1$

$\hat{c} = 2 + (0 - 1) = 1$

$\Phi = 0$

$\hat{c} = 1 + (10 - 0) = 11$ → k-2

☹

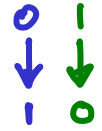
$\Phi = 10$

$\hat{c} = 11 + (0 - 10) = 1$

$\Phi = 0$

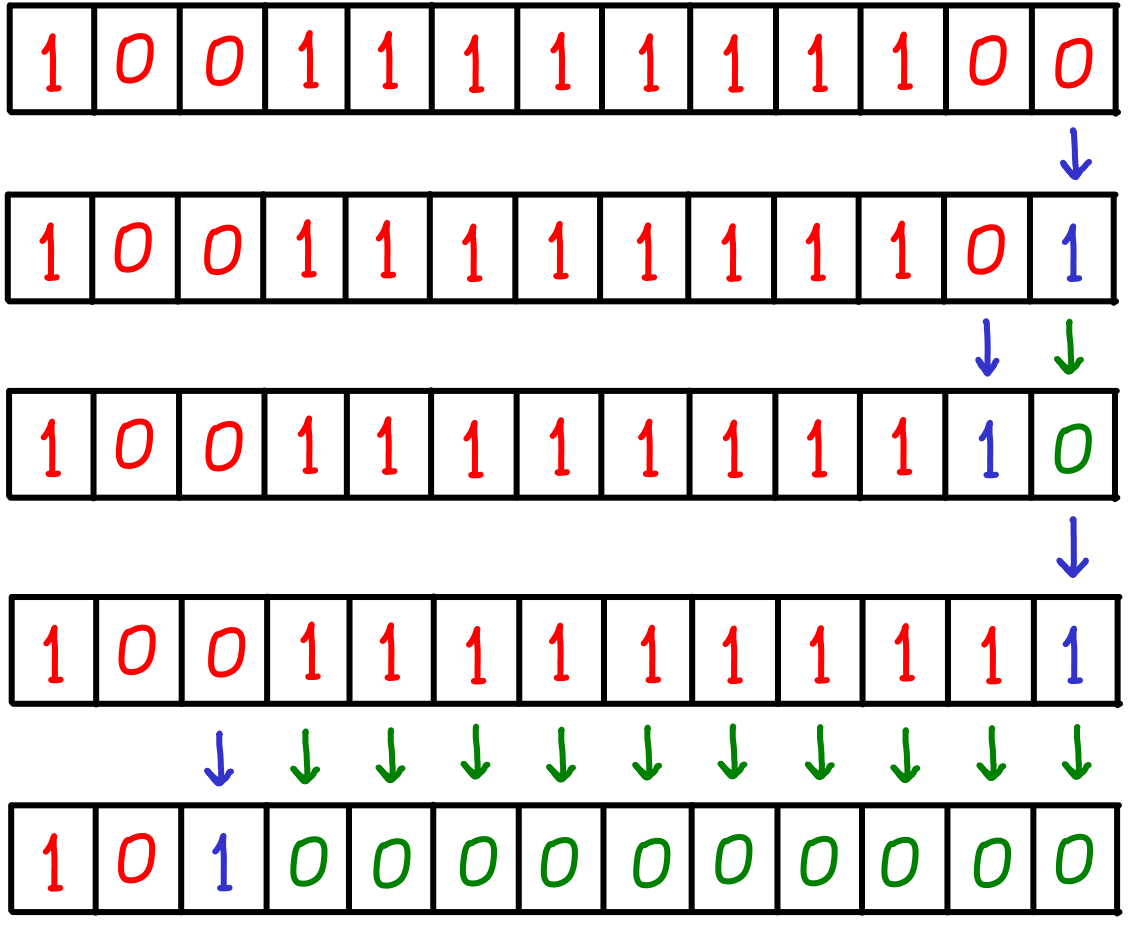
Incrementing a k-bit counter: cost of n increments? (assume max = 111111...)

cost = # bits flipped.



($< 2^k$)

Φ: #leading 1's



$\Phi = 0$

$\hat{c} = 1 + (1 - 0) = 2$

$\Phi = 1$

$\hat{c} = 2 + (0 - 1) = 1$

$\Phi = 0$

$\hat{c} = 1 + (10 - 0) = 11$ → allowed small c to grow a lot ☹

$\Phi = 10$

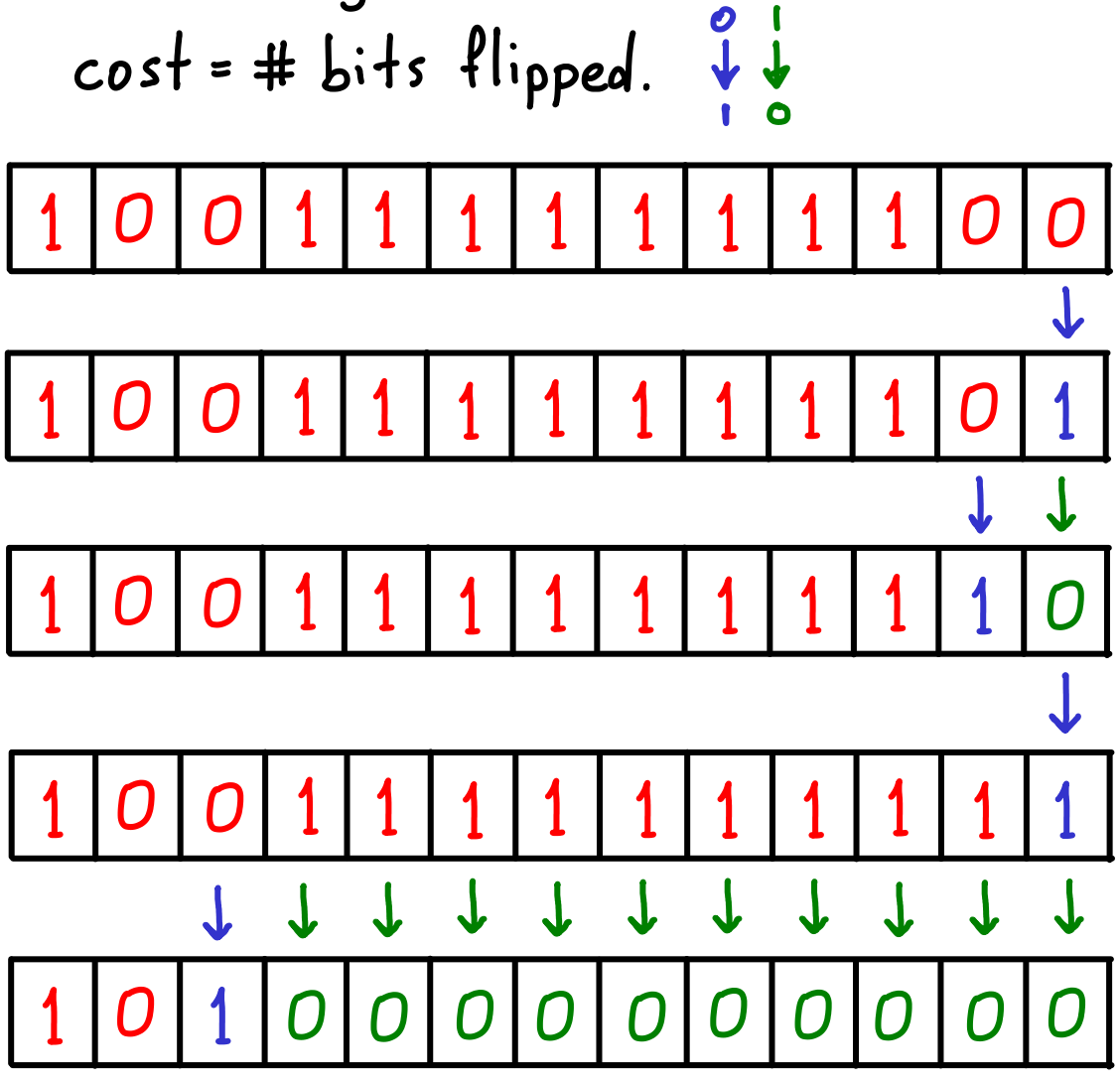
$\hat{c} = 11 + (0 - 10) = 1$ → killed large c ☺

$\Phi = 0$

Incrementing a k-bit counter: cost of n increments? (assume max = 111111...)

cost = # bits flipped.

($< 2^k$)



Φ: #leading 1's

Φ = 0

$\hat{c} = 1 + (1 - 0) = 2$

Φ = 1

$\hat{c} = 2 + (0 - 1) = 1$

Φ = 0

$\hat{c} = 1 + (10 - 0) = 11$ ☹

Φ = 10

$\hat{c} = 11 + (0 - 10) = 1$

Φ = 0

Φ = total #1's

Φ = 9

Φ = 10

Φ = 10

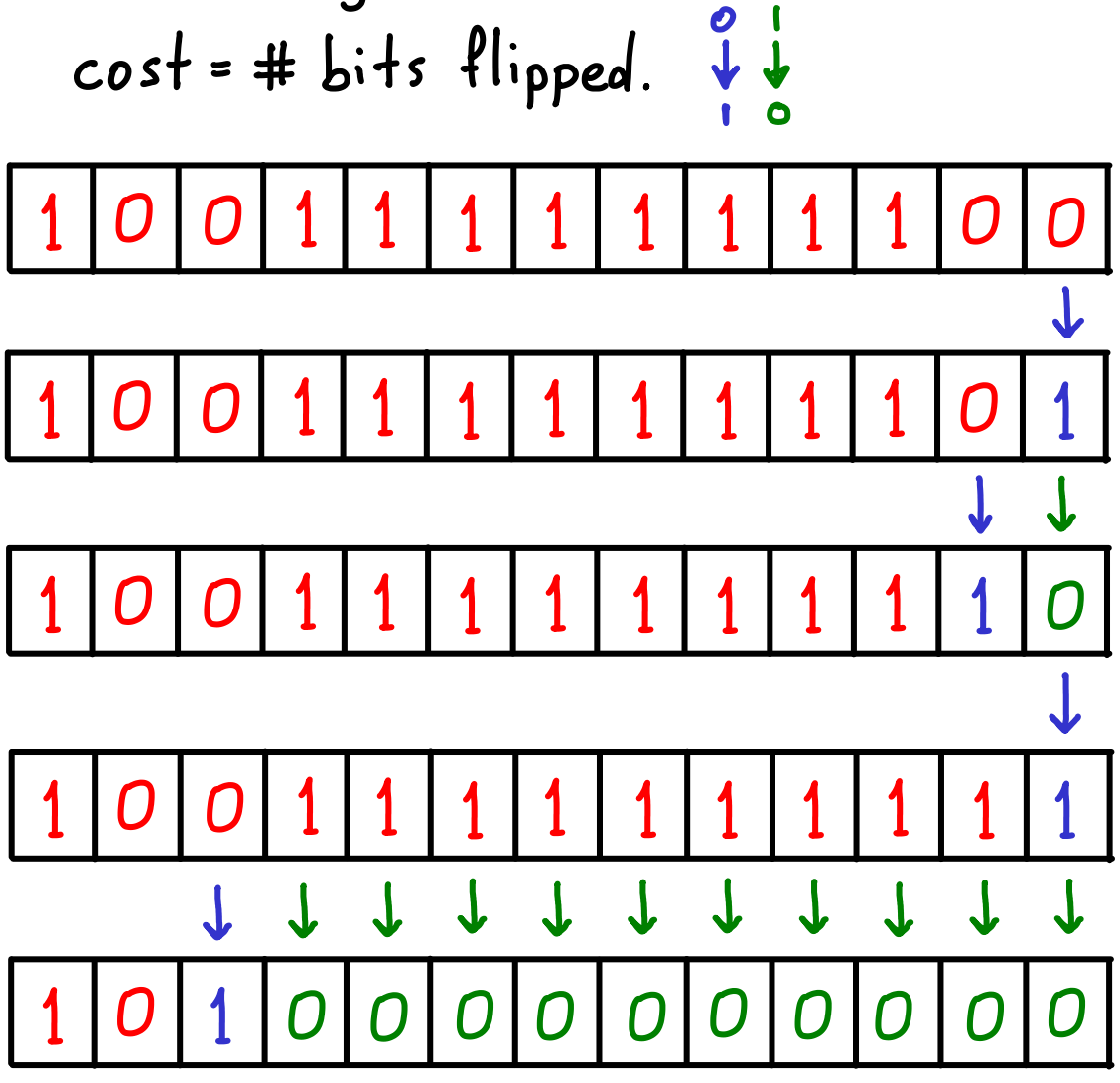
Φ = 11

Φ = 2

Incrementing a k-bit counter: cost of n increments? (assume max = 111111...)

cost = # bits flipped.

($< 2^k$)



Φ: #leading 1's

Φ = 0

$\hat{c} = 1 + (1 - 0) = 2$

Φ = 1

$\hat{c} = 2 + (0 - 1) = 1$

Φ = 0

$\hat{c} = 1 + (10 - 0) = 11$ ☹

Φ = 10

$\hat{c} = 11 + (0 - 10) = 1$

Φ = 0

Φ = total #1's

Φ = 9

$\hat{c} = 1 + (10 - 9) = 2$

Φ = 10

$\hat{c} = 2 + (10 - 10) = 2$

Φ = 10

$\hat{c} = 1 + (11 - 10) = 2$

Φ = 11

$\hat{c} = 11 + (2 - 11) = 2$ ☺

Φ = 2

Φ: #leading 1's $\longrightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

$\sum \hat{c} \leq n \cdot \max(\hat{c}) = n(k-1)$

$\hookrightarrow \leq k-1$ per increment

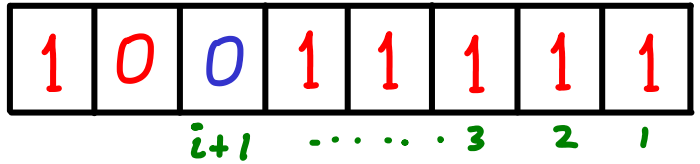
Φ: #leading 1's $\longrightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

$\sum \hat{c} \leq n \cdot \max(\hat{c}) = n(k-1)$

$\hookrightarrow \leq k-1$ per increment ☹

Φ = total #1's

First 0 at position i+1



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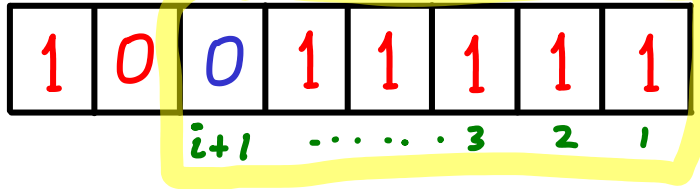
$\sum \hat{c} \leq n \cdot \max(\hat{c}) = n(k-1)$

$\hookrightarrow \leq k-1$ per increment ☹

Φ = total #1's

First 0 at position i+1

$c = i+1$



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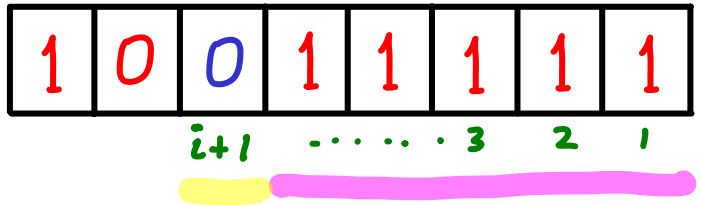

$c = i+1$

$\Delta\Phi = 1 - i$

Φ: #leading 1's $\longrightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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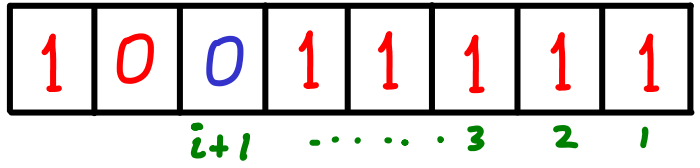

$c = i+1$

$\Delta\Phi = 1-i$

$\hat{c}$: 2 (always)

$\sum \hat{c} \leq n \cdot \max(\hat{c}) = 2n$ ☺

Φ: #leading 1's $\rightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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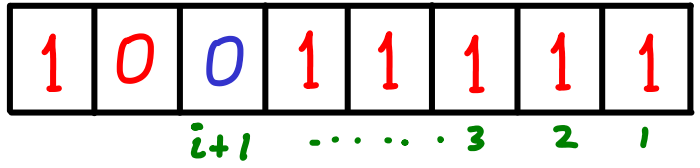

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But is $\sum c \leq \sum \hat{c}$? $\rightarrow$ what do we check?

Φ: #leading 1's $\longrightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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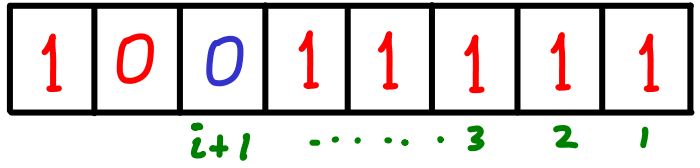
Φ = total #1's

First 0 at position i+1



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But is $\sum c \leq \sum \hat{c}$? $\left(\sum \hat{c} = \sum c + \Phi_n - \Phi_0\right)$

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Φ = total #1's

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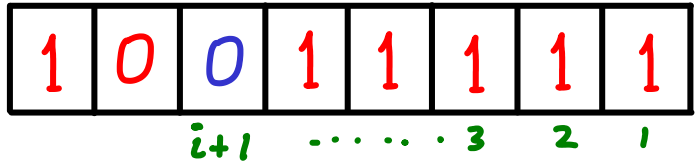

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$\Phi_n \geq 0$ but Φ_0 not necessarily 0, so $\Phi_n - \Phi_0$ could be negative ×

Φ: #leading 1's $\longrightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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Φ = total #1's

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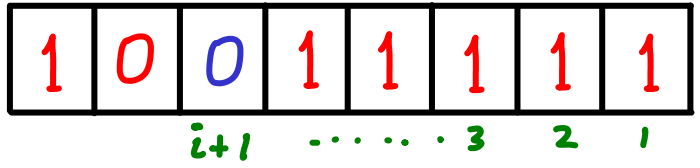

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$\Phi_n \geq 0$ but Φ_0 not necessarily 0, so $\Phi_n - \Phi_0$ could be negative ✗

$\hookrightarrow = 0$ if we start count up from zero ✓

Φ: #leading 1's $\longrightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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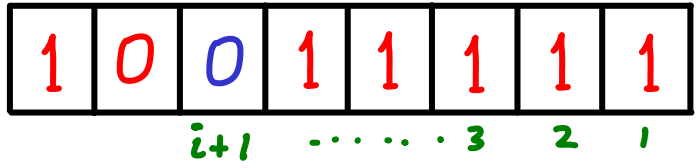
↳ $\leq k-1$ per increment ☹

Φ = total #1's

First 0 at position i+1



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$\Phi \leq k$

Φ: #leading 1's $\rightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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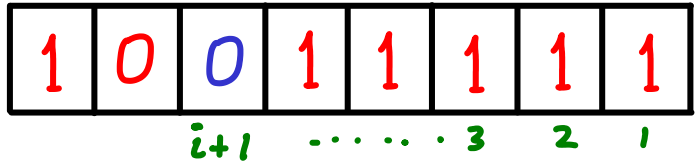

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$\Phi \leq k \rightarrow \Phi_n - \Phi_0 \geq -k$

Φ: #leading 1's $\rightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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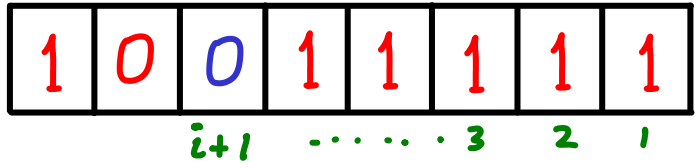
Φ = total #1's

First 0 at position i+1



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$\Phi \leq k \rightarrow \Phi_n - \Phi_0 \geq -k \rightarrow \sum \hat{c} \geq \sum c - k$

Φ: #leading 1's $\rightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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↳ $\leq k-1$ per increment ☹

Φ = total #1's

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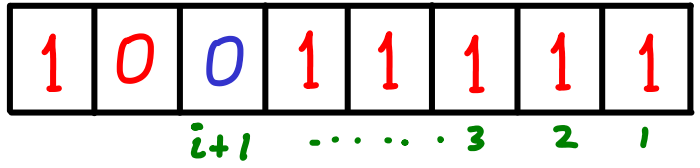

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$\Phi \leq k \rightarrow \Phi_n - \Phi_0 \geq -k \rightarrow \sum \hat{c} \geq \sum c - k \rightarrow \sum c \leq k + \sum \hat{c}$

Φ: #leading 1's $\rightarrow$ $\hat{c}$: 2, 1, 11, 1 ... (up to k-1)

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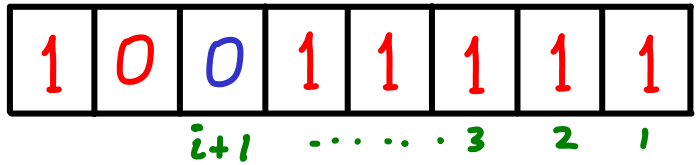
$\hookrightarrow \leq k-1$ per increment ☹

Φ = total #1's

First 0 at position i+1



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$\Phi \leq k \rightarrow \Phi_n - \Phi_0 \geq -k \rightarrow \sum \hat{c} \geq \sum c - k \rightarrow \sum c \leq k + \sum \hat{c} \rightarrow$

$\rightarrow \sum c \leq k + 2n$

Φ: #leading 1's → $\hat{c}$: 2, 1, 11, 1 ... (up to k-1) $\sum \hat{c} \leq n \cdot \max(\hat{c}) = n(k-1)$

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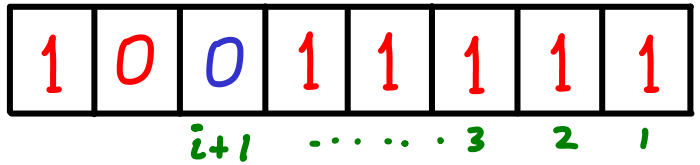
Φ = total #1's First 0 at position i+1



$c = i+1$, $\Delta\Phi = 1-i$ } $\hat{c}$: 2 (always)

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$\Phi \leq k \rightarrow \Phi_n - \Phi_0 \geq -k \rightarrow \sum \hat{c} \geq \sum c - k \rightarrow \sum c \leq k + \sum \hat{c} \rightarrow$

$\rightarrow \sum c \leq k + 2n \rightarrow$ if $n \geq k$, $\sum c \leq 3n$

Incrementing a k-bit counter

Accounting

Incrementing a k-bit counter

Accounting

Every $0 \to 1$ will cost $\hat{c} = 2$ (instead of $c = 1$)

Use the extra 1 to pay for $1 \to 0$ for the same bit (later)

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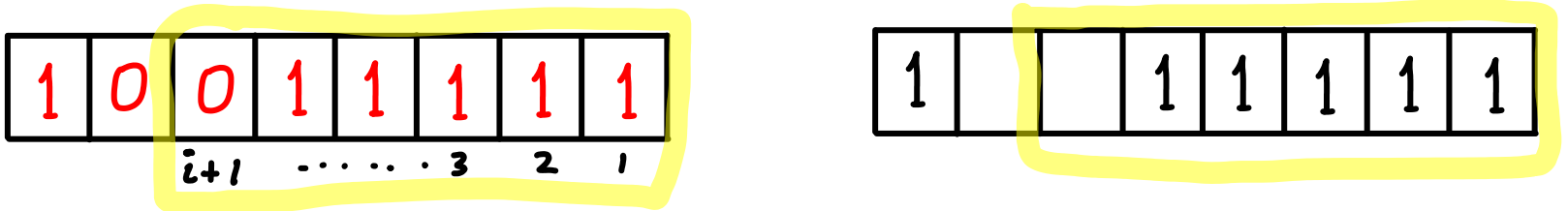
Every $0 \to 1$ will cost $\hat{c} = 2$ (instead of $c = 1$)

Use the extra 1 to pay for $1 \to 0$ for the same bit (later)

↳ $1 \to 0$ will cost $\hat{c} = 0$ ... paid for with savings



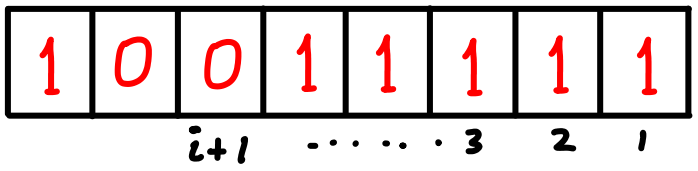
Incrementing a k-bit counter

Accounting

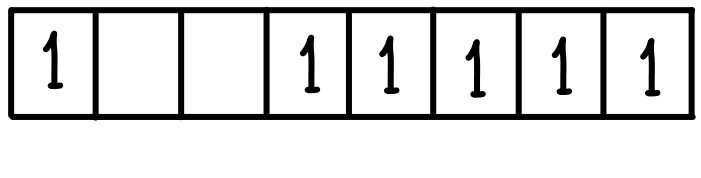
Every $0 \to 1$ will cost $\hat{c} = 2$ (instead of $c = 1$)

Use the extra 1 to pay for $1 \to 0$ for the same bit (later)

↳ $1 \to 0$ will cost $\hat{c} = 0$

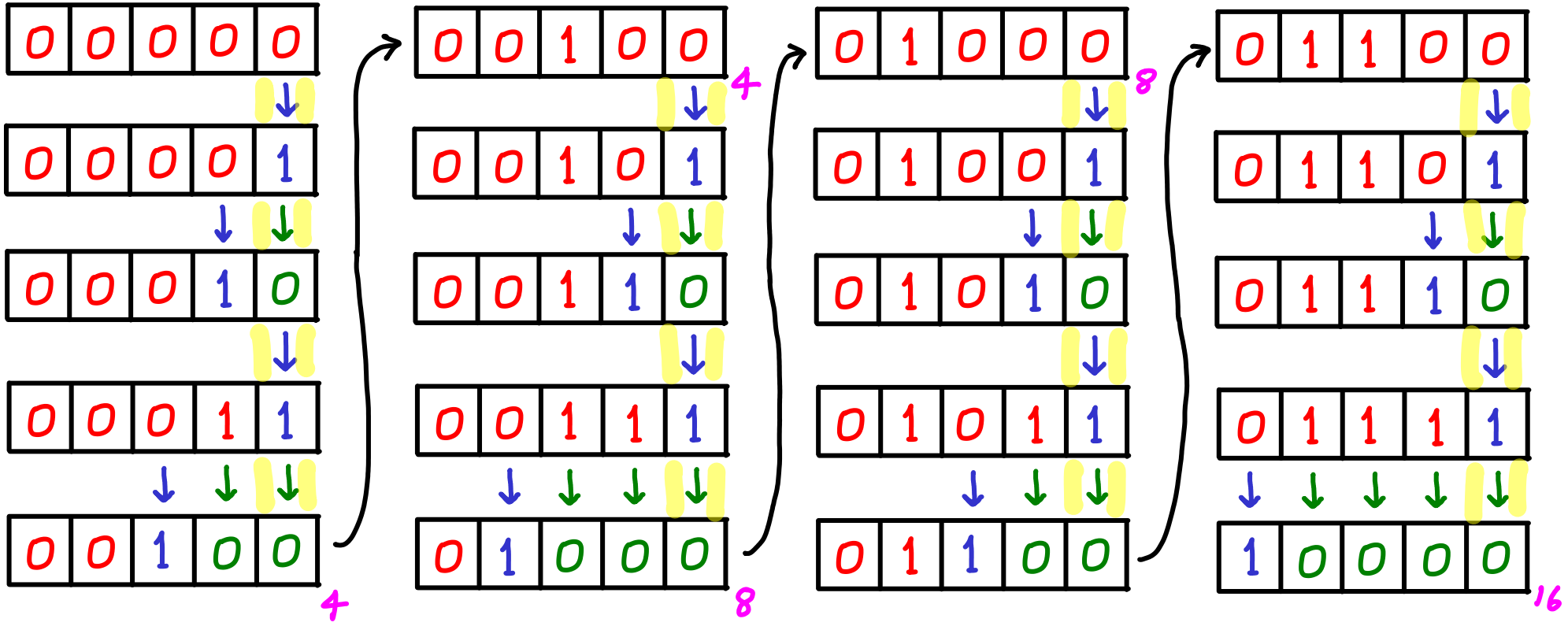


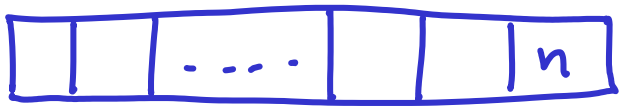
BANK



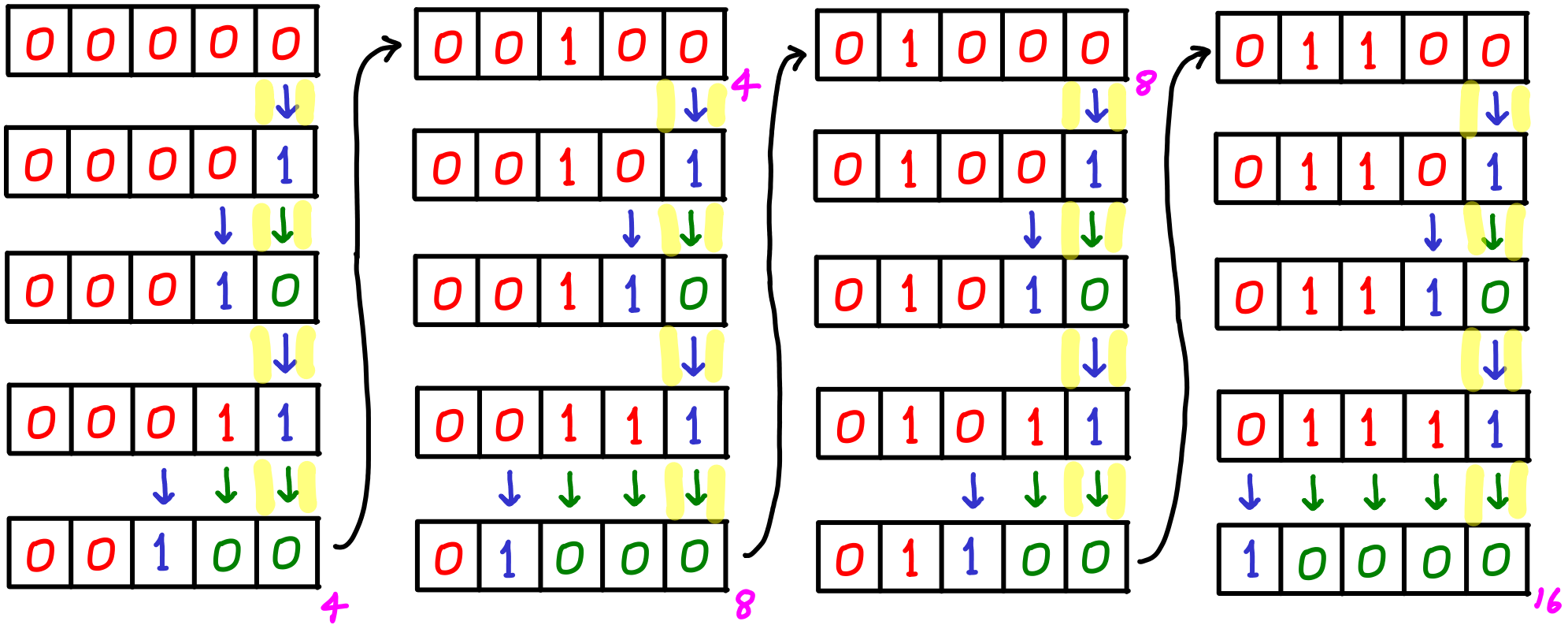
Aggregate method: Just count everything

* relies on full understanding of operation types, frequencies, costs
* not always possible

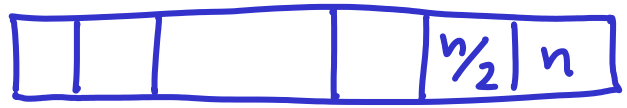


Incrementing a k-bit counter (n times) – aggregate: cost = 

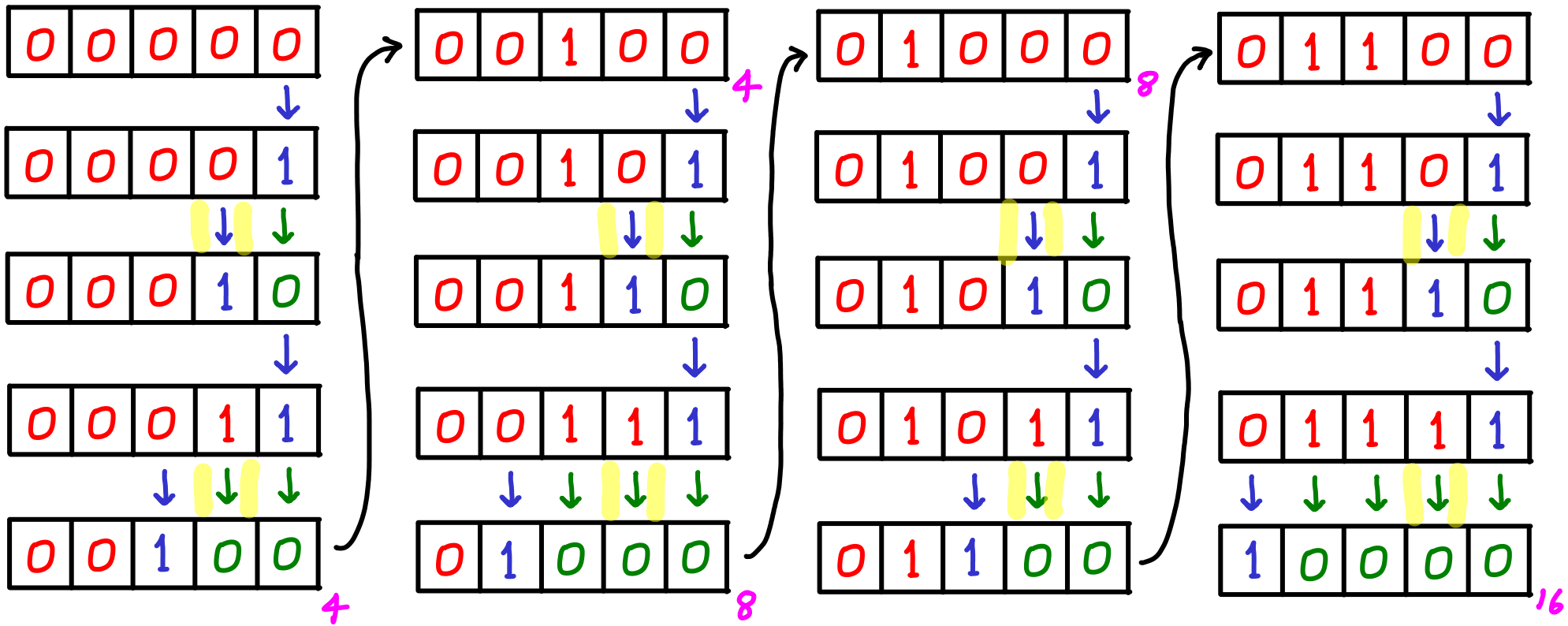
index 0 gets flipped every iteration



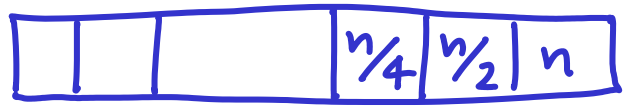
Incrementing a k-bit counter (n times) – aggregate: cost =



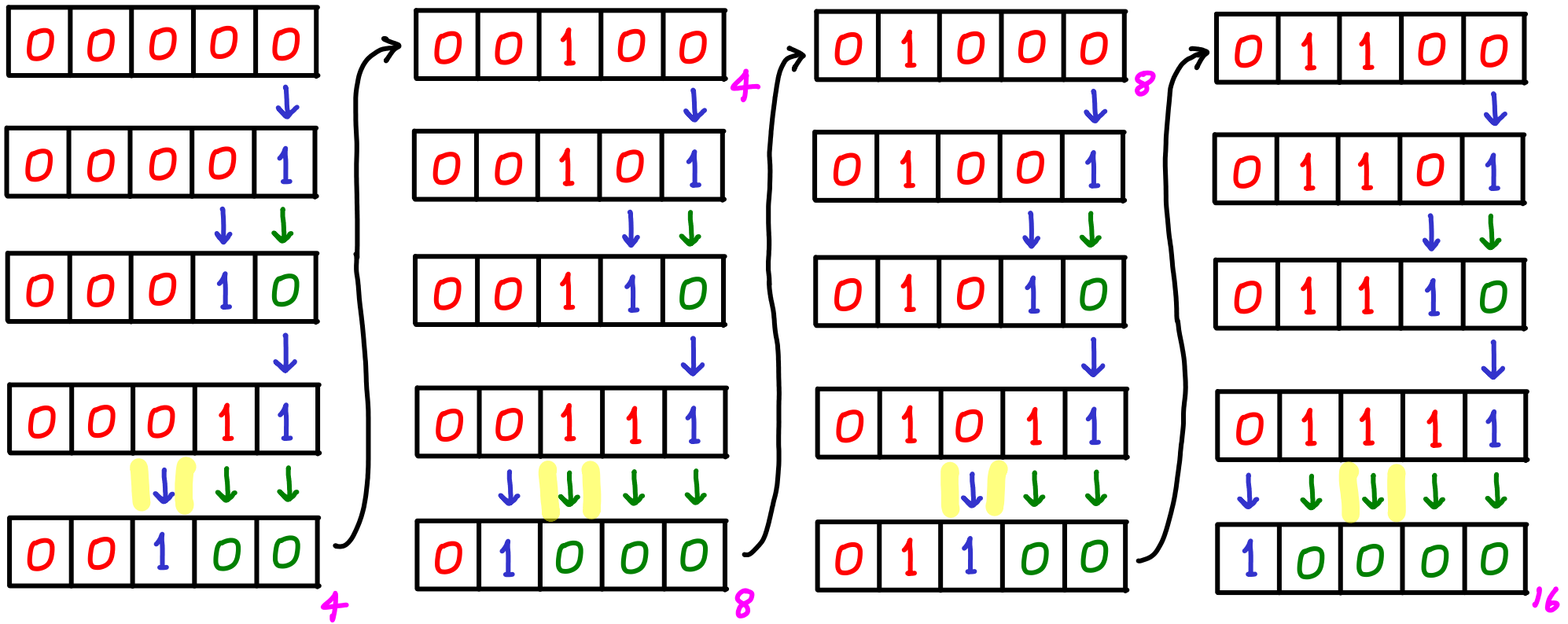
index 1 gets flipped every 2 iterations



Incrementing a k-bit counter (n times) – aggregate: cost =



index 2 gets flipped every 2^2 iterations



Incrementing a k-bit counter (n times) – aggregate: cost = $n/2^i$ | ... | $n/4$ | $n/2$ | n = $2n$

index i gets flipped every 2^i iterations: total $\sum_{i=0}^{n} \frac{n}{2^i}$