

RANDOM VARIABLES

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Think of these as functions, mapping sample space to a number.

e.g., $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \dots\}$

\vdots

$\{(6,1), (6,2), \dots, (6,6)\}$

$\left. \begin{array}{l} X \rightarrow 2 \dots 12 \\ Y \rightarrow 0, 1 \end{array} \right\}$

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$$E[X] = \frac{0 + 10 + 16 + 18 + 16 + 10}{36} \sim 1.944$$

LINEARITY OF EXPECTATION

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$$E[c_1X_1 + c_2X_2 + \dots + c_nX_n] = c_1E[X_1] + c_2E[X_2] + \dots + c_nE[X_n]$$

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independence \longrightarrow

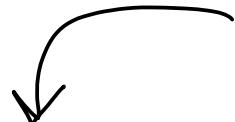
$$P(X=a \& Y=b) = P(X=a) \cdot P(Y=b)$$

for all $a, b \dots$

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1000 dice, expected value of sum $= 1000 \cdot 3.5$

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but $E[X \cdot Y] = E[X] \cdot E[Y] \rightarrow \text{NOT always true}$

If X & Y are independent, then $E[X \cdot Y] = E[X] \cdot E[Y]$

However, $E[X \cdot Y] = E[X] \cdot E[Y]$ does NOT imply
 X & Y are independent.

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(taking value 0 or 1)

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Another example: flip a coin 10 times.

$X = \#$ times we see pattern HT

$$E[X] = ?$$

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$$\begin{aligned} E[X] &= E[X_1 + X_2 + \cdots + X_9] \\ &= E[X_1] + E[X_2] + \cdots + E[X_9] \end{aligned} \quad \text{linearity of expectation}$$

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Notice X_1 & X_2 are
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$$= 9 \cdot \frac{1}{4}$$

INDICATOR RANDOM VARIABLES

The hat-check problem (a.k.a. coat-check)

- ◆ n people at a party leave their hats with an attendant
- ◆ The attendant gives hats back randomly.

How many people do we expect to get their own hats back?

The hat-check problem

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$$= \sum_{k=1}^n \frac{1}{n} = \textcircled{1}$$

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- ↳ any time you interview someone better than all previous, you hire the new person & fire the current assistant.

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$$E[X_k] = ? \quad \begin{cases} \text{person } k \text{ is hired} \\ \text{iff better than} \\ \text{all } k-1 \text{ previous} \end{cases}$$

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$$E[X_k] = \frac{1}{k} \begin{cases} \text{person } k \text{ is hired} \\ \text{iff better than} \\ \text{all } k-1 \text{ previous} \end{cases}$$

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$$= \sum_{k=1}^n \frac{1}{k} \leq 1 + \ln n$$

linearity of expectation

$$E[X_k] = \frac{1}{k}$$

person k is hired
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The birthday problem

How many people do we need in a room so that
we expect to have (at least) one birthday match?

The birthday problem

$X = \# \text{ birthday matches among } n \text{ people}$

For what n do we get $E[X] \geq 1$?

↳ Set up $E[X]$ as function of n

The birthday problem

$X = \#$ birthday matches among n people

What should our I.R.V. be ? $X_?$

The birthday problem

$X = \#$ birthday matches among n people

$X_{ij} = ?$

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$X = \# \text{ birthday matches among } n \text{ people} = ?$

$$X_{ij} = \begin{cases} 1 & \text{if persons } i \& j \text{ match} \\ 0 & \text{otherwise} \end{cases}$$

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all $\binom{n}{2}$ pairs

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□

$P(\geq 2 \text{ people in a group of } k \text{ have same birthday})$

(new problem)

Not about
expected value or IRV

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Person #1

$$= 1 - \frac{365}{365}$$

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Person #1 Person #2

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(Person #1) (Person #2) (Person #3)

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Person #1 Person #2 Person #3 ... Person #K

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Person #1 Person #2 Person #3 ... Person #K

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$(k > 365 \rightarrow P=1)$

