Edge Coloring Using a Constructive Proof of Vizing's Theorem

Samuel Rabinowitz

August 7, 2020

1 Introduction

Edge coloring is a useful and practical application of computational hardware to solve real world problems. Many systems that have limited amounts of resources stand to benefit from applications of edge coloring, which will constrain the amount of time the system needs to perform a given set of tasks. Consider a factory where we have some set amount of machines and products that need to be processed by these machines. Each machine can process only one product at a time, and each product can only be operated on by one machine at a time. If we make a graph where products and machines are vertices and edges between products and machines represent how we need that machine to operate on that product, an optimal edge coloring will allow us to limit the amount of time necessary to complete a production cycle by correlating each edge color with a time. The critical piece of information in this case is how many edge colors we need. Where Δ is the maximum degree of any vertex in our graph, we can trivially say that we will need at least Δ colors. Vizing's Theorem allows us to limit the number of colors we will need to $\Delta + 1$. This paper will present a proof of Vizing's Theorem (see last page for source) with detailed explanations and illustrations. To accompany the paper, I provide an implementation of an edge coloring algorithm in Python 3 that uses this proof as framework for constructing an edge coloring of a given graph.

2 Introduction to the Proof

For a simple graph G, we want to pick a vertex v where v and all of its neighbors have degree at most k and only one neighbor can actually have degree k. If $k = \Delta(G) + 1$, then all vertices in G have degree less than k, so we can pick any vertex as v in this case. We want to prove that if $G - v$ is k edge colorable, so is G. Our inductive hypothesis is that if $G - v$ is $k - 1$ edge colorable, so is G'. Note that we are performing induction on k, and not on edges or vertices. This proves Vizing's Theorem because once we are able to show that $G - v$ being

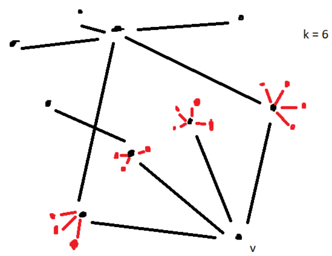

Figure 1: Red vertices are dummies

k-edge colorable implies that G is k-edge colorable, we can perform a second induction on the vertices of G so that we can find a k-edge coloring of G.

3 Proof

Our inductive hypothesis is that for any simple graph G, if $G - v$ is $k - 1$ edge colorable, then G is $k - 1$ edge colorable. So, we take a simple graph G where $G - v$ is k-edge colorable. We want to prove that we can add back v and still have a k-edge colorable graph. This would allow us to claim that G is $\Delta(G) + 1$ edge colorable, which would prove Vizing's Theorem.

For our inductive case, we pick any v in G. We know that if $k = \Delta(G) + 1$, we can say that both v and all neighbors u of v have degrees less than k. That means that we can make all the neighbors of v have degrees of exactly $k - 1$ save for one neighbor which we will give degree k, by adding new vertices to our graph and connecting these vertices to the neighbors until we have the desired result (fig. 1).

We know that we can k-color $G - v$, so pick a coloring and label each of the k colors with the numbers $i = 1, \ldots, k$. X_i will be the set of vertices that neighbor v but are not adjacent to an edge with color i. For each neighbor of v with degree $k - 1$ in G, in $G - v$ that vertex has degree $k - 2$, so it must be missed by two colors in our k coloring of $G - v$, and thus be in two X_i's. For the one neighbor of v with degree k in G, that neighbor has degree $k - 1$ in $G - v$ so it is in one X_i. That means $\sum_{i=1}^{k} |X_i| = 2 * degree(v) - 1 < 2k$ (since $degree(v) \leq k$).

There may be many ways to k color $G-v$, but we want to make sure that we use the coloring that minimizes $\sum_{i=1}^{k} |X_i|^2$'s. This minimization should ensure that $|X_a|$ and $|X_b|$ are within 2 of each other. That is because if we consider a coloring where $|X_a| > |X_b| + 2$, we could hypothetically decrease $|X_a|^2 + |X_b|^2$ by transferring one or two vertices X_a to X_b. In fact, this transfer is always possible when $|X_a| > |X_b| + 2$. If we look at the structure of our edge colorings, we can find sets of vertices that are connected by only colors a and b. We will call this a bicolor subgraph. A vertex in one of these bicolor subgraphs can either be connected to just an edge of color a, just an edge of color b, or an

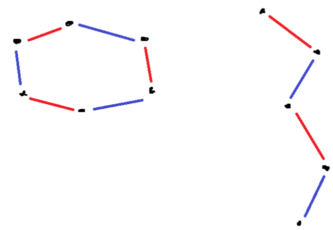

Figure 2: A bicolor cycle and path

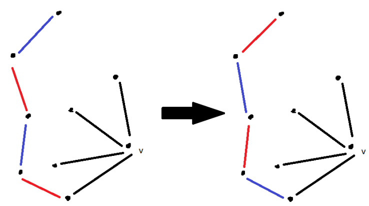

Figure 3: Switching the colors on a path

edge of color a and an edge of color b. That means, each bicolor subgraph can have one of two structures. It can either be a path of vertices where the edges connecting these vertices alternate between colors a and b, or it can be a cycle of vertices connected by edges of alternating colors a and b (fig. 2).

Note that in the cycle, we have just as many a-edges as we have b-edges, and in the line, the number of a-edges and b-edges can differ by at most one. Also note that we can recolor any of these bicolor subgraphs individually by switching which edges are colored a and which edges are colored b, which will not affect the rest of the graph because the edges connected to a vertex that are neither color a nor color b do not care if we switch the a's and b's in a component (fig. 3).

Since $|X_a| > |X_b| + 2$, we know that there must be at most one bicolor subgraph with at least one more vertex in $|X_a|$ than $|X_b|$. In fact, the only way a vertex can be in $|X_a|$ but not $|X_b|$ is if that vertex is at the end of a path-shaped bicolor subgraph. At the other end of that path, we either have a vertex that is also in $|X_a|$, or a vertex that is in neither $|X_a|$ nor $|X_b|$ by not being connected to v. Each vertex along the path is in neither $|X_a|$ nor $|X_b|$ because each is connected to both an a-color and b-color edge. Now, we can switch all of the a and b edge colors on that path, which will transfer either one or two vertices from $|X_a|$ to $|X_b|$ depending on whether one end of the path is in X_a or both ends of the path are in X_a respectively (fig. 4).

Now we know that we can in fact create a coloring for $G - v$ where each $|X_a|$ is within two of each $|X_b|$. Recall that $\sum_{i=1}^{k}|X_i| < 2k$, meaning that

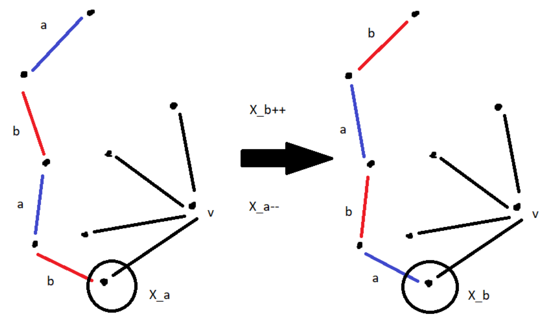

Figure 4: Switching color a and color b on a path; X_a decreases by one, X_b increases by one

$E[|X_i|] < 2$. Therefore, we either have an $|X_i| = 1$ or 0. If we have $|X_a| = 0$, then all the other $|X_i|$'s must be either $0, 1, 2$ so that they are within 2 of $|X_a|$. Since we know that $\sum_{i=1}^{k} |X_i| = 2 * degree(v) - 1$, which is odd, we must have at least one $|X_i| = 1$, so no matter what, we will have an $|X_i| = 1$. Then, let $|X_k| = 1$, where the one vertex in $|X_k|$ we label u.

Make G' the subgraph of G where we remove edge $\overleftrightarrow{vu}$ and all edges colored k in $G - v$. Trivially, this gives us a $k - 1$ edge coloring of $G' - v$, which by our inductive hypothesis means that G' is $k - 1$ edge colorable. Now we can add back $\overleftrightarrow{vu}$, which we color k, and add back all the k colored edges we removed from G. Now we have a k coloring of G. Since we initally chose $k = \Delta + 1$, we now have a $\Delta + 1$ coloring of G. Using this method, we can construct G by iterating over each vertex of G and successively showing that we can k-color that graph.

References

[1] Proof of Vizing's Theorem,
https://homepages.cwi.nl/~lex/files/vizing.pdf