Pragmatists and Purists on CPT Invariance in Relativistic Quantum Field Theories

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1 Introduction

Pragmatist approaches to relativistic quantum field theories (RQFTs) trade mathematical rigor for the ability to derive predictions from realistic interacting theories. Examples include the Lagrangian approach found in most textbooks, and Weinberg's approach. *Purist* approaches to RQFTs trade the ability to formulate realistic interacting theories for mathematical rigor. Examples include the axiomatic and algebraic formalisms. Philosophers are split on whether foundational issues related to RQFTs should be framed within pragmatist or purist approaches. Wallace (2011), for instance, has argued that cutoff quantum field theory (CQFT), a particular pragmatist approach, has been successful at resolving the problems associated with renormalized perturbation theory, while axiomatic and algebraic quantum field theory (AQFT), which epitomize purist approaches, have not; and this indicates that CQFT is the correct framework for philosophy of QFT. Fraser (2011), on the other hand, argues that renormalization techniques indicate how CQFT and AQFT are empirically indistinguishable, and that AQFT is to be preferred for its mathematical rigor.

This essay probes this debate by viewing it through the lens of the CPT theorem. This theorem entails that the state of a physical system described by an RQFT must possess CPT invariance; i.e., invariance under the combined transformations of charge conjugation C, space inversion P, and time reflection T. There are both pragmatist and purist versions of this theorem (Bain 2013). While all versions apply unproblematically to non-interacting states, and some unrealistic interacting states,

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extending them to realistic interacting states is problematic: For both pragmatists and purists, to do so requires confronting foundational problems. Greenberg (2002), however, claims that a violation of CPT invariance in an interacting RQFT, appropriately construed, entails a violation of Lorentz invariance. This claim is surprising not only since it purports to cover interacting theories in one fell swoop, but also because standard proofs of CPT invariance (both purist and pragmatist) require more than just the assumption of Lorentz invariance. Greenberg's claim has been influential in the physics literature since it suggests a test for violations of Lorentz invariance *via* experiments that measure CPT violation. Moreover, in apparently linking Lorentz invariance with CPT invariance, it suggests the latter is mysterious; in particular, some philosophers have wondered how the charge conjugation transformation C can arise from a purely spatiotemporal symmetry (Greaves 2010).

This essay analyzes Greenberg's claim in the context of the debate between pragmatists and purists. Section 2 reviews two formulations of the CPT theorem, one purist and the other pragmatist. Section 3 uses the problems these formulations face to inform a characterization of the distinction between pragmatism and purity based on the sense in which an RQFT can be said to exist. This distinction is then applied in Sect. 3 to a critique of Greenberg's claim. It will be seen that Greenberg's claim can be interpreted in either a purist or a pragmatist sense, and in either case, it fails to address the associated foundational problems.

2 Pragmatism Versus Purity on CPT Invariance

2.1 The Axiomatic CPT Theorem

The first example of a formulation of the CPT theorem I'd like to consider is the purist Wightman axiomatic approach (see, e.g., Streater and Wightman 1964).[1] The basic objects are vacuum expectation values of unordered products of fields, referred to as Wightman functions, $W^{(n)}(x_1, \ldots, x_n) \equiv \langle 0 | \phi(x_1) \ldots \phi(x_n) | 0 \rangle$, where $\phi(x)$ is a generic quantum field (technically defined as an operator-valued distribution), and $|0\rangle$ is its vacuum state. Wightman functions are required to satisfy a number of axioms, and it is the goal of this approach to construct models of these axioms that represent interacting RQFTs. For the purposes of deriving CPT invariance, the following three assumptions suffice.

(i) *Restricted Lorentz invariance* (RLI). The fields are invariant under the restricted Lorentz group $L_+^\uparrow$ (the subgroup of the Lorentz group connected to the identity that consists of Lorentz boosts but not parity or time reversal transformations).

[1] Another purist approach is the algebraic formalism which will not be discussed in this essay. CPT theorems have been proven in the algebraic approach by Borchers and Yngvason (2001) and Guido and Longo (1995). For a brief discussion of the latter, see Bain (2013).

(ii) *Spectrum Condition* (SC). The fields possess positive energy, in the sense that the spectrum of the momentum operator associated with $L_{+}^{\uparrow}$ is confined to the forward lightcone.

(i) and (ii) entail that Wightman functions can be extended to complex-analytic functions that are invariant under the proper complex Lorentz group. Moreover, the extended domain contains real points of analyticity referred to as Jost points.[2] The third assumption refers to these latter:

(iii) *Weak Local Commutativity* (WLC). At (or in the neighborhood of) a Jost point the fields satisfy $\langle 0 |\phi(x_1)\ldots\phi(x_n)| 0\rangle = i^K \langle 0 |\phi(x_n)\ldots\phi(x_1)| 0\rangle$, where K is the number of fermionic fields.

Jost (1957) showed that the conjunction of (i), (ii), (iii) entails the existence of an anti-unitary operator that combines the actions of C, P, and T transformations on fields, leaving them invariant (Streater and Wightman 1964, p. 150). The axiomatic CPT theorem thus states:

$$[(\text{RLI of fields}) \ \& \ \text{SC} \ \& \ \text{WLC}] \Rightarrow (\text{CPT invariance of fields})$$

This axiomatic understanding of CPT invariance faces what might be called the *Problem of Empirical Import*: No "realistic" interacting models of the Wightman axioms currently exist; i.e., no interacting models exist for theories (like QED and QCD) from which empirical predictions have been derived and confirmed. On the other hand, non-interacting models, and "unrealistic" interacting models of the Wightman axioms have been constructed (the latter are discussed by Fraser 2011, p. 127). This suggests that the axiomatic CPT theorem (currently) restricts CPT invariance to non-interacting, or unrealistic interacting RQFT states. This is problematic, since the evidence for CPT invariance in particular, and for the reliability of RQFTs in general, invariably comes from successful predictions made by realistic interacting RQFTs.

2.2 Weinberg's CPT Theorem

I'd now like to consider Weinberg's derivation of the CPT theorem as an example of a pragmatist approach (Weinberg 1995). The basic object of this approach is the S-matrix, which satisfies three assumptions:

(i) *Perturbation Theory*. The S-matrix is given by a power series expansion in time-ordered products of an interaction Hamiltonian density $\mathcal{H}_{int}(x)$:

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[2] A Jost point $(x_1, \ldots, x_n)$ is a convex set of points that are spacelike separated from each other. In other words, the difference variables $\xi_i \equiv x_{i-1} - x_i$ satisfy $\left(\sum \lambda_j \xi_j\right)^2 < 0$, for$\lambda_{\mathrm{j}} \geq 0, \sum \lambda_{\mathrm{j}} > 0$ (Streater and Wightman 1964, p. 71).

$$S_{\beta\alpha} = \sum_{n=0}^{\infty} \frac{-i^n}{n!} \int \langle \beta \left| T \left\{ \mathcal{H}_{int}(x_1) \ldots \mathcal{H}_{int}(x_n) \right\} \right| \alpha \rangle \, d^4x_1 \ldots d^4x_n \qquad (1)$$

where $|\beta\rangle, |\alpha\rangle$ are asymptotic multi-particle states, and the time-ordered product $T\{\mathcal{H}_{int}(x_1) \ldots \mathcal{H}_{int}(x_n)\}$ orders the $\mathcal{H}_{int}(x_i)$ according to $t_1 > \cdots > t_n$.

(ii) *Lorentz Invariance*. The S-matrix is invariant under restricted Lorentz transformations.

(iii) *Cluster Decomposition* (CD). The S-matrix satisfies cluster decomposition (briefly, correlations between scattering experiments decrease to zero as their separation distance increases to space-like infinity).

Weinberg shows that a sufficient condition for CD to be compatible with (ii) is that $\mathcal{H}_{int}(x)$ be a functional of fields that satisfy RLI and local commutativity (i.e., the fields commute or anti-commute at spacelike separated distances), and that are linear combinations of Fock space creation and annihilation operators for non-interacting particle states. Weinberg (1995, p. 198) then argues that if these fields carry a conserved charge, then anti-particle states must be posited. CPT invariance of the full Hamiltonian density then follows from a consideration of how the relevant creation and annihilation operators transform under C, P, and T separately (1995, pp. 244–246). The CPT theorem thus takes the following form:

$$[(\text{RLI of } S\text{-matrix}) \;\; \& \; \text{CD} \; \& \;\; (\text{existence of conserved charges})]$$
$$\Rightarrow (\text{CPT invariance of } \mathcal{H}(x))$$

where $\mathcal{H}(x)$ is the full Hamiltonian density.

In Weinberg's approach, one might claim that CPT invariance is a property of both interacting and non-interacting states, insofar as the demonstration of CPT invariance of $\mathcal{H}(x)$ rests on CPT invariance of the creation and annihilation operators of non-interacting multi-particle states that transform, under the S-matrix, into interacting multi-particle states. However, lest one think that this is an improvement over the axiomatic understanding of CPT invariance, the rigor of this approach faces the following problems:

(a) Expression (1) assumes that multi-particle states at asymptotic times are non-interacting, and can be unitarily related to interacting states at finite times. This is made problematic by Haag's theorem, which indicates that, under reasonable assumptions, the Hilbert spaces for interacting and non-interacting states belong to unitarily inequivalent representations of the canonical (anti-)commutation relations, thus a unitary S-matrix operator that transforms non-interacting states into interacting states does not exist (Duncan 2012, pp. 359–370).

(b) For many of the types of interacting QFTs of interest, the terms in the power series (1) diverge at high energies. This is referred to as the *UV* (*ultra-violet*) *Problem*.

(c) For the types of interacting QFTs of interest, there is a consensus that the power series (1) does not converge. Call this the *Convergence Problem*.

A few qualifications are in order at this point. First, these problems are not unique to Weinberg's approach, but rather afflict pragmatist approaches in general. Second, some interacting QFTs of interest, quantum chromodynamics (QCD) for instance, do not suffer the *UV Problem*; it is generally thought that QCD has an ultra-violet fixed point (more on this in Sect. 3 below). Third, problem (a) is implicitly addressed in pragmatist approaches by employing renormalization. In order to further distinguish pragmatists from purists, it will help to review where in pragmatist approaches renormalization occurs. This will be done in Sect. 2.3 below, but before this discussion, a final concern should perhaps be addressed involving the extent to which the Wightman axiomatic CPT theorem differs from Weinberg's CPT theorem.

In many textbooks, one finds pragmatist proofs of the CPT theorem followed by the advice that if one seeks a more rigorous proof, one should consult the Wightman axiomatic approach (see, e.g., Weinberg 1995, p. 245; Duncan 2012, pp. 479–483). There is a limited sense in which such pragmatist appeals to purist proofs of the CPT theorem seem justified. In the cases of non-interacting theories, and in some unrealistic interacting theories, one can argue that purist approaches and pragmatist approaches are intertranslatable. In these cases, purists do not face the *Problem of Empirical Import*, and pragmatists do not face the *Convergence Problem* (in these cases, expressions like (1) converge, and, moreover, as explained below, problems (a) and (b) can be effectively addressed). When intertranslatability holds, a pragmatist might be excused for appealing to Jost's proof, for instance, to explain CPT invariance. However, even in such cases, it seems to me that if we take pragmatist and purist approaches literally, we should hesitate to "mix and match" proofs of CPT invariance. In particular, in Weinberg's approach, the basic objects are the *S*-matrix and particle states, and fields and field equations are purely instrumental devices. Weinberg makes it clear that fields are introduced only to guarantee that the *S*-matrix satisfies restricted Lorentz invariance and Cluster Decomposition.[3] Moreover, to accomplish this task, fields are introduced in a particular format; i.e., as linear combinations of creation and annihilation operators that act on a multiparticle Fock space. In the Wightman axiomatic approach, on the other hand, the basic objects, arguably, are fields which need not be expressible as linear combinations of Fock space creation and annihilation operators.[4] Thus, one reason an advocate of the Weinberg approach should be hesitant in adopting Jost's proof of CPT invariance is that the latter is more general than Weinberg's

[3]This is reflected in Weinberg's (1995, p. 198) view of the axiomatic assumption of local commutativity (i.e., fields at spacelike separated points commute): "The point taken here is that [local commutativity of fields] is needed for the Lorentz invariance of the *S*-matrix, without any ancillary assumptions about measurability or causality."

[4]In other words, a model of the Wightman axioms need not take the form of a Fock space representation of the canonical (anti-) commutation relations. Note that the basic objects of the Wightman approach are tempered distributions (i.e., Wightman functions), but Wightman's (1956) reconstruction theorem indicates that these can be interpreted as vacuum expectation values of unordered products of fields.

proof, and suggests the theory is about more than a pragmatist intends it to be about. Moreover, keeping purist and pragmatist proofs of the CPT theorem separate may be important when it comes to addressing the question of why the theorem fails for non-relativistic quantum field theories (and non-relativistic quantum mechanics in general). For instance, Lévy-Leblond (1967) appeals to Weinberg's proof to explain why the CPT theorem fails for axiomatic Galilei-invariant QFTs (i.e., Galilei-invariant QFTs formulated *via* a set of axioms similar to the Wightman axioms).[5] Given the conceptual differences between these approaches, and in particular, in those cases of physical interest in which intertranslatability fails, it seems more appropriate to frame an explanation of the failure of the CPT theorem in axiomatic non-relativistic QFTs in terms of Jost's axiomatic proof, as opposed to Weinberg's proof.

2.3 Pragmatism and the Renormalization Problem

Typical pragmatist approaches simplify (1) by reducing it to an expression that involves vacuum expectation values of time-ordered products of fields, $\langle 0 | T\{\phi(x_1), \ldots, \phi(x_n)\} | 0\rangle$, referred to as τ-functions. One can distinguish between non-interacting and interacting τ-functions, depending on whether the fields are non-interacting or interacting (i.e., satisfy homogeneous or inhomogeneous field equations, respectively). The initial goal of pragmatist approaches is to reduce (1) to an expression that only involves non-interacting τ-functions (this subsequently facilitates the calculation of (1) *via* Feynman diagrams). This goal is achieved by the following:

(i) One first uses the LSZ reduction formula to relate S-matrix elements to interacting τ-functions (e.g., Duncan 2012, p. 286). This formula comes in many flavors, one per type of field. For instance, the LSZ formula for a scalar field of mass m is given by:

$$
\begin{aligned}
&{}_{out}\Big\langle \mathbf{p}_1, \ldots, \mathbf{p}_n \Big| \mathbf{k}_1, \ldots, \mathbf{k}_\ell \Big\rangle_{in} \\
&= \left(i/\sqrt{Z}\right)^{n+\ell} \int d^4x_1 \ldots d^4y_\ell e^{-ip_ix_i + ik_jy_j} \prod\nolimits_i \left(\partial^2_{x_i} + m^2\right) \prod\nolimits_j \left(\partial^2_{y_j} + m^2\right) \\
&\quad \times \langle 0 | T\{\varphi(x_1) \ldots \varphi(x_n)\, \varphi(y_1) \ldots \varphi(y_\ell)\} | 0\rangle \,.
\end{aligned} \tag{2}
$$

[5] Lévy-Leblond (1967, p. 165) explains the failure of the CPT theorem in GQFTs as due to the fact that GQFTs do not satisfy local commutativity: "This situation [i.e., the GQFT case] is to be contrasted with the relativistic case where the requirements of local commutativity on a free field... impose both the existence of a TCP [i.e., CPT] operation... and the spin-statistics relation, as has been shown in a very illuminating way, for this free-field case, by Weinberg..."

The left-hand side of (2) represents an S-matrix element for ℓ incoming particles with momenta k_i and n outgoing particles with momenta p_i. The right-hand side indicates how this can be calculated in terms of an interacting τ-function, where $\varphi(x)$ is an interacting field (i.e., a solution to the inhomogeneous Klein-Gordon equation).

(ii) One then assumes a perturbative split of the Hamiltonian, $H = H_0 + H_{int}$, into a non-perturbed piece H_0 and a piece H_{int} encoding small perturbations away from H_0.[6] The following Gell-Mann/Low formula then relates interacting τ-functions to non-interacting τ-functions (e.g., Duncan 2012, p. 246):

$$\langle 0 \left| T \left\{ \varphi\left(x_1\right) \ldots \varphi\left(x_n\right)\right\} \right| 0 \rangle = \frac{\left\langle 0 \left| T \left\{ \phi_I\left(x_1\right) \ldots \phi_I\left(x_n\right) e^{-i \int H_I dt} \right\} \right| 0 \right\rangle}{\left\langle 0 \left| T \left\{ e^{-i \int H_I dt} \right\} \right| 0 \right\rangle}. \tag{3}$$

In (3), $\varphi(x)$ is an interacting field, $\phi_I(x)$ is a non-interacting field in the interaction picture, and $H_I \equiv e^{iH_0} H_{int} e^{-iH_0}$ is the interaction picture representation of H_{int}.[7]

In the LSZ formula (2), Z is a renormalization constant. Its purpose is to relate the interacting field $\varphi(x)$ to non-interacting fields $\phi_{in}(x)$, $\phi_{out}(x)$ at asymptotic times. One assumes,

$$\langle \beta \left| \varphi(x) \right| \alpha \rangle \underset{t \to -\infty}{\longrightarrow} \sqrt{Z} \langle \beta \left| \phi_{in}(x) \right| \alpha \rangle, \langle \beta \left| \varphi(x) \right| \alpha \rangle \underset{t \to +\infty}{\longrightarrow} \sqrt{Z} \langle \beta \left| \varphi_{out}(x) \right| \alpha \rangle \tag{4}$$

where $|\beta\rangle, |\alpha\rangle$ are non-interacting multi-particle states. This assumption may be motivated by considering the action of a non-interacting asymptotic field on the vacuum with respect to a single-particle state (Duncan 2012, p. 282). If $|\mathbf{k}\rangle$ is a normalized single-particle state, then $\langle \mathbf{k} \left| \phi_{in}(x) \right| 0 \rangle = 1$. An interacting field $\varphi(x)$ cannot, in general, be decomposed into creation and annihilation operators, thus one sets $\langle \mathbf{k} \left| \varphi(x) \right| 0 \rangle = \sqrt{Z}$, for some constant Z. (4) may be considered a generalization of this. Formally, the constant Z can be removed from the LSZ formula by replacing the "bare" interacting field with a renormalized interacting field defined by $\varphi_r(x) \equiv Z^{-1/2}\varphi(x)$. This assignment guarantees that the renormalized interacting field behaves like the non-interacting field with respect to single-particle states; namely, $\langle \mathbf{k} \left| \varphi_r(0) \right| 0 \rangle = 1$.

Renormalization also enters into the derivation of the Gell-Man/Low formula (3). In particular, (3) assumes $H_0 |0\rangle = 0 = H |0\rangle$. The first equality entails $|0\rangle$ is the vacuum state of the non-interacting fields. Since H is a functional of interacting fields which cannot, in general, be decomposed into creation and annihilation operators, the second equality is typically not guaranteed. To enforce it, one defines a renormalized Hamiltonian $H_r \equiv H - \Delta$. This corresponds to renormalizing the

[6]The Hamiltonian is related to the Hamiltonian density by $H(t) = \int d^3\mathbf{x}\mathcal{H}(\mathbf{x}, t)$.

[7]$\phi_I(x)$ is defined by $\phi_I(\mathbf{x}, t) \equiv e^{iH_0(t-t_0)}\phi(\mathbf{x}, t_0)\, e^{-iH_0(t-t_0)}$, where $\phi(\mathbf{x}, t_0)$ is a non-interacting field at time t_0.

mass that appears in H. If this is given by m_B (the "bare" mass), and the shift corresponding to Δ is given by δm, then the renormalized mass m_r (the "physical" mass) is given by $m_r{}^2 \equiv m_B{}^2 + \delta m^2$

In these examples, renormalization is imposed to force the interacting theory to behave like the non-interacting theory, as far as the vacuum and single-particle states are concerned. This solves Problem (a) in the following sense: The renormalized field and the renormalized Hamiltonian are not self-adjoint operators (for typical interacting theories, the constant Z and the mass shift δm are infinite). This entails, for instance, that H_r does not implement unitary time translations, contrary to one of the assumptions of Haag's theorem (Fraser 2009, p. 547). Whether this constitutes an *adequate* solution to Problem (a) will depend on one's mathematical proclivities. The fact that renormalized parameters are, typically, infinite may upset purists. For such purists, the renormalization procedure simply replaces Problem (a) with another problem, call it the *Renormalization Problem*.

Note that renormalization is independent of perturbation theory as evidenced by its appearance in the non-perturbative LSZ formula. Thus the *Renormalization Problem* is independent of the *UV* and *Convergence Problems*.[8] At this point, it will be instructive to review how renormalization group (RG) techniques address these pragmatist problems. Wallace (2011) argues that such techniques underwrite heuristic (i.e., pragmatist) approaches, whereas Fraser (2011) claims they support rigorous (i.e., purist) approaches. The next section addresses this issue, as well as the general concern of how best to distinguish purity from pragmatism.

3 Distinguishing Purity from Pragmatism

The goal of the RG approach to renormalization is to determine how a theory's low-energy degrees of freedom depend on its high-energy degrees of freedom. Towards this end, the coupling constants g that appear in the interaction Hamiltonian (or Lagrangian) density, are defined as functions $g(\Lambda(\mu))$ of a scale-dependent cutoff $\Lambda(\mu)$. Changing the scale (by integrating out high-energy degrees of freedom with respect to Λ) generates a flow in the theory's parameter space. Couplings can then be characterized by how they behave as the scale is lowered: relevant couplings increase, irrelevant couplings decrease, and marginal couplings remain constant. One can show that, for a $(3+1)$-dim weakly coupled theory, there are a finite number of relevant and marginal couplings, and any irrelevant couplings are suppressed at a given energy scale μ by powers of μ/Λ(e.g., Duncan 2012, pp. 652–660).[9] In such a theory, the low-energy degrees of freedom depend on the

[8]As Weinberg (1995, p. 441) states, "...the renormalization of masses and fields has nothing directly to do with the presence of infinities, and would be necessary even in a theory in which all momentum space integrals were convergent."

[9]An important exception to this is QCD, which is not weakly coupled.

high-energy degrees of freedom through a finite number of parameters (the relevant and marginal couplings), and while the theory may still contain parameters that become infinite at high energies (the irrelevant couplings), it is still predictive in the sense that its predictions will be finite if constrained to a given scale. At this scale, the theory is *effectively* renormalizable insofar as any irrelevant couplings it may possess cannot be experimentally detected. With respect to Sect. 2.3's discussion of renormalization, an effectively renormalizable interacting theory requires a finite number of parameters to empirically imitate the behavior of the corresponding non-interacting theory, and these parameters, as functions of a finite cutoff, are finite.

In this effective field theory approach, the *Renormalization Problem* is addressed by adopting effective renormalizability, and the *UV Problem* is addressed by using the cutoff Λ to regulate divergent terms in expressions like (1) in Sect. 2.2. The cutoff serves to freeze out the high energy degrees of freedom of the theory, and one then adopts an agnostic attitude about what happens at energy scales above Λ. According to Wallace,

> This, in essence, is how modern particle physics deals with the renormalization problem: it is taken to presage an ultimate failure of quantum field theory at some short lengthscale, and once the bare existence of that failure is appreciated, the whole of renormalization theory becomes unproblematic, and indeed predictively powerful in its own right. (Wallace 2011, p. 119.)

While this appeal to RG techniques allows a pragmatist to address the *Renormalization* and *UV Problems*, the *Convergence Problem* still remains (with the qualifications noted at the end of Sect. 2.2). Moreover, Fraser (2011) suggests that RG techniques support purity, as opposed to pragmatism. In particular, the RG flow of the type of theory described above indicates an underdetermination of the theory's high-energy content by low-energy experiments. The latter fix the values of the theory's finite relevant and marginal couplings at the experimental energy scale, but fail to fix the values of the theory's irrelevant couplings. These latter determine how the theory behaves at high-energies. This implies that the successful predictions made by a realistic interacting RQFT (of this type) fail to determine the form it takes at high-energies. This suggests to Fraser that axiomatic and algebraic RQFT (AQFT) on the one hand, and Wallace's "cutoff" QFT (CQFT) on the other, are empirically indistinguishable at the energy scales currently probed by experiments:

> The upshot of the application of RG methods is that a range of Lagrangians at short distance scales each yield approximately the same predictions for relatively low energies. ... This lends support to the claim that the theoretical framework of QFT is underdetermined by the empirical evidence. AQFT and [CQFT] should be viewed as alternative theoretical frameworks for QFT which approximately agree in their empirical predictions. (Naturally, subject to the qualification that the construction of models of AQFT is still in progress). (Fraser 2011, p. 135.)

The qualification at the end of this quote is important. It acknowledges that the purist's *Problem of Empirical Import* is a potential obstruction to the claim that RG underdetermination holds between AQFT and CQFT. This obstruction takes the form of the question of whether there are AQFTs that can be "RG-related" to the appropriate low-energy experiments (as CQFTs can be).

These considerations suggest that an appeal to RG techniques is not decisive in adjudicating between pragmatists and purists. Both pragmatists and purists can make such an appeal, and such appeals fail to completely address foundational issues: the *Convergence Problem* remains for the RG pragmatist (with the requisite qualifications), and the *Problem of Empirical Import* remains for the RG purist. The discussion in Sect. 2.3 also indicates that an appeal to perturbation theory won't help either. On the one hand, pragmatists can employ non-perturbative techniques (the LSZ formula, for example; and lattice techniques in theories like QCD that are not weakly coupled). On the other hand, purists can employ perturbative techniques, as evidenced by "perturbative" AQFT which seeks to combine techniques from causal perturbation theory with AQFT (see, e.g., the review in Summers 2012, pp. 45–48).

The diversity of these methods allowed by both pragmatists and purists also suggests that a general appeal to mathematical rigor may not be enough to make the distinction as clear as it could be. Note first that the distinction between non-perturbative and perturbative methods does not necessarily map onto a distinction between rigorous and non-rigorous methods. In particular, the use of perturbative methods need not signal a relaxation of rigor. For instance, causal perturbation theory has been viewed by its advocates as providing a rigorous mathematical foundation for perturbative techniques, and these advocates include both purists and pragmatists.[10] Arguably, the lack of rigor that purists have traditionally associated with pragmatists' use of perturbation theory ultimately manifests itself in the pragmatists' *Convergence Problem*.

Thus what remains to distinguish pragmatists from purists are the *Convergence Problem* for the former (with the requisite qualifications), and the *Problem of Empirical Import* for the latter. These problems are concerned with the sense in which realistic interacting RQFTs can be said to exist. Call this basic foundational concern common to both purity and pragmatism, the *Existence Problem*. As Bouatta and Butterfield (2014, p. 16) suggest, this problem can be addressed in a number of ways. Purists, perhaps, can be essentially characterized by their demand for a strong notion of existence; namely, existence of a model of an appropriate set of axioms. Pragmatists, perhaps, can be essentially characterized by their adoption of a weaker notion of existence. One might require existence of a theory to entail the convergence of power series expansions like (1) (in which case interacting QED probably does not exist, whereas interacting QCD probably does). Alternatively, pragmatists might settle for existence defined in terms of renormalizability (in which case both interacting QED and QCD exist), or in terms of the existence of

[10]Causal perturbation theory consists of both a regularization scheme to address UV divergences in power series expansions, and an axiomatic scheme underwriting such expansions. These schemes can be separated; in particular, the regularization scheme can be adopted by pragmatists independently of the axiomatic scheme (Helling 2012; Falk et al. 2010). Conversely, the axiomatic scheme can be adopted by purists to extend purist axiomatic systems to include perturbative techniques (Brunetti and Fredenhagen 2000).

a UV fixed point in an RG flow (in which case interacting QED does not exist, but asymptotically free and/or safe theories like interacting QCD do, as well as conformally invariant theories).

Thus, distinguishing purity from pragmatism on the basis of the *Existence Problem* addresses the fact that both purists and pragmatists make use of similar methods, perturbative and non-perturbative, as well as the concern that mathematical rigor may be in the eye of the beholder. Moreover, it addresses the concern that the types of interactions described by realistic interacting RQFTs differ in essential ways. The next section will put this distinction to work.

4 Greenberg on Relativity and CPT Invariance

Greenberg (2002, p. 1) claims: "If CPT invariance is violated in an interacting quantum field theory, then that theory also violates Lorentz invariance." This claim is both influential and puzzling. In the physics literature it is cited for statements like the following:

> Note that Lorentz violation does not imply CPT violation for local EFTs, while CPT violation does imply Lorentz violation in local EFTs. (Liberati 2013, p. 12.)

> In all proofs of the CPT theorem Lorentz symmetry is the basic hypothesis, and indeed a theorem states that if CPT symmetry is violated then Lorentz symmetry must be violated, too . . . (Sozzi 2008, p. 198.)

> In realistic field theories, CPT violation is always accompanied by Lorentz violation, but not vice versa. (Berger 2011, p. 180.)

While Greenberg is not directly cited in the philosophy literature, one does find the following statements:

> . . . the CPT theorem . . . says that violations of CPT symmetry imply violations of Lorentz invariance, but not vice versa. (Hagar 2009, p. 261.)

> How can it come about that one symmetry (e.g., Lorentz invariance) entails another (e.g., CPT) *at all*? (Greaves 2010, p. 28)

> The CPT theorem says that any (restricted) Lorentz invariant quantum field theory must also be invariant under the combined operation of [CPT]. (Arntzenius 2011, p. 633.)

Greenberg's claim is puzzling for two reasons. First, in both the purist and pragmatist proofs of the CPT theorem reviewed in Sect. 2, more than just Lorentz invariance was needed to derive CPT invariance. Second, both proofs showed that, given appropriate assumptions, CPT invariance holds for non-interacting fields, and certain unrealistic interacting fields. In order to extend the proofs to realistic interacting fields, both the purist and the pragmatist need to confront Sect. 3's *Existence Problem*. This problem intimately depends on the type of interaction, and this suggests that a demonstration of CPT invariance for realistic interacting fields may have to be done on a case by case basis. Thus, on the surface, Greenberg's claim seems to both simplify the assumptions needed to derive CPT invariance, and address the issue of the extent of its applicability in one fell swoop.

Greenberg begins with the following assertions:

> To calculate the S matrix, we need τ functions, or similar functions, such as retarded or advanced products (r functions or a functions). We require covariance of a quantum field theory both in and out of cone as the condition for Lorentz invariance of the theory; thus both Wightman functions and the τ (or r or a) functions must be covariant for the theory to be Lorentz invariant. (Greenberg 2002, p. 1.)

He then provides the following expression for an n-point τ-function:

$$\tau^{(n)}\left(x_1,\ldots,x_n\right) \equiv \sum\nolimits_p \theta\left(t_{p_1}-t_{p_2}\right)\ldots\theta\left(t_{p_{n-1}}-t_{p_n}\right)W^{(n)}\left(x_{p_1},\ldots,x_{p_n}\right) \tag{5}$$

where the product of Heaviside functions $\theta\left(t_{p_1}-t_{p_2}\right)\ldots\theta\left(t_{pn_{-1}}-t_{pn}\right)$ enforces the time ordering $t_{p_1} > \cdots > t_{pn}$ on the Wightman function $W^{(n)}$, and the sum is over all permutations of the indices. Greenberg now argues that, at a Jost point, restricted Lorentz invariance of $\tau^{(n)}$ entails that $W^{(n)}$ satisfies weak local commutativity (WLC).[11] Thus if we require Wightman functions to satisfy RLI and SC, then a violation of CPT invariance of Wightman functions entails a violation of RLI of τ-functions. Schematically,

$$\begin{aligned}&\left(\text{RLI of } \tau^{(n)}\text{at Jost points}\right) \Rightarrow \left(\text{WLC of } W^{(n)}\text{at Jost points}\right)\\ &\quad\Rightarrow \left(\text{CPT invariance of } W^{(n)}\text{that satisfy RLI and SC}\right)\end{aligned} \tag{6}$$

where the second entailment follows from the axiomatic proof of CPT invariance (Sect. 2.1). Given the assumption that a theory is RLI only if both its Wightman and τ-functions are RLI, Greenberg concludes that a violation of CPT invariance of a theory's Wightman functions entails the theory does not satisfy RLI. No mention of an *interacting* theory has occurred at this point. However Greenberg (2002, p. 1) now states: "[t]his argument does not apply to a non-interacting theory for which τ functions need not be considered". This suggests the view that τ functions are a necessary ingredient in interacting QFTs, but not in non-interacting QFTs. The complete argument may thus be schematically represented by the following:

I. RLI violation of τ-functions entails RLI violation of the corresponding interacting QFT.
II. CPT violation of Wightman functions entails RLI violation of the corresponding τ-functions.
∴ Therefore, CPT violation of Wightman functions entails RLI violation of the corresponding interacting QFT.

Dütsch and Gracia-Bondía (2012, p. 429) observe that Greenberg's argument depends on the assumption that expression (5) exists for realistic interacting theories. This of course faces the purist's *Problem of Empirical Import*. It appears

[11] The proof of this claim rests on the fact that it is always possible to choose two Lorentz transformations that time-order a Jost point $(x_1, \ldots, x_n)$ in opposite ways.

explicitly in Premise II, which relies on the axiomatic proof of CPT invariance. Recall that this proof assumes (among other things) that Wightman functions satisfy the Spectrum Condition, which is essential to establish that complex extensions of Wightman functions are analytic. Dütsch and Gracia-Bondía (2012, p. 429) then observe: "... to the best of our knowledge, for non-trivial realistic models one cannot ascertain analyticity of Wightman-like functions; hence the argument *a la* Jost in [Greenberg 2002] flounders." They conclude with the following remarks:

> While the assertion that PCT conservation holds for everyday interacting relativistic theories remains plausible, to the question whether it has been proven at the required level of rigour, the clear and present answer is: only for a class of models ... and for none by Greenberg's argument. (Dütsch and Gracia-Bondía 2012, p. 429.)

Thus, as a *purist* attempt to extend CPT invariance to realistic interacting RQFTs, Greenberg's argument fails, to the extent that it fails to address the obstacle to extending the standard axiomatic proof of CPT invariance to realistic interacting RQFTs; namely, the *Problem of Empirical Import*. One way to express this failure is the observation that simply replacing Wightman functions with τ-functions does not automatically convert a theory that satisfies CPT invariance according to the axiomatic proof into a realistic interacting theory.

Does Greenberg's argument fair any better as a *pragmatist* attempt to extend CPT invariance to realistic interacting RQFTs? It appears that pragmatists have good reason to reject both Premises I and II. Consider, first, how a pragmatist might view Premise I. In Weinberg's approach, for instance, we have the following implications (Weinberg 1995, pp. 144–145)s:

$$(\mathcal{H}_{int}(x) \text{ is RLI and commutes at spacelike separations}) \Rightarrow (\tau\text{-functions of } \mathcal{H}_{int}(x) \text{are RLI})$$
$$\Rightarrow (\text{RLI of } S\text{-matrix})$$

where $\mathcal{H}_{int}(x)$ is the theory's interaction Hamiltonian density.[12] Thus if an RQFT is identified with its S-matrix, then a violation of RLI of its τ-functions does not necessarily entail a violation of RLI of the theory. This immediately blocks Greenberg's argument without further discussion. On the other hand, if an RQFT is identified with its Hamiltonian density, then a violation of RLI of its τ-functions entails either the theory violates RLI, or it is nonlocal (in the sense that its Hamiltonian density does not commute at spacelike separations). Thus a way is still open for this type of pragmatist to avoid Greenberg's argument, too.[13]

[12]The first entailment is based on the fact that the time-ordering of two points is RLI unless the points are spacelike separated. Thus if a field is RLI, then so are time-ordered products of it, except when it is evaluated at spacelike separated points. But if the field commutes when it is evaluated at spacelike separated points, then time-ordering will not violate RLI even at such points. This also holds for sums of products of fields, and hence for $\mathcal{H}_{int}(x)$. The second entailment follows since if time-ordered products of $\mathcal{H}_{int}(x)$ are RLl, then so is the S-matrix in the form (1), since all other quantities in (1) are manifestly RLI.

[13]Chaichian et al. (2011, p. 178) provide examples of non-local interaction Hamiltonian densities that are restricted Lorentz invariant and violate CPT invariance (thanks to a referee for pointing this out).

Two observations perhaps should be made at this point. First, note that one can adopt either of these options (i.e., identifying an RQFT with its *S*-matrix or with its Hamiltonian density) and still be faced with the pragmatist's *Existence Problem*. One is still faced with the question of whether a given *S*-matrix is well-defined in any of the pragmatist senses listed in Sect. 3, or if a given Hamiltonian density entails a well-defined *S*-matrix in these senses. Second, one might argue that the type of RQFTs of interest should be local (in the sense that their Hamiltonian densities commute at spacelike separations), and/or should be such that the condition of RLI of time-ordered products of their Hamiltonian densities is both necessary and sufficient for RLI of their *S*-matrix. But more must be said on both points for Greenberg's argument to gain initial traction for pragmatists.[14]

With respect to Premise II, pragmatists can justify the existence of realistic interacting τ-functions, not by providing provisos concerning the possibility of constructing realistic interacting models of a set of axioms, but rather by employing the Gell-Mann/Low formula (3). However, this confronts them with the *Existence Problem*. In particular, the Gell-Mann/Low formula requires a perturbative power series expansion of relevant quantities, and this expansion, even after it has been regularized and renormalized, fails to converge for the theories of interest. Thus, with respect to Premise II, Greenberg's argument is on the same shaky foundations for pragmatists as it is for purists. This problem makes its explicit appearance for a pragmatist in the second entailment in Greenberg's derivation (6) of Premise II. The technical difficulty in this case is that realistic interacting τ-functions obtained from the Gell-Mann/Low formula do not satisfy the Spectrum Condition.[15]

The upshot of this discussion is that, considered as either a purist or a pragmatist attempt to extend CPT invariance to realistic interacting fields, Greenberg's claim faces the *Existence Problem*. While his demonstration is insightful in uncovering connections between Lorentz invariance and CPT invariance in abstract objects like τ-functions and Wightman functions, both purists and pragmatists should be hesitant in extending these observations to concrete things like realistic interacting RQFTs.

[14]Here is another concern about the feasibility of Premise I in pragmatist approaches. If an interacting RQFT is in the business of calculating *S*-matrix elements, then τ-functions play an important role, as the discussion of the LSZ and Gell-Mann/Low formulas indicated, and this seems to make Premise I initially plausible. However, if there are other methods for calculating *S*-matrix elements that do not rely on τ-functions, and, moreover, if there are other testable predictions of RQFTs that can be derived without the use of τ-functions, then Premise I will again loose traction with pragmatists.

[15]For the purist, this problem manifested itself in the fact that currently there are no examples of realistic interacting τ-functions in the form of well-defined analytic functions. For the pragmatist who allows τ-functions to take the form of divergent power series expansions obtained *via* the Gell-Mann/Low formula, the problem is that such expressions do not satisfy the Spectrum Condition (in the sense that the fields that occur in them do not satisfy the Spectrum Condition).

5 Conclusion

This essay has used the debate between purists and pragmatists to critically examine Greenberg's (2002) claim that a violation of CPT invariance in an interacting RQFT entails a violation of Lorentz invariance. Section 2 revealed the extent to which purist and pragmatist versions of the CPT theorem extend to realistic interacting RQFTs. In both cases, this extent is constrained by what Sect. 3 called the *Existence Problem*; namely, the problem of articulating an appropriate notion of existence for a QFT, and then demonstrating that this notion holds for realistic interacting RQFTs. Purists can be characterized by their adoption a notion of existence that requires the existence of a model of an appropriate set of axioms, and the *Existence Problem* then becomes the task of constructing such a model for realistic interacting RQFTs. Pragmatists can be characterized by their adoption of a weaker notion of existence (convergence, renormalizability, existence of a UV fixed point, *etc.*), and the *Existence Problem* then becomes the task of demonstrating that their preferred notion holds for the types of realistic interacting RQFTs of interest. Greenberg's claim was shown in Sect. 4 to suffer from a failure to address the *Existence Problem*, in either its purist or its pragmatist form.

References

Arntzenius, F. (2011). The CPT theorem. In C. Callender (Ed.), *The Oxford handbook of philosophy of time* (pp. 633–646). Oxford: Oxford University Press.

Bain, J. (2013). CPT invariance, the spin-statistics connection, and the ontology of relativistic quantum field theories. *Erkenntnis, 78*, 797–821.

Berger, M. (2011). Lorentz violation in top-quark production and decay. In V. Kostelecky' (Ed.), *Proceedings of the 5th meeting on CPT and Lorentz Symmetry* (pp. 179–183). Singapore: World Scientific.

Borchers, H., & Yngvason, J. (2001). On the PCT-theorem in the theory of local observables. In R. Longo (Ed.), *Mathematical physics in mathematics and physics: Quantum and operator algebraic aspects* (pp. 39–64). Providence: American Mathematical Society. Available online as arXiv:math-ph/0012020v1.

Bouatta, N., & Butterfield, J. (2014). On emergence in gauge theories at the 't Hooft limit. *European Journal for Philosophy of Science*. doi:10.1007/s13194-014-0098-1. Preprint. Available at http://philsci-archive.pitt.edu/id/eprint/9288.

Brunetti, R., & Fredenhagen, K. (2000). Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Communications in Mathematical Physics, 208*, 623–661.

Chaichian, M., Dolgov, A., Novikov, V., & Tureanu, A. (2011). CPT violation does not lead to violation of lorentz invariance and vice versa. *Physics Letters, B699*, 177–180.

Duncan, A. (2012). *The conceptual framework of quantum field theory*. Oxford: Oxford University Press.

Dütsch, M., & Gracia-Bondía, J. (2012). On the assertion that PCT violation implies Lorentz non-invariance. *Physics Letters, B711*, 428–433.

Falk, S., Häußling, R., & Scheck, F. (2010). Renormalization in quantum field theory: An improved rigorous method. *Journal of Physics, A43*, 035401.

Fraser, D. (2009). Quantum field theory: Underdetermination, inconsistency, and idealization. *Philosophy of Science, 76*, 536–567.

Fraser, D. (2011). How to take particle physics seriously: A further defense of axiomatic quantum field theory. *Studies in History and Philosophy of Modern Physics, 42*, 126–135.

Greaves, H. (2010). Towards a geometrical understanding of the CPT theorem. *British Journal for the Philosophy of Science, 61*, 27–50.

Greenberg, O. (2002). CPT violation implies violation of Lorentz invariance. *Physical Review Letters, 89*, 231602.

Guido, D., & Longo, R. (1995). An algebraic spin and statistics theorem. *Communications in Mathematical Physics, 172*, 517–533.

Hagar, A. (2009). Minimal length in quantum gravity and the fate of Lorentz invariance. *Studies in History and Philosophy of Modern Physics, 40*, 259–267.

Helling, R. (2012). *How I learned to stop worrying and love QFT* (preprint), arXiv:1201.2714v2 [math-ph].

Jost, R. (1957). Eine Bemerkung zum CTP theorem. *Helvetica Physica Acta, 30*, 409–416.

Lévy-Leblond, J.-M. (1967). Galilean quantum field theories and a ghostless Lee model. *Communications in Mathematical Physics, 4*, 157–176.

Liberati, S. (2013). Tests of Lorentz invariance: A 2013 update. *Classical and Quantum Gravity, 30*, 133001.

Sozzi, M. (2008). *Discrete symmetries and CP violation*. Oxford: Oxford University Press.

Streater, R., & Wightman, A. (1964). *PCT, spin and statistics, and all that*. Princeton: Princeton University Press.

Summers, S. (2012). *A perspective on constructive quantum field theory* (preprint), arXiv:1203.3991v1 [math-ph].

Wallace, D. (2011). Taking particle physics seriously: A critique of the algebraic approach to quantum field theory. *Studies in History and Philosophy of Modern Physics, 42*, 116–125.

Weinberg, S. (1995). *The quantum theory of fields* (Vol. 1). Cambridge: Cambridge University Press.

Wightman, A. (1956). Quantum field theory in terms of vacuum expectation values. *Physical Review, 101*, 860–866.