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The Coordinate-Independent 2-Component Spinor Formalism and the Conventionality of Simultaneity

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In recent articles, Zangari (1994) and Karakostas (1997) observe that while an ε-extended version of the proper orthochronous Lorentz group $O^{\uparrow}_{+}$ (1,3) exists for values of ε not equal to zero, no similar ε-extended version of its double covering group SL(2, *C*) exists (where $\varepsilon = 1 - 2\varepsilon_R$, with ε_R the non-standard simultaneity parameter of Reichenbach). Thus, they maintain, since SL(2, *C*) is essential in describing the rotational behaviour of half-integer spin fields, and since there is empirical evidence for such behaviour, ε-coordinate transformations for any value of $\varepsilon \neq 0$ are ruled out empirically. In this article, I make two observations:

(a) There is an isomorphism between even-indexed 2-spinor fields and Minkowski world-tensors which can be exploited to obtain generally covariant expressions of such spinor fields.
(b) There is a 2-1 isomorphism between odd-indexed 2-spinor fields and Minkowski world-tensors which can be exploited to obtain generally covariant expressions for such spinor fields up to a sign. Evidence that the components of such fields do take unique values is not decisive in favour of the realist in the debate over the conventionality of simultaneity in so far as such fields do not play a role in clock synchrony experiments in general, and determinations of the one-way speed of light in particular.

I claim that these observations are made clear when one considers the coordinate-independent 2-spinor formalism. They are less evident if one restricts oneself to earlier coordinate-dependent formalisms. I end by distinguishing these conclusions from those drawn by the critique of Zangari given by Gunn and Vetharaniam (1995). © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Spacetime; Conventionalism; Simultaneity; Spinors; Formalism

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1. Introduction[1]

Recently a new twist has been given to a long-standing debate in philosophy of spacetime, the debate over the conventionality of simultaneity (CS, hereafter) in Special Relativity. In recent articles, Zangari (1994) and Karakostas (1997) observe that while an ε-extended version of the proper orthochronous Lorentz group $O^{\uparrow}_{+}(1, 3)$ exists for values of ε not equal to zero, no similar ε-extended version of its double covering group SL(2, C) exists (where $\varepsilon = 1 - 2\varepsilon_{\mathbf{R}}$, with $\varepsilon_{\mathbf{R}}$ the non-standard simultaneity parameter of Reichenbach). Thus, they maintain, since SL(2, C) is essential in describing the rotational behaviour of half-integer-spin fields, and since there is empirical evidence for such behaviour, ε-coordinate transformations for any value of $\varepsilon \neq 0$ are ruled out empirically. Gunn and Vetharaniam (1995) respond to Zangari by claiming that the existence of half-integer-spin fields does not force restriction to standard coordinate charts. They substantiate this claim by formulating the Dirac equation in ε-coordinates. Furthermore, they suggest that Zangari conflates the internal symmetries of spin space with the spacetime symmetries of the Poincaré group. Finally, they claim that, regardless, the SL(2, C) 'complex representation' of spacetime points, while simpler than the $O^{\uparrow}_{+}(1, 3)$ representation, does not admit a parity operator. What the latter requires is the full Lorentz group O(1, 3).

In this article, I contend first that Zangari and Karakostas base their critique of the CS thesis on a coordinate-dependent description of 2-component spinors (found in most of the early literature on the subject) and fail to do justice to coordinate-independent techniques. This is significant in so far as the coordinate-independent 2-component spinor (alternatively '2-spinor') formalism indicates three points. First, 2-spinor fields, just like tensor fields, can be considered geometrical objects independent of particular coordinate representations of them. Second, the 2-spinor formalism makes clear the relation between 2-spinor fields and general tensor fields: even-indexed 2-spinor fields can be put into a 1-1 correspondence with Minkowski world-tensors; whereas odd-indexed 2-spinor fields can be put into a 2-1 correspondence with Minkowski world-tensors. Since Minkowski world-tensors can always be extended to general tensors, it is evident that the information contained in even-indexed 2-spinor fields can be represented in arbitrary general linear coordinates (i.e. such fields can be given tensorial, generally covariant expressions), whereas the information contained in odd-indexed 2-spinor fields can only be given in arbitrary coordinates up to a sign.

Third and finally, the coordinate-independent description of 2-spinors makes clear the basis of Zangari's and Karakostas's critique of the CS thesis; namely,

[1] In this article, the abstract index notation is in force with the following qualifications: lower-case Latin indices denote tensor fields, and lower-case Greek indices denote their components. Upper-case Latin indices denote spinor fields, and upper-case Greek indices denote their components.

that the existence of 2-spinor fields requires spacetime to possess a particular structure, viz that given by a global Minkowski tetrad field. This last point is obscured in the coordinate-dependent description of 2-spinors. I will claim that it is not decisive in the CS debate, in so far as it amounts to a variant of a typical realist argument against the CS thesis that appeals to criteria of epistemic warrant (unifying power, simplicity, etc.) that a traditional conventionalist will not accept. I thus agree with Gunn and Vetharaniam's overall conclusion, but for different reasons. Moreover, I shall argue that some of the particulars of Gunn and Vetharaniam's critique are a bit misleading.

In Section 2, I review the debate over the Conventionality of Simultaneity. In Sections 3 and 4, I review the significance of the group SL(2, *C*) in physical theories and indicate the nature of the relation between 2-spinors and tensors. In Sections 5 and 6, I criticise Zangari's and Karakostas's argument against CS and indicate the options for the conventionalist. Finally, in Section 7, I criticise Gunn and Vetharaniam's contribution to this debate.

2. The CS Debate

Recall that the CS debate centres on the value of Reichenbach's simultaneity parameter ε_R.[2] Spacetime realists claim that the value $\varepsilon_R = 1/2$ is uniquely specified by (at the least) the conformal structure of Minkowski spacetime.[3] Hence two clocks A and B may be judged to be in synchrony just when their readings are identical when both clocks lie on the same spacelike hypersurface. For the realist, such synchrony may be established operationally in the following manner: emit a light signal from clock A to clock B at time t_a (as judged by A). Upon reception at clock B at reading t_b (as judged by B), reflect it back to A. Record the reception time $t_{a''}$ at A. Then clocks A and B are judged to be in synchrony just when

$$t_b = t_a + \tfrac{1}{2}(t_{a''} - t_a). \tag{1}$$

Conventionalists, on the other hand, insofar as they reject *a priori* ontological commitments to spacetime structure, claim that equation (1) presupposes that the one-way speed of light is independent of direction; i.e. that the speed c_+ of light from clock A to clock B is identical to the speed of light c_- from B to A;

[2] The conventionality of simultaneity thesis should be made distinct from the relativity of simultaneity. The latter follows explicitly from the metrical structure of Minkowski spacetime (M, η_{ab}) and describes the fact that the relation of simultaneity between events in (M, η_{ab}) is defined uniquely only relative to a given inertial reference frame. In general, observers in different inertial frames will not agree on judgements of simultaneity. The conventionality thesis claims that the simultaneity relation is not uniquely specified *within* a given inertial frame and can only be given by imposing one of a continuum of conventional choices (viz $0 < \varepsilon_R < 1$). For recent reviews of the debate see Janis (1998), Anderson *et al.* (1998), and Norton (1992).

[3] Malament (1977) demonstrates that standard simultaneity is the only non-trivial equivalence relation that can be implicitly defined in terms of the light cone structure of Minkowski spacetime and the temporally-oriented worldline of an inertial observer. Sarkar and Stachel (1999) demonstrate that additional 'null-cone' simultaneity relations can be defined if one gives up time symmetry.

and this cannot be ascertained without a prior determination of synchrony between A and B. Without this presupposition, (1) must be written as

$$t_b = t_a + \varepsilon_{\mathrm{R}}(t_{a''} - t_a), \tag{2}$$

with the value of ε_{R} being chosen by convention and having the value 1/2 just when it is stipulated that $c_+ = c_-$.[4]

Equation (2) defines an ε-coordinate transformation given by

$$\begin{aligned} x_\varepsilon^i &= x^i, & i &= 1,2,3, \\ x_\varepsilon^0 &= x^0 - \tfrac{1}{c}x^i(1 - 2\varepsilon_{\mathrm{R}}^i), & \varepsilon_{\mathrm{R}}^i &= (\varepsilon_{\mathrm{R}}^1, \varepsilon_{\mathrm{R}}^2, \varepsilon_{\mathrm{R}}^3), \end{aligned} \tag{3}$$

where, in general, ε_{R} depends on spatial direction. Defining $\varepsilon^i = (1 - 2\varepsilon_{\mathrm{R}}^i)$, (3) can be written as (Zangari, 1994, p. 269)

$$x_\varepsilon^\mu = N(\varepsilon^i)^\mu{}_\nu x^\nu = \begin{pmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cx^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{4}$$

with $|\vec{\varepsilon}| < 1$. If the $N(\varepsilon^i)$ are composed with a Lorentz transformation, one obtains the 'ε-extended' homogeneous Lorentz group, call it ε-O(1, 3), with elements[5]

$$\Lambda_\varepsilon = N\Lambda N^{-1}. \tag{5}$$

This amounts to an ε-coordinate-induced similarity transformation on Λ. If the invariants of a transformation group of a theory are taken as the observables of the theory, the conventionalist can claim that the invariants of ε-O(1, 3) are observationally indistinguishable from the invariants of O(1, 3). For example, under (5), all invariants of O(1, 3) map onto invariants of ε-O (1, 3). In particular, we have $\mathrm{d}s^2 = c^2(\mathrm{d}x_0)^2 - (\mathrm{d}x^i)^2 = \mathrm{d}s_\varepsilon^2$. Hence, making use of (3),

$$\mathrm{d}s_\varepsilon^2 = (\mathrm{d}x_\varepsilon^0)^2 + \frac{2\varepsilon^i}{c}\mathrm{d}x_\varepsilon^i \mathrm{d}x_\varepsilon^0 + \frac{(\varepsilon^i)^2 - 1}{c^2}(\mathrm{d}x_\varepsilon^i)^2, \tag{6}$$

with associated metric

$$(g_\varepsilon)_{00} = 1, \quad (g_\varepsilon)_{i0} = \varepsilon^i, \quad (g_\varepsilon)_{ij} = \varepsilon^i\varepsilon^j\delta^{ij} - \delta^{ij}. \tag{7}$$

The realist in the CS debate will view the ε-coordinate transformation (3)–(5) as a passive coordinate transformation, and the 'ε-extended' Lorentz group ε-O(1, 3) obtained by the induced similarity transformation (5) as just O(1, 3) in kooky coordinates.

[4] Suppose the distance between clocks A and B is d. Then the round-trip speed of light c is given by $c = 2d/(t_{a''} - t_a)$. Hence $c_+ = d/(t_b - t_a) = c/2\varepsilon_{\mathrm{R}}$, and $c_- = d/(t_{a''} - t_b) = c/2(1 - \varepsilon_{\mathrm{R}})$. Explicitly, then, knowledge of $c_\pm$ requires prior knowledge of ε_{R}.

[5] Suppressing indices, a homogeneous Lorentz transform Λ acts on 4-vectors x according to $x' = \Lambda x$. Hence $x'_\varepsilon = Nx' = N\Lambda x \equiv \Lambda_\varepsilon x_\varepsilon = \Lambda_\varepsilon Nx$, for any x. Thus $\Lambda_\varepsilon = N\Lambda N^{-1}$.

In general, the conventionalist is a type of semantic anti-realist. Such a creature holds that, in the light of multiple intertranslatable descriptions of a physical system P that agree on all observational aspects of P, but do not agree on in-principle unobservable aspects of P, any particular descriptive framework can only be chosen by convention. In the CS context, the multiple descriptive frameworks in question are coordinate charts, the physical system P consists of the determination of clock synchrony, the in-principle unobservable aspect of P is the one-way speed of light, and intertranslatability is secured by (3)–(5).

3. SL (2, C) and the Lorentz Group O(1, 3)

In this section, I review the relation between the group SL(2, C) (generated by linear 2×2 complex matrices with determinant $= 1$) and the Lorentz group O(1, 3). Readers already familiar with this standard material may skip directly to Section 4.

I first describe how the group SL(2, C) forms a 2-1 faithful representation of the restricted Lorentz group $\mathrm{O}_+^\uparrow(1, 3)$.[6] In other words, there exists a group homomorphism ρ: $\mathrm{SL}(2, C) \to \mathrm{O}_+^\uparrow(1, 3)$ such that to every element $\Lambda_A \in \mathrm{O}_+^\uparrow(1, 3)$ there corresponds exactly two elements $\pm A \in \mathrm{SL}(2, C)$. I now briefly indicate how ρ is constructed.

Let M^4 be Minkowski vector space and Herm(2) be the collection of all 2×2 complex Hermitian matrices. Any matrix $H \in \mathrm{Herm}(2)$ can be expanded as

$$H = x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = x^\mu\sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}, \tag{8}$$

where the x^μ are real and the σ_μ are the Pauli matrices.[7] This defines a map S: $x^\mu \mapsto x^\mu\sigma_\mu$, from R^4 to Herm(2). Under this correspondence, the determinant of H is just the Lorentz length of x^μ:

$$\det \mathrm{S}(x^\mu) = g_{\mu\nu}x^\mu x^\nu, \tag{9}$$

where $g_{\mu\nu}$ are Minkowski metric components. The left-hand side of (9) is preserved under transformations of the type L_A: $\mathrm{Herm}(2) \to \mathrm{Herm}(2)$ given

[6] Recall that the Lorentz group O(1, 3) with elements Λ^a_b has 4 components, $\mathrm{O}(1, 3) = \mathrm{O}_+^\uparrow(1,3) \cup \mathrm{O}_+^\downarrow(1, 3) \cup \mathrm{O}_-^\uparrow(1, 3) \cup \mathrm{O}_-^\downarrow(1, 3)$, where $\pm$ indicates a determinant of ± 1 and $\uparrow(\downarrow)$ indicates $\Lambda^0_0 >$ $(<)\, 0$. Those components with det $= +1$ are referred to as proper (and alternatively denoted by SO(1, 3)). The proper components are connected with the identify. Time- and space-reflections are given by elements of $\mathrm{O}_+^\downarrow(1, 3)$ and $\mathrm{O}_+^\uparrow(1, 3)$, respectively.

[7] Explicitly,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These form a basis for Herm(2). Hence (8) is basis-dependent description of the elements of Herm(2).

explicitly by

$$L_A(H) \equiv H' = AHA^\dagger, \tag{10}$$

where $A \in \mathrm{SL}(2, C)$ is a 2×2 complex matrix with unit determinant.[8] The map L_A thus induces a transformation on x^a that preserves its length; i.e. it induces a homogeneous Lorentz transformation $\Lambda_A\colon M^4 \to M^4$ given by $\Lambda_A = S^{-1}L_AS$ and acting on elements of M^4 according to $H' = x'^\mu\sigma_\mu$, where $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. The explicit form of Λ_A is given by[9]

$$(\Lambda_A)^\mu{}_\nu = \mathrm{tr}(\sigma^\mu A\sigma_\nu A^\dagger). \tag{11}$$

This defines the map $\rho\colon \mathrm{SL}(2, C) \to \mathrm{O}^\uparrow_+(1, 3)$. It is to the proper orthochronous component of O(1, 3) since SL(2, C) is connected to the identity and, upon explicit calculation using (11), $(\Lambda_A)^0{}_0 > 0$ for any $A \in \mathrm{SL}(2, C)$. It is a group homomorphism (i.e. it preserves group multiplication) since $\Lambda_A\Lambda_B = S^{-1}L_AL_BS = S^{-1}L_{AB}S = \Lambda_{AB}$. Moreover, it is onto (viz, faithful), since both SL(2, C) and $\mathrm{O}^\uparrow_+(1, 3)$ are six-parameter groups. Finally, it is 2-1, since for any $A, B \in \mathrm{SL}(2, C)$, if $\Lambda_A = \Lambda_B$, then $A = \pm B$. (To see this, suppose $A, B \in \mathrm{SL}(2, C)$ and $\Lambda_A = \Lambda_B$. Then $AB^{-1} \in \mathrm{SL}(2, C)$ and $\Lambda_{AB^{-1}} = \Lambda_A\Lambda_{B^{-1}} = \Lambda_A(\Lambda_B)^{-1} = \Lambda_A(\Lambda_A)^{-1} = I$ (the identity element of $\mathrm{O}^\uparrow_+(1, 3)$). By (11), it follows that $AB^{-1} = \pm I$ (identity on SL(2, C)). Hence $A = \pm B$.) Thus elements of SL(2, C) form a 2-valued faithful representation of the proper orthochronous Lorentz group.[10]

Furthermore, since SL(2, C) is simply connected and $\mathrm{O}^\uparrow_+(1, 3)$ is doubly connected, the homomorphism ρ is a covering map, identifying SL(2, C) as the universal covering group of $\mathrm{O}^\uparrow_+(1, 3)$.[11] This topological property turns out to have physical significance. It entails that, under a rotation of 2π, carriers of representations of SL(2, C) change sign, whereas carriers of representations of $\mathrm{O}^\uparrow_+(1, 3)$ do not, as can be seen by the following considerations.

From (11), a path from positive identity I to $-I$ in SL(2, C) corresponds to a path from I to I in $\mathrm{O}^\uparrow_+(1, 3)$. In SL(2, C), such a path is not homotopic (i.e. continuously deformable) to the trivial curve (since it does not close). If the path is continued back to I in SL(2, C), then, since SL(2, C) is simply connected, the

[8] Suppose $H \in \mathrm{Herm}(2)$. Then $L_A(H)^\dagger = (AHA^\dagger)^\dagger = (A^\dagger)^\dagger H^\dagger A^\dagger = AHA^\dagger = L_A(H)$, hence $L_A(H) \in \mathrm{Herm}(2)$. Furthermore, L_A preserves the determinant, $\det(AHA^\dagger) = \det(A)\det(H)\det(A^\dagger) = \det(H)$.

[9] To derive this, note that, from (8), $H' = x'^\mu\sigma_\mu = (\Lambda_A)^\mu{}_\nu x^\nu\sigma_\mu = Ax^\mu\sigma_\mu A^\dagger = x^\mu A\sigma_\mu A^\dagger$. Identifying coefficients of x^μ yields $(\Lambda_A)^\mu{}_\nu\sigma_\mu = A\sigma_\nu A^\dagger$. Multiplying both sides of this latter with σ_ρ and taking the trace then produces $\mathrm{tr}(\sigma_\rho(\Lambda_A)^\mu{}_\nu\sigma_\mu) = (\Lambda_A)^\mu{}_\nu\mathrm{tr}(\sigma_\rho\sigma_\mu) = (\Lambda_A)^\mu{}_\nu(2\delta_{\rho\mu}) = \mathrm{tr}(\sigma_\rho A\sigma_\nu A^\dagger)$. Equation (11) then follows.

[10] Let G and H be groups. Then a *representation* of G is a map $D\colon G \to H$ which is a group homomorphism; i.e. $D(g)D(g') = D(gg')$, $\forall g, g' \in G$, $D(e)$ = identity on H, for e the identity on G. Furthermore, for later reference, if G and H are matrix groups, and H is of order n and V^n is an n-dimensional vector space, then by selecting a basis for V^n, elements of H can be considered linear transformations (for changes of basis matrices) on V^n. V^n is then referred to as the *representation space* of G and elements of V^n are referred to as *carriers of the representation* of G.

[11] For lucid accounts of the topological concepts involved, see the discussions in Wald (1984, pp. 344–346), Naber (1992, Appendix B), and Penrose and Rindler (1984, pp. 41–46).

path is now homotopic to the trivial curve. The corresponding path in $O^{\uparrow}_{+}(1, 3)$ retraces itself. To see this concretely, note that a series of rotations given by

$$\pm A = \begin{pmatrix} e^{i\frac{1}{2}\theta} & 0 \\ 0 & e^{-i\frac{1}{2}\theta} \end{pmatrix}, \tag{12}$$

as θ ranges from 0 to 2π, corresponds to the first path described above, from $I_{SL(2,\, C)}$ to $-I_{SL(2,\, C)}$ and from $I_{O^{\uparrow}_{+}(1,\, 3)}$ to $I_{O^{\uparrow}_{+}(1,\, 3)}$. (Acting on (8) via (10), the matrix A above rotates x^a by θ in the (x_1, x_2)-plane.) Extending the range to $0 \leqslant \theta \leqslant 4\pi$ corresponds to the second path, homotopic to the trivial curve in both SL(2, C) and $O^{\uparrow}_{+}(1, 3)$. Hence carriers of representations of SL(2, C) change sign under 2π rotations, whereas carriers of $O^{\uparrow}_{+}(1, 3)$ do not.

Note that the carriers of the representations of SL(2, C) and $O^{\uparrow}_{+}(1, 3)$ are the main concern here (see footnote 10).[12] They are the physical/geometrical objects of concern to the physicist. It is the components of these objects with respect to a given coordinate chart that are measured. Ontologically, they may be construed in one of two ways. One may consider the carriers to be real, independently of coordinate chart representation or decomposition in a given frame field. Alternatively, one may consider only the values of the components of the carriers in a given coordinate chart to be real (i.e. what is real is what actually gets measured). In this coordinate-dependent reading, the carrier itself is considered no more than an array of the values of its components relative to a given coordinate chart that transforms between charts under a particular transformation rule. To see this more concretely, I will now indicate how an intrinsic, coordinate-independent description of the carriers of SL(2, C), i.e. 2-spinors, may be given. This will then be compared to a coordinate-dependent description.

[12] More abstractly, one need not restrict talk to matrix representations of SL(2, C) and carriers of matrix representations; one can talk of general representations. Recall first that the homogeneous Lorentz group SO(1, 3) is generated by 3 boosts $\mathbf{K}$ and 3 rotations $\mathbf{J}$. Defining new generators $\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K})$, $\mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K})$, it follows that $[A_i, A_j] = i\varepsilon_{ijk}A_k$, $[B_i, B_j] = i\varepsilon_{ijk}B_k$, $[A_i, B_j] = 0$ $(i, j = 1, 2, 3)$. Hence $\mathbf{A}$, $\mathbf{B}$ each generate SU(2). Thus any representation of SO(1, 3) can be labelled by two angular momenta (A, B), the first associated with $\mathbf{A}$ and the second with $\mathbf{B}$. (Note, however, that since SO(1, 3) is non-compact, its finite-dimensional representations are not unitary; so while the 4-dimensional compact rotation group SO(4) decomposes as SO(4) = SU(2)$\otimes$SU(2), the homogeneous Lorentz group decomposes as SO(1, 3) = SL(2, C)$\otimes$SL(2, C).) In general, a field that transforms according to the (A, B) representation of SO(1, 3) has components that rotate like objects with spins $j = A + B, A + B - 1, \ldots, |A - B|$. In particular, a scalar field transforms like the (0, 0) representation, a 4-vector field transforms according to $(\frac{1}{2}, \frac{1}{2})$ (since it has a scalar (spin = 0) time component plus a 3-vector (spin = 1) component), and a 2-component spinor field transforms either according to the $(\frac{1}{2}, 0)$ or the $(0, \frac{1}{2})$ representation. When SO(1, 3) is extended by adding to it a parity transformation, these two representations can no longer be considered separately (parity transformations interchange them). One is thus led to the direct sum $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$, which is invariant under parity and constitutes an irreducible representation of SO(1, 3) extended by parity (i.e. ISO(1, 3)). This latter is the Weyl representation of a Dirac 4-spinor (see footnote 13 below).

4. Coordinate-Independent Description of 2-Spinors

The representation space of the matrix representations of SL(2, C) considered above is a 2-dimensional complex linear vector space S endowed with a bilinear skew-symmetric 2-form $\in$. The structure $(S, \in)$ is often referred to as spin space and abbreviated simply by S. An element κ^A of S (i.e. a carrier of the 2×2 matrix representations of SL(2, C)) is referred to as a 2-component spinor.[13] Its Hermitian conjugate $\bar{\kappa}^{A'}$ is an element of the 2-dimensional complex vector space denoted S'. An element $\mu^{A'}$ of S' is distinguished from an element of S by the primed index convention. The spaces S and S' are distinct: Hermitian conjugation does not identify S with S' insofar as it defines an anti-isomorphism. For $\kappa^A, \mu^A \in S$ and $\lambda \in C$, conjugation takes $\kappa^A + \lambda\mu^A$ into $\bar{\kappa}^A + \bar{\lambda}\bar{\mu}^A$, rather than $\bar{\kappa}^A + \lambda\bar{\mu}^A$. In addition to the vector spaces S, S', one can form the dual spaces S^*, S'^* with elements ω_A, $\pi_{A'}$, distinguished by lowered indices. They are anti-linear maps from S and S' into C.

The bilinear skew-symmetric 2-form $\in$ is a map from $S \times S \to C$ and can be identified with an element $\in_{AB}$ of $S^* \times S^*$. It amounts to a skew-symmetric 'metric' on S, identifying S with its dual S^* by means of the mappings $\kappa^B \mapsto \kappa_B = \in_{AB}\kappa^A$ and $\kappa_C \mapsto \kappa^C = \in^{CB}\kappa_B$, where $\in_{AB}\in^{BC} = \delta_A{}^C$ is the identity element on S. Hence the $\in$-symbol raises and lowers spinor indices (the standard convention uses contraction over the first index of $\in_{AB}$ to lower indices and contraction over the second index to raise indices). The linear transformations $A\colon S \to S$ that preserve $\in_{AB}$ are elements of SL(2, C).[14] They act on elements of S via $\kappa'^A = A^A{}_B\kappa^B$. A general spinor of valence $(p, q; r, s)$ is an element of the tensor product of the four spaces S, S^*, S', S'^*:

$$\Phi^{A\ldots B}{}_{C\ldots D}{}^{A'\ldots B'}{}_{C'\ldots D'} \in (S)^p \times (S^*)^q \times (S')^r \times (S'^*)^s, \tag{13}$$

where $(S)^p$ is the tensor product of p copies of S, etc. The standard techniques of tensor algebra can now be employed to generate relations between spinors of various valences.

[13] Alternatively, a univalent spinor, or a Weyl spinor. Such 2-spinor elements of S should be distinguished from the 2-spinors that appear in non-relativistic quantum mechanics and that encode the two degrees of freedom of the non-relativistic wave function associated with its spin along a particular axis ('up' or 'down'). The 2-component spinors of S are carriers (of matrix representations) of SL(2, C); non-relativistic 2-component spinors are carriers of SU(2), the 2-fold covering group of SO(3). The move from SU(2) spinors to SL(2, C) spinors may be associated heuristically with the move from Galilean-invariant quantum mechanics to Lorentz-invariant quantum mechanics. The 'relativistic' spinors that usually appear in the latter are Dirac spinors ψ^a, $a = 0, 1, 2, 3$. These are 4-component elements of the direct sum space $S'^* \oplus S$: $\psi^a = (\kappa^{A'}, \mu_A)$ (see equations (22) below). Note, finally, that there is nothing inherently quantum mechanical about spinors (of either the SU(2) or the SL(2, C) variety). Indeed, SL(2, C) spinors have been used to great extent in classical general relativity.

[14] In particular, if the determinant of A is defined by $\det(A) = \in_{AB}\in^{CD}A^A{}_C\, A^B{}_D$, the $\det(A) = 1$ entails $A^A{}_C A^B{}_D \in_{AB} = \in_{CD}$. This is analogous to the defining property of the homogeneous Lorentz group O(1, 3) given by $\Lambda^a{}_c\, \Lambda^b{}_d\, g_{ab} = g_{cd}$, where $\Lambda \in \mathrm{O}(1, 3)$ and g_{ab} is the Minkowski metric.

No mention of coordinates has been made up to this point. Equation (13) defines a general valence spinor as a definite geometrical object in a coordinate-independent manner.[15] Just as with the standard tensor formalism, a coordinate-dependent description of spinors can also be used. A spin basis for S consists of two orthonormal (with respect to $\in_{AB}$) univalent 2-spinors (o^A, ι^A). Taken together, they are sometimes referred to as a spin basis and can be schematically represented by $\{\in_\Sigma\}^A$, where $\Sigma = 0, 1$ labels the basis and A is the abstract index label for a (1, 0; 0, 0) 2-spinor.[16] The components of a $(p, q; r, s)$ 2-spinor then can be obtained by expanding the 2-spinor in the appropriate spin basis:

$$\Phi^{\Sigma_1 \ldots \Sigma_p}{}_{\Gamma_1 \ldots \Gamma_q}{}^{\Sigma'_1 \ldots \Sigma'_r}{}_{\Gamma'_1 \ldots \Gamma'_s} = \Phi^{A \ldots B}{}_{C \ldots D}{}^{A' \ldots B'}{}_{C' \ldots D'} \{\in^{\Sigma_1}\}_A \cdots \{\in_{\Gamma'_s}\}^{D'}. \quad (14)$$

Note here that the components on the left are, in general, complex-valued functions; i.e. they are scalar quantities that take values in the coordinate chart adapted to the bases. The 2-spinor $\Phi^{A \cdots}{}_{C \ldots}{}^{A' \cdots}{}_{C' \ldots}$ and the bases, on the other hand, are definite, coordinate-independent geometrical objects. Since the bases are invariant under SL(2, C), the components transform between spin bases according to the rule

$$A(\Phi)^{\Sigma_1 \ldots \Sigma_p \Sigma'_1 \ldots \Sigma'_r}{}_{\Gamma_1 \ldots \Gamma_q \Gamma'_1 \ldots \Gamma'_s} = A^{\Sigma_1}{}_{\Delta_1} \ldots \bar{A}^{\Pi'_s}{}_{\Gamma'_s} \Phi^{\Delta_1 \ldots \Delta_p \Delta'_1 \ldots \Delta'_r}{}_{\Pi_1 \ldots \Pi_q \Pi'_1 \ldots \Pi'_s}, \quad (14')$$

where the A are SL(2, C) transformations.

Similarly, a time-oriented orthonormal basis in Minkowski vector space is called a tetrad basis and consists of a set of 4 mutually orthonormal (with respect to η_{ab}) vectors. Such a tetrad is given by $\{e_\mu\}^a$, where $\mu = 0$–3 labels the basis and the abstract index 'a' indicates a vector. The components of a (p, q) Minkowski world-tensor $T_w{}^{a \cdots}{}_{c \ldots}$ (to be distinguished from a general tensor[17]) relative to a tetrad basis are given by expanding the world-tensor in the basis:

$$T_w{}^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_q} = T_w{}^{a \ldots b}{}_{c \ldots d} \{e^{\nu_1}\}_a \cdots \{e_{\mu_q}\}^d. \quad (15)$$

They transform between tetrad bases according to the rule

$$\Lambda(T_w)^{\nu_1 \cdots \nu_p}{}_{\mu_1 \ldots \mu_q} = \Lambda^{\nu_1}{}_{\rho_1} \cdots (\Lambda^{-1})^{\sigma_q}{}_{\mu_q} T_w{}^{\rho_1 \cdots}{}_{\ldots \sigma'_q}, \quad (15')$$

where the Λ are homogeneous Lorentz transformations. Again, one should clearly distinguish between the coordinate representation of the world-tensor,

[15] Alternatively, the general valence spinor $\Phi^{A \ldots B}{}_{C \ldots D}{}^{A' \ldots B'}{}_{C' \ldots D'}$ may be viewed as a multilinear map Φ: $(S)^p \times (S^*)^q \times (S')^r \times (S'^*)^s \to C$ (for spinor fields, replace C with the space of complex-valued functions). Note that, strictly speaking, spinor fields cannot be considered geometrical object fields on the standard definition of the latter (see e.g. Trautman, 1965, p. 86). I am considering a widening of this definition to allow in general cross sections of relevant fiber bundles to count as geometrical object fields (see for instance Torretti (1983, p. 98) and Stachel (1992, 1986).

[16] Hence $\in_0{}^A = o^A$, $\in_1{}^A = \iota^A$. The normalisation condition requires $\in_{AB} o^A \iota^B = 1$. Penrose and Rindler (1984) use the term 'spin basis' to refer to such a pair of 2-spinors, reserving the term 'dyad basis' for an unnormalised pair that satisfies $\in_{AB} o^A \iota^B = \lambda$, for arbitrary complex λ.

[17] Here and below I distinguish a world-tensor with a 'w' subscript. World-tensors are carriers of matrix representations of $O^\uparrow_+(1, 3)$. General (4-dimensional) tensors are carriers of matrix representations of GL(4, R), the 4×4 general linear group over the reals.

given by its components $T_w{}^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_q}$ relative to coordinates adapted to a tetrad basis; and the coordinate-independent geometrical objects $T_w{}^{a \cdots}{}_{c \ldots}$ and $\{e_\mu\}^a$.[18]

The early literature on spinors defined a 2-spinor simply as an array of complex numbers that transforms according to (14′). Such literature often stresses that, in general, the A's in (14′) cannot be replaced by arbitrary non-degenerate linear transformations, whereas the Λ's in (15′) can. Hence, whereas a Minkowski world-tensor can be extended to a general tensor whose components can take values in arbitrary, general linear coordinate charts, a 2-spinor cannot be so-extended (e.g. Gel'fand *et al.* (1963, p. 252), Cartan (1966, pp. 150–151)). This is a consequence of the fact that, while $O_+^\uparrow(1, 3)$ can be embedded in the general linear group GL(4, R), SL(2, C) cannot, due to its topological properties, as Karakostas (1997, p. 261) points out.

At this point, three things should be made clear concerning the fact that SL(2, C) cannot be embedded in GL(4, R).

First, the fact that SL(2, C) cannot be embedded into GL(4, R) does not entail that GL(4, R) has no spinorial (i.e. double-valued) representations. Ne'eman and Sijacki (1987) have shown that GL(4, R) does have a double covering, denoted by $\overline{\mathrm{GL}}(4, R)$, which admits infinite dimensional representations.[19] Ne'eman and Sijacki (1997) and Sijacki (1998) are recent reviews canvassing ways in which physical fields can be described using such infinite dimensional spinorial representations of GL(4, R). Their work indicates two options for the physicist:

[18] Note that the distinction between a coordinate chart and a basis (or frame) should be made here, and in the spinor case above. In the general case of an n-dimensional manifold M, a coordinate chart is a pair (U, ϕ) where $U \subset M$ is a region of M and $\phi: U \to R^n$ is a map that labels each point in U with an n-tuple of real numbers $\{x^\mu\}$, $\mu = 1 \ldots n$. A frame field, on the other hand, is in general a set of linearly independent vector fields on M. (In the spinor case above, the frame field consists of 2-spinor fields). If the fields of the set mutually commute, then a coordinate chart can be adapted to them by taking their integral curves as coordinate curves (if they do not mutually commute, they constitute a non-coordinate, or non-holonomic, basis). Conversely, to every coordinate chart characterised by the n-tuple $\{x^\mu\}$ one can adapt a frame field by the identification $e_\mu{}^a = (\partial/\partial_x{}^\mu)^a$. The coordinates adapted to a Minkowski tetrad field are Minkowski inertial (standard) coordinates.

[19] I thank Michael Redhead for bringing this reference to my attention. That GL(4, R) admits (infinite dimensional) spinorial representation is not made all that clear in the physics literature. Two representative statements are:

> Like SO(n), the general linear group GL_n is not simply connected. However, its universal covering group has no linear representation other than GL_n representations. This is why physicists tried in vain for some time to define spinors in curved space using Einstein's gauge (Glöckeler and Schücker, 1987, p. 191).

> Since a general, curved spacetime possesses no isometries or any other preferred classes of diffeomorphisms and since even in Minkowski spacetime there is no natural action of the full group of diffeomorphisms on spinor fields, we cannot expect to define a 'transformation law' of the type (2.2.10) [i.e. a general linear transformation] under diffeomorphisms for spinor fields in curved spacetimes (Wald, 1984, p. 360).

These statements are correct if by 'spinor' is meant 'finite dimensional representation of SL(2, C)'. Cartan's (1996, p. 151) 'no-go' theorem is in fact restricted to the finite representation case.

(a) Stick with finite dimensional representations to describe physical fields. Thus we may use finite dimensional representations of GL(4, *R*) (i.e. standard general tensor fields) for integer-spin fields, and finite dimensional representations of SL(2, *C*) (i.e. 2-spinor fields) for half-integer-spin fields. The components of 2-spinor fields cannot take values in general linear coordinate charts because SL(2, *C*) cannot be embedded in GL(4, *R*).
(b) Use infinite dimensional representations of $\overline{\mathrm{GL}}(4, \mathrm{R})$ (what Ne'emen and Sijacki call 'manifields') to represent physical fields of both integer and half-integer spin. Since there is a natural action of Diff(4, *R*) (the group of diffeomorphism of R^4) on these, we can use arbitrary general linear coordinates to describe them.

Option (b) effectively refutes Zangari and Karakostas's argument against the CS thesis. In brief (see Sections 5 and 6 below), they may be taken to argue that:

(1) The components of 2-spinor fields (i.e. finite dimensional representations of SL(2, *C*)) do not take values in ε-coordinates;
(2) Half-integer-spin fields must be represented by 2-spinor fields;
(3) Therefore, the existence of half-integer-spin fields prohibits the use of ε-coordinates.

Option (b) indicates that (2) is false. In the rest of this paper, I shall argue that even if one adopts option (a) and allows that 2-spinor fields are the best way to describe half-integer-spin fields, the CS thesis is not thereby put in jeopardy.

The second point concerning the fact that SL(2, *C*) cannot be embedded into GL(4, *R*) is that it does not follow from this that 2-spinor components *only* take values in spin bases. The consequence of replacing the *A*'s in (14′) with general linear transformations is essentially to extend the group SL(2, *C*) to the general linear group of 2×2 complex matrices GL(2, *C*). Mathematically, there is nothing to stop one from doing this, thereby obtaining values for the components of 2-spinors in general linear 2-dimensional complex coordinates. What is prohibited is a physical interpretation of such coordinates; and this prohibition rests on a mathematical result; namely, that carriers of GL(2, *C*), which might be called GL(2, *C*)-'spinors', are not related to carriers of GL(4, *R*) (i.e. general tensors) in the same way (detailed below) that SL(2, *C*) 2-spinors are related to carriers of $O^{\uparrow}_{+}(1, 3)$ (i.e. world-tensors). GL(2, *C*) cannot be embedded into GL(4, *R*); hence general tensors cannot be decomposed into products of GL(2, *C*)-'spinors'. (For this reason, calling such general linear 2-component complex vectors 'GL(2, *C*)-*spinors*' is an abuse of standard terminology, which restricts the term 'spinor' to a carrier of a double-valued representation. In this sense, saying that 2-spinor components can only take values in spin bases is true by definition (in the same way that saying world-tensor components can only take values in tetrad bases is true by definition).)

Thus carriers of GL(2, *C*) do not admit physical interpretations; there are no physical phenomena that they can be said to describe. But this does not prevent one from viewing the extension to GL(2, *C*) as a passive coordinate re-description of a carrier of SL(2, C). Again, such an extension is mathematically possible; the obstacle it faces is interpretational; what is prohibited are physical interpretations of general linear 2-dimensional complex coordinates. (Of course this is just what underlies the CS debate; namely, the physical interpretation of coordinate charts, and not just their mere existence. I would claim that emphasis on coordinate-dependent techniques obscures this.)

The third and final point concerning the non-embeddability of SL(2, C) in GL(4, R) is the following: it turns out that even-indexed 2-spinors are isomorphic with world-tensors. Hence, since there are generally covariant expressions of world-tensors, there are generally covariant expressions of even-indexed 2-spinors. Thus the *information content* of even-indexed 2-spinors *can* be expressed in arbitrary general linear coordinates. (Recall that an expression is generally covariant just when it is given purely in terms of general tensors; such an expression can be expanded in arbitrary coordinate bases.) Moreover, the information content of odd-indexed 2-spinors can be expressed in arbitrary charts up to a sign, in so far as odd-indexed 2-spinors can be given world-tensor expressions up to a sign. This will be made clear in discussion of the relation between spinors and tensors in the coordinate-independent approach, to which I now turn.

4.1. Isomorphism between M^4 and $Re(S \times S')$

To relate 2-spinors and tensors, one can construct an isomorphism between the real subspace $Re(S \times S')$ of $S \times S'$ and Minkowski vector space M^4 in the following manner. Let $(o^A\bar{o}^{A'}, o^A\bar{\iota}^{A'}, \iota^A\bar{o}^{A'}, \iota^A\bar{\iota}^{A'})$ be a basis for $S \times S'$, where (o^A, ι^A) and $(\bar{o}^{A'}, \bar{\iota}^{A'})$ are spin bases for S and S'. Then $Re(S \times S')$ is a real 4-dimensional vector space.[20] Moreover, it can be shown that $Re(S \times S')$ is endowed with a Lorentz metric of signature (1, 3). To see this, note that a suitable orthonormal basis for $Re(S \times S')$ is given by[21]

$$t^{AA'} = \frac{1}{\sqrt{2}}(o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}), \quad y^{AA'} = \frac{i}{\sqrt{2}}(o^A\bar{\iota}^{A'} - \iota^A\bar{o}^{A'}),$$

$$x^{AA'} = \frac{1}{\sqrt{2}}(o^A\bar{\iota}^{A'} + \iota^A\bar{o}^{A'}), \quad z^{AA'} = \frac{1}{\sqrt{2}}(o^A\bar{o}^{A'} - \iota^A\bar{\iota}^{A'}). \tag{16}$$

This can be checked by forming the (0, 2; 0, 2) spinor $\eta_{AA'BB'} = \in_{AB} \in_{A'B'}$ with respect to which the above basis has the desired properties:

(i) $\eta_{AA'BB'}t^{AA'}x^{BB'} = 0$, and likewise for other combinations; and,

[20] To see this, note that any spinor $\tau^{AA'} \in S \times S'$ can be expanded as $\tau^{AA'} = \alpha o^A\bar{o}^{A'} + \beta o^A\bar{\iota}^{A'} + \gamma\iota^A\bar{o}^{A'} + \delta\iota^A\bar{\iota}^{A'}$, where $\alpha, \beta, \gamma, \delta \in C$. $\tau^{AA'}$ is real just when $\bar{\tau}^{AA'} = \tau^{AA'}$. This entails that $\alpha, \beta \in R$ and $\gamma = \bar{\delta}$. Hence the real elements of $S \times S'$ can be parameterised by four real constants each and thus form a 4-dimensional real subspace.

[21] An alternative orthonormal basis for $Re(S \times S')$ is given by $l^{AA'} = o^A\bar{o}^{A'}, m^{AA'} = o^A\bar{\iota}^{A'}, n^{AA'} = \iota^A\bar{\iota}^{A'}, \bar{m}^{AA'} = \iota^A\bar{o}^{A'}$. The corresponding basis $\{l^a, m^a, n^a, \bar{m}^a\}$ in M^4 is referred to as a null tetrad.

(ii) $\eta_{AA'BB'}t^{AA'}t^{BB'} = 1 = -\eta_{AA'BB'}x^{AA'}x^{BB'} = -\eta_{AA'BB'}y^{AA'}y^{BB'}$

$$= -\eta_{AA'BB'}z^{AA'}z^{BB'}; \tag{17}$$

hence $\eta_{AA'BB'}$ is a Lorentz metric on $\mathrm{Re}(S \times S')$.

Now suppose (t^a, x^a, y^a, z^a) is a tetrad basis for M^4. Define the map σ: $\mathrm{Re}(S \times S') \to M^4$ by

$$\sigma^a{}_{AA'} = t^a t_{AA'} - x^a x_{AA'} - y^a y_{AA'} - z^a z_{AA'}. \tag{18}$$

Then $\sigma^a{}_{AA'}$ is an isomorphism preserving the Lorentz metric $\eta_{ab} = \sigma_a{}^{AA'}\sigma_b{}^{BB'}\eta_{AA'BB'}$. In components relative to a spin basis, $\sigma^\mu{}_{\Sigma\Sigma'} = (1/\sqrt{2})\sigma^\mu$, where σ^μ are the Pauli matrices. In such a basis, the components of the image $v^{AA'}$ of a world-tensor (4-vector) $v_w{}^a \in M^4$, with components (v^0, v^1, v^2, v^3) relative to the basis (t^a, x^a, y^a, z^a), are given by

$$v^{\Sigma\Sigma'} = \frac{1}{\sqrt{2}}\begin{pmatrix} v^{00'} & v^{01'} \\ v^{10'} & v^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} v^0 + v^3 & v^1 + iv^2 \\ v^1 - iv^2 & v^0 - v^3 \end{pmatrix}. \tag{19}$$

This is the coordinate representation of the 2-spinor $v^{AA'}$ in the basis $(o^A\bar{o}^{A'}, o^A\bar{\iota}^{A'}, \iota^A\bar{o}^{A'}, \iota^A\bar{\iota}^{A'})$. Again, it should be stressed that $v^{AA'}$ is a well-defined mathematical object independent of this particular coordinate representation.

To summarise, the isomorphism σ: $\mathrm{Re}(S \times S') \to M^4$

(1) is defined in terms of a tetrad basis in M^4; and
(2) only relates even-indexed 2-spinors with world-tensors (the extension of σ to isomorphisms between tensor product spaces $M^4 \times M^4 \times \ldots$ and $\mathrm{Re}(S \times S') \times \mathrm{Re}(S \times S') \times \ldots$ is trivial).

This last fact should be emphasised. It indicates that the isomorphism σ just is an isomorphisms between even-indexed 2-spinors (carriers of SL(2, C)) and world-tensors (carriers of $\mathrm{O}^\uparrow_+(1, 3)$). This is what the explicit appearance of standard coordinates in its construction indicates. To extend the isomorphism to general tensors, one needs to explicitly include a tetrad basis. Hence under σ, the (1, 0; 1, 0) 2-spinor $v^{AA'}$ corresponds uniquely to the world-tensor $v_w{}^a$ and to the general tensor pair $(v^a, \{e_\mu\}^a)$, where v^a is a general tensor and $\{e_\mu\}^a$ is a tetrad field. This general tensor pair is the generally covariant expression of the even-indexed 2-spinor.

4.2. Odd-indexed spinors

Again, the isomorphism (18) effects a 1-1 translation between even-indexed 2-spinors and Minkowski world-tensors. For odd-indexed 2-spinors, no such 1-1 translation exists owing to the topological properties of SL(2, C).[22]

[22] Recall that univalent 2-spinors, as carriers of representations of SL(2, C), change sign under 2π rotations, whereas 4-vectors, as carriers of representations of $\mathrm{O}^\uparrow_+(1, 3)$, do not. Any odd-indexed 2-spinor is, in general, the sum of outer products of an odd number of univalent 2-spinors, hence will also change sign under 2π rotations.

However, a 2-1 translation can be obtained, which I now review (cf. Penrose and Rindler, 1984, pp. 125–129).

Every univalent 2-spinor κ^A defines a null 4-vector $l_w{}^a$ by $l_w{}^a = \kappa^A\bar{\kappa}^{A'}$.[23] But $e^{i\theta}\kappa^A$ defines the same null vector $l_w{}^a$ for real θ. This phase is encoded up to sign in the bivector defined by

$$(F_w)_{ab} = \kappa_A\kappa_B \in_{A'B'} + \kappa_{A'}\kappa_{B'} \in_{AB}. \tag{20}$$

To see this, one first completes κ^A to a spin basis (κ^A, τ^A), where $\tau^A\tau^A$ is unique up to an additive multiple of κ^A. The bivector $(F_w)_{ab}$ can then be written as

$$(F_w)_{ab} = l_a m_b - l_b m_a + l_a \bar{m}_b - l_b \bar{m}_a = l_{[a} X_{b]}, \tag{21}$$

where $(l^a, m^a, n^a, \bar{m}^a)$ is null tetrad (footnote 21) and $X_b = (1/\sqrt{2})(m_b + \bar{m}_b)$ is real, spacelike and orthogonal to l^a.[24] Hence $(F_w)_{ab}$ contains $l_w{}^a$ and lies in the spacelike 2-plane spanned by X_a and $Y_a = (1/\sqrt{2})(m_b - \bar{m}_b)$. The pair $(l_w{}^a, (F_w)_{ab})$ is called a 'null flag', consisting of a flagpole $l_w{}^a$ with flag $(F_w)_{ab}$. Under the phase change $\kappa^A \mapsto e^{i\theta}\kappa^A$, we have $\tau^A \mapsto e^{-i\theta}\tau^A$ and $m^a \mapsto e^{2i\theta}m^a$. Hence $X_a \mapsto \cos 2\theta X_a + \sin 2\theta Y_a$ and $Y_a \mapsto -\sin 2\theta X_a + \cos 2\theta Y_a$. Thus under a phase change of θ, the flagpole $l_w{}^a$ remains invariant but the flag $(F_w)_{ab}$ rotates about $l_w{}^a$ by 2θ. In particular, under a phase change of π, the spinor κ^A changes sign, while the null-flag remains invariant. In this sense, a null-flag only encodes a univalent 2-spinor up to sign.

Note that such null-flags are world-tensors; they assume the existence of a tetrad field. A generally covariant expression up to sign of a univalent 2-spinor κ^A is obtained by replacing the world-tensor pair $(l_w{}^a, (F_w)_{ab})$ with a general tensor triple $(l^a, F_{ab}, \{e_\mu\}^a)$ satisfying the appropriate conditions.

4.3. *Fields with half-integer spin as carriers of SL(2, C)*

The above correspondences between 2-spinors and world-tensors may be extended to Minkowski spacetime (M, η_{ab}) (for M a differential manifold and η_{ab} a Minkowski metric) by replacing M^4 with the tangent spaces $T_p(M)$ at points p of M, and defining a 2-spinor field of valence $(p, q; r, s)$ in (M, η_{ab}) as a map from points in M to the space $(S)^p \times (S^*)^q \times (S')^r \times (S'^*)^s$. Spinor field equations can now be written by introducing a spinor derivative, and translations for various field equations can be given (see e.g. Wald (1984), Penrose and Rindler (1984)). One finds, in general, that fields with spin s may be represented by 2-spinors with $2s$ indices. Hence, in the context of Minkowski spacetime, the 2-spinor formalism is expressively equivalent to the tensor formalism (insofar as,

[23] To see this, note that, under the isomorphism σ, such an l^a is indeed null: $\eta_{ab}l^a l^b = \in_{AB} \in_{A'B'} \kappa^A\bar{\kappa}^{A'}\kappa^B\bar{\kappa}^{A'} = 0$.

[24] To derive this, use is made of the relation $\kappa_A\tau_B - \tau_A\kappa_B = \in_{AB}$ which holds in general for any spin basis (κ_A, τ_A). X_a is unique up to an additive multiple of l_a determined by the choice of τ^A. Under $\tau^A \mapsto \tau^A + \lambda o^A$, $\lambda \in C$, we have $m_a \mapsto m_a + \lambda m_a$, $\bar{m}_a \mapsto \bar{m}_a + \bar{\lambda}\bar{m}_a$, hence $X_a \mapsto X_a + (\lambda + \bar{\lambda})l_a$.

for any theory T restricted to Minkowski spacetime, dynamically possible models of T expressed in the tensor formalism can be expressed in the 2-spinor formalism).

However, as we have seen with odd-indexed 2-spinors, the converse does not in general hold. For half-integer-spin fields given by odd-indexed 2-spinor fields, no fully equivalent tensor expressions can be constructed; rather, one must be content with tensor expressions up-to-a-sign. For example, in the coordinate-independent 2-spinor formalism, the Dirac equation is given by

$$\partial_{AA'}\psi^{A'} = \mu\varphi_A, \quad \partial^{AA'}\varphi_A = \mu\psi^{A'}, \tag{22}$$

where $\mu = -i2^{-1/2}m\hbar^{-1}$ and the 2-spinors $\psi^{A'}, \varphi_A$ form the 4-component Dirac spinor $\Psi = (\psi^{A'}, \varphi_A)$, appearing in the standard expression $(i\hbar\gamma_a\partial^a - m)\Psi = 0$.[25] Equation (22) can be translated into the tensor formalism by (Penrose and Rindler, 1984, pp. 221–222):

$$\begin{aligned} F_{ab}\nabla_d F_c{}^d + F_{ad}\nabla_c F_b{}^d &= -2\mu F_{ab}C_c, \\ G_{ab}\nabla_d G_c{}^d + G_{ad}\nabla_c G_b{}^d &= -2\mu G_{ab}C_c, \end{aligned} \tag{22$'$}$$

where $F_{ab} = \varphi_A\varphi_B \in_{A'B'}, G_{ab} = \psi_{A'}\psi_{B'} \in_{AB'}$ and $C_a = \varphi_A\psi_{A'}$. These translations require F_{ab}(resp. G_{ab}) to be null, skew, and anti-self-dual (resp. self-dual):[26]

$$\begin{aligned} &\text{(a) } F_{ab}F^{ab} = G_{ab}G^{ab} = 0, \\ &\text{(b) } F_{ab} = -F_{ba};\ G_{ab} = -G_{ba}, \\ &\text{(c) } {}^*F_{ab} = -iF_{ab};\ {}^*G_{ab} = iG_{ab}. \end{aligned} \tag{23}$$

Note that the null and (anti-) self-dual conditions can be satisfied by introducing a tetrad $\{e_\mu\}^a$ (in the case of condition (c), such a tetrad provides the orientation 4-form e_{abcd}). Hence a Dirac 4-spinor Ψ corresponds up to a sign to the general tensor 4-tuple $(F_{ab}, G_{ab}, C_{a'}, \{e_\mu\}^a)$ satisfying conditions (a)–(c) and the field equations (22′). (The translation is essentially based on the one between a univalent 2-spinor and a null-flag, taking into account that the Dirac 4-spinor is composed of two univalent 2-spinors. Penrose and Rindler (1984, p. 222) give the extension of (22′) for nth-order differential equations involving odd-indexed 2-spinors).

At this point, it should be noted that there is empirical evidence for the rotational behaviour of carriers of SL(2, C). The mere existence of half-integer-spin fields does not in and of itself provide reason to use carriers of SL(2, C)

[25] Equations (22) are obtained from the standard expression of the Dirac equation by identifying γ_a with the map $\gamma_a{}^{A\oplus A'}: M^4 \mapsto \mathrm{Hom}(S\oplus S'^*)$ given by $\gamma_a{}^{A\oplus A'}v^a(\psi^{A'}, \varphi_A) = \sqrt{2}(v^{AA'}\varphi_{A'}, v_{AA'}\psi^{A'})$, where $v^a \in M^4$ and $\mathrm{Hom}(S\oplus S'^*)$ is the set of linear transformations from $S\oplus S'^*$ to itself. The effect of the linear transformation $\gamma_a{}^{A\oplus A'}v^a$ is to first exchange the 2-spinor components of a Dirac 4-spinor and then contract them with the σ-image of the 4-vector.

[26] The $*$-operator in condition (c) is the hodge-dual operator, defined for bivectors by ${}^*F_{ab} = \frac{1}{2}e_{abcd}F^{cd}$, where e_{abcd} is a volume element 4-form.

to describe them. It might be argued that the phase change associated with SL(2, C)-induced rotations is unobservable (insofar as such phases are, from the standpoint of quantum mechanics, unobservable). Hence one might conclude that representations of half-integer-spin fields that are unique up to a sign are sufficient. However, macroscopic spinorial behaviour has been observed.[27] This indicates that the additional degree of freedom associated with carriers of SL(2, C) is physically significant, hence carriers of SL(2, C) are better suited to describe half-integer-spin fields than are carriers of $O^{\uparrow}_{+}(1, 3)$.

Finally, it should be mentioned that 2-spinor fields may be defined on curved spacetimes as sections of an appropriate spinor bundle. This is the vector bundle associated with a principal SL(2, C)-bundle (or dyad bundle). The latter may be heuristically considered as the 'double covering' of the bundle of time-oriented and oriented orthonormal frames (more precisely, the bundle space of a dyad bundle is the double covering of the bundle space of a tetrad bundle). One might expect then that 2-spinor fields, as cross sections of spinor bundles, can only be defined on manifolds that support a global tetrad field. Indeed, Geroch (1968) has shown that, for non-compact spacetime manifolds, a necessary and sufficient condition for the existence of spinor fields is the existence of a global tetrad field (see Wald (1984, pp. 365–369) for a lucid discussion of this and other conditions for the existence of spinor bundles).

One point to be kept in mind here is that the existence of a global tetrad field does not, in and of itself, force us to use a standard Minkowski inertial coordinate chart adapted to it. Indeed, the arena of the traditional CS debate is Minkowski spacetime which trivially admits a global tetrad field, and this has not prevented conventionalists from renouncing standard coordinate charts. It is a realist interpretation of the global tetrad field that can be defined in Minkowski spacetime to which the conventionalist objects. Again, more specifically, her objection is to a realist interpretation of the isotropic temporal structure provided by such a tetrad. She claims that the temporal structure of spacetime is an in-principle unobservable object; and, as I have described it above, her conventionalism is motivated by skepticism towards such objects. However, as described above, her conventionalism is only possible in so far as there are multiple intertranslatable descriptions of the phenomena that agree on all observable aspects. Zangari and Karakostas in effect claim that this is not the

[27] Aharonov and Susskind (1967, p. 1237) proposed the following thought experiment to detect the rotational behaviour of half-integer-spin fields:

> Imagine two systems having free electrons and exhibiting tunnelling current when they are close together. We then separate the systems spatially and rotate one of them n times relative to the other and bring them together so that tunnelling current can flow again. It turns out that the direction of current flow depends on n modulo 2.

In Werner *et al.* (1975), a similar effect was observed for neutrons. A neutron beam is split and one component in directed through a magnetic field that can be adjusted in order to force the magnetic moments of neutrons passing through it to precess by a given amount (2π, say). The beams are recombined and relative phase shifts can then be observed in the form of interference patterns. See also Weingard and Smith (1982) for a discussion of the interpretation of such experiments.

case; that the wealth of inter-translatable descriptions reduces to just a single description when one attempts to account for the empirically-grounded rotational behaviour of half-integer-spin fields. I turn now to their critique.

5. Zangari and Karakostas on SL(2, C) and ε-Coordinates

Zangari (1994, p. 273) and Karakostas (1997, p. 260) observe that, while ε-extended Lorentz coordinate transformations can be defined via (5), there are no corresponding ε-extended SL(2, C) coordinate transformations.[28] Using (8) and (10), such a transformation M would satisfy

$$H_\varepsilon = \mathrm{MHM}^\dagger = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} ct + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & ct - x^3 \end{pmatrix}\begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \tag{24}$$

where

$$H_\varepsilon = x_\varepsilon{}^\mu \sigma_\mu = \begin{pmatrix} (ct + \varepsilon_i x^i) + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & (ct - \varepsilon_i x^i) - x^3 \end{pmatrix}, \tag{25}$$

and a solution M exists only for $\varepsilon^i = 0$. Zangari and Karakostas conclude that the possibility of defining ε-coordinates vanishes in the 2-spinor formalism:

> Thus, contrary to what the CS thesis (and even its opponents) assert, it is not always possible to define coordinates with [$\varepsilon_R \neq 1/2$] on which representations of the Lorentz group can act. Spinor representations do not have Lorentz transformations in non-standard coordinates systems, and one cannot then formulate STR [special theory of relativity] (Zangari, 1994, p. 273).
>
> Hence the possibility of recoordinatising the temporal component of a local coordinate system according to t_ε-synchrony relations no longer exists, if the well-defined transformation properties of the spinor representation of the Lorentz group are to be preserved. Thus the standard synchronisation relation, contrary to the CS thesis, seems to be singled out by the very nature of things when the spinor (universal covering) group of Lorentz transformations is taken into consideration (Karakostas, 1997, p. 260).

What the non-existence of an ε-version of M indicates is that an ε-transformation Λ_ε cannot be decomposed into the product of two Herm(2) matrices, and, in

[28] Zangari (1994, pp. 270–271) interprets (8) as a complex representation of a spacetime point (Gunn and Vetharaniam (1995, p. 604) follow his lead). This is a bit misleading. Such an interpretation conflates the notion of a vector space (M^4) with that of an affine space (Minkowski spacetime). Equation (8) establishes a correspondence between Minkowski 4-vectors and 2×2 complex Hermitian matrices. At the most, this can be interpreted as a correspondence between tangent vectors at a point in Minkowski spacetime and elements of Herm(2); and not as a correspondence between Minkowski spacetime points and elements of Herm(2). The significance of the group SL(2, C) is not that it provides 'complex representations' of Minkowski spacetime points, but rather it is the dynamical symmetry group for half-integer spin fields (see footnote 12).

particular, into the product of two SL(2, C) matrices. This means that SL(2, C) cannot be embedded in ε-O(1, 3), and this, again, is a special consequence of the fact that SL(2, C) cannot be embedded in GL(4, R). Hence the components of 2-spinor fields cannot take values in ε-coordinates. From this Zangari and Karakostas conclude that, once we require the use of 2-spinor fields in descriptions of physical phenomena, we cannot employ ε-coordinates.

While this claim is straightforwardly correct from a mathematical point of view, its significance to the CS debate is questionable. Before canvassing possible conventionalist options, a few initial qualifications should be made explicit.

(a) First, arguably, 2-spinors (even and odd-indexed) can be expanded in arbitrary GL(2, C) coordinates (see Section 4 above). So while ε-coordinates are ruled out by 2-spinors, standard coordinates, *qua* coordinates, are not thereby given priority.

(b) Again as indicated in Section 4, one may adopt a coordinate-dependent definition of 2-spinors that restricts their components, by definition, to spin bases. However, given this choice, not only are ε-coordinates and 4-dimensional real general linear coordinates prohibited; so also are standard Minkowski inertial coordinates. The only coordinates permitted are those adapted to spin-bases.[29]

(c) While even-indexed 2-spinor components cannot take values in ε-coordinates, the information content they contain *can* be expressed in ε-coordinates; and the information content contained in odd-indexed 2-spinors can be expressed in ε-coordinates up to a sign. These facts follow from the ability to construct generally covariant expressions for even-indexed 2-spinors, and generally covariant expressions up to a sign for odd-indexed 2-spinors.

(d) Finally, note that what the non-existence of an ε-version of M establishes is that the isomorphism between M^4 and $\mathrm{Re}(S \times S')$ is by definition one between world tensors and even-indexed 2-spinors. The appearance of standard charts in the construction of the isomorphism entails that an even-indexed 2-spinor can only be translated into a general tensor (as opposed to a world-tensor) in the presence of a tetrad basis. In other words, if we want a generally covariant description of even-indexed 2-spinor fields, then we cannot do without a tetrad basis. Hence the charge against the conventionalist can be phrased in terms of the following (Spinor Realism) claim.

(SR) The existence of half-integer-spin fields requires using odd-indexed 2-spinor fields to describe them. The existence of odd-indexed 2-spinor

[29] Of course these are just 'spin-entangled' Minkowski inertial coordinates (i.e. inertial coordinates with two extra degrees of freedom). Such spin-coordinates do provide the temporal structure needed to define the standard simultaneity relation. Still, the point remains that if the concern is solely with what coordinate charts are permitted by the existence of 2-spinor fields, then we see that such fields prevent the use of *both* non-standard ε-coordinates and standard coordinates, as the latter are normally defined. I agree that the moral at this point is that coordinates are not what are at stake, but rather a realist interpretation of temporal structure. This further urges the adoption of (SR) below as the point of debate for a spinor realist.

> fields as *geometrical objects* requires the existence of a global Minkowski tetrad field. In particular, it entails the existence of an isotropic temporal structure on spacetime that *cannot* be done away with, if we wish to account for all the observable phenomena.

This, I would submit, is the substantive charge that can be levelled against the conventionalist by a spinor realist. It indicates how the conventionalist's semantic anti-realism in undercut by the privileging of one descriptive framework over competing alternatives. Note that (SR) can be justified simply by reference to Geroch's (1968) theorem that the existence of 2-spinor fields on a spactime manifold M requires the existence of a global Minkowski tetrad field on M. One need not bring coordinates into the discussion at all. I now consider ways the conventionalist can respond to (SR).

6. Options for the Conventionalist

If the conventionalist is read as claiming that there is no fact of the matter as to the local metrical structure of spacetime (in particular, the local temporal structure), i.e. there is no fact of the matter that determines whether, locally, the metric of spacetime is the Minkowski metric η_{ab} or an ε-metric $(g_\varepsilon)_{ab}$, then a Spinor Realist will counter with the claim that the existence of half-integer-spin fields provides the fact of the matter that privileges a preferred temporal structure. And this indicates that there is a preferred simultaneity relation; namely, the standard one.

In response, the conventionalist can first and foremost claim that the existence of half-integer-spin fields does not require using odd-indexed 2-spinor fields to describe them. As indicated in Section 4, we can use spinorial representations of GL(4, R) in descriptions of half-integer-spin fields; and such representations do admit ε-coordinates.

Furthermore, even if the realist is granted the use of 2-spinors in descriptions of half-integer-spin fields, the conventionalist can simply claim that anything you can do with even-indexed 2-spinor fields, you can do just as well with general tensors. And, while what you can do with odd-indexed 2-spinor fields, you can only do up to a sign with general tensor fields, this is fine, since odd-indexed 2-spinor fields have nothing to do with determinations of clock synchrony. Simply put, clocks and measuring rods are not spinorial (i.e. half-integer-spin) objects. Hence, for the purposes of describing procedures involving the determination of clock synchrony, the tensor formalism is adequate. To argue otherwise is to engage the conventionalist in an argument the premisses of which she will reject; namely, that unifying power, or simplicity, count as criteria when it comes to justifying belief in in-principle unobservable objects.

Recall that the conventionalist's position is a particular type of anti-realist's position; specifically, the conventionalist is an anti-realist with respect to the in-principle unobservable quantity given by the speed of light in a given direction. As indicated in Section 2, any value (constrained by $0 < \varepsilon < 1$) is consistent with the observational consequences of special relativity. Hence any simultaneity relation defined by a given value for ε is observationally adequate. In the past, realists have argued that the standard simultaneity relation is privileged because it is more simple or more unifying than non-standard relations in so far as it is adapted to the conformal structure of Minkowski spacetime; or because we should be realists with respect to Robertson–Walker spacetimes (or, in general, globally hyperbolic spacetimes) with privileged time coordinates (cf. Friedman (1983), p. 320, Torretti (1983), p. 230). But the die-hard conventionalist will claim these responses beg the question over the in-principle unobservable one-way speed of light. In this context, the claim that half-integer-spin fields privilege a given time coordinate is just another argument that begs this same question. Half-integer-spin fields have nothing to do with determining clock synchrony, and, in particular, the one-way speed of light. Hence their existence should do nothing to convince an anti-realist who claims the one-way speed of light has no determinate value.

Furthermore, it is not all that apparent that arguments for using the 2-spinor formalism based on simplicity and/or unifying power should be convincing even to a spacetime realist. The existence of physical phenomena that can be described by the 2-spinor formalism, and that cannot be described by the general tensor formalism, should not automatically force us to use the former formalism. Note that, while the 2-spinor formalism is expressively equivalent to the world-tensor formalism, both are not expressively equivalent to the general tensor formalism. There are models of GR that can only be expressed using general tensors; that cannot be expressed, in particular, using world-tensors or 2-spinors. To restrict all descriptions of physical phenomena to the 2-spinor formalism would require viewing models of GR that do not admit global tetrad fields as unphysical. This seems a bit hasty. Part of the central problem facing contemporary theoretical physics is to reconcile phenomena best described by the 2-spinor formalism (viz, half-integer-spin fields) with phenomena best described by the general tensor formalism (viz, phenomena displaying diffeomorphism invariance). It seems the lesson to be learned, at least until a quantum theory of gravity is established, is that which formalism one should use will depend on what type of phenomena is being described. Hence the conventionalist would be warranted in refusing to use (even-indexed) 2-spinors in descriptions of clock synchrony procedures.

I submit, therefore, that the adoption of the 2-spinor formalism in descriptions of half-integer spin fields does not require adopting coordinates adapted to spin bases, let alone standard (Minkowski inertial) coordinates, in determinations of clock synchrony.

7. Gunn and Vetharaniam (1995)

In this section, I assess three points of criticism that Gunn and Vetharaniam (1995) level against Zangari (1994).

7.1.

First, Gunn and Vetharaniam claim that the Dirac equation governing the dynamical behaviour of a massive spin-1/2 field can be given an ε-coordinate covariant expression. In particular, they write,

$$(i\hbar(\gamma_\varepsilon)^\mu\partial_\mu - m)\psi^a = 0, \tag{26}$$

where ψ^a are the components of a Dirac 4-spinor and the ε-versions of the γ matrices are given by

$$(\gamma_\varepsilon)^0 = \begin{pmatrix} 1 & n_i\sigma^i \\ -n_i\sigma^i & 1 \end{pmatrix}, \quad (\gamma_\varepsilon)^i = \begin{pmatrix} 1 & \sigma^i \\ \sigma^i & 1 \end{pmatrix}. \tag{27}$$

They thus conclude that, *pace* Zangari, the SL(2, *C*) 'complex representation' of spacetime points (see footnote 28) is not necessitated, in particular, by the existence of spin-1/2 fields.

Karakostas (1997, p. 271) observes that the γ_ε do not form a Clifford algebra since $(\gamma_\varepsilon)_a(\gamma_\varepsilon)_b + (\gamma_\varepsilon)_b(\gamma_\varepsilon)_a \neq 2\eta_{ab}$. However, the γ_ε *are* closed under Clifford multiplication defined by $(\gamma_\varepsilon)_a(\gamma_\varepsilon)_b + (\gamma_\varepsilon)_b(\gamma_\varepsilon)_a = 2(g_\varepsilon)_{ab}$, with $(g_\varepsilon)_{ab}$ defined by (7). This is perfectly consistent with the definition of a Clifford algebra.[30] What is problematic with (26) in the context of the CS debate is the fact that the Dirac 4-spinor components in (26) are undefined in general linear coordinates. In particular, the components of a Dirac 4-spinor *cannot* take values in ε-coordinates. Recall that in the 2-spinor formalism, a Dirac 4-spinor is simply the direct sum of two univalent 2-spinors, and the components of the latter cannot take values in ε-coordinates, as discussed above.[31]

To be fair to Gunn and Vetharaniam, they treat the ψ in (26) as a scalar with respect to diffeomorphisms on *M*. (This follows the standard practice of constructing a curved-space version of the Dirac equation by gauging the Lorentz group. Karakostas (1997, pp. 273–274) provides a good exposition. See also Kaku (1993, p. 641).) Doing so, I would argue, does not directly address the objection to the CS thesis given by Zangari and Karakostas, which is that

[30] In general, if *V* is a vector space over a commutative field *K* with unit element 1 and equipped with a quadratic form $q: V \mapsto K$, then the Clifford algebra $C(V, q)$ associated with *V* is defined as the quotient $C(V, q) \equiv T(V)/J$, where $T(V)$ is the tensor algebra over *V*, and *J* is the two-sided ideal in $T(V)$ generated by elements of the form $x \otimes x + 2q(x)1$, for $x \in V$. The Clifford product in $C(V, q)$ is then the product induced by the tensor product in $T(V)$. In terms of the associated metric *g* (the bilinear form defined by $g(x, y) = q(x + y) - q(x) - q(y)$), the Clifford product is given by $xy + yx = 2g(x, y)1$, for $x, y \in V$. *Only* in an orthonormal basis is this product anticommutative. In general, a Dirac spinor is an irreducible representation of a Clifford algebra.

[31] This essentially means that the Clifford algebra $C(R^4, g_\varepsilon)$ generated by the γ_ε's does not have irreducible representations that are, in addition, representations of the direct sum SL(2, *C*)⊕SL(2, *C*). Arguably, ε-Dirac spinors can simply be defined as the irreducible representations of $C(R^4, g_\varepsilon)$, but such mathematical objects cannot be physically interpreted (in the same sense that GL(2, *C*) '2-spinors' cannot be physically interpreted): such ε-Dirac 'spinors' cannot encode the rotational behaviour of half-integer-spin fields.

2-spinor components cannot take values in ε-coordinates.[32] This is trivially the case for ψ when it is treated as a scalar with respect to coordinate transformations on M; as such, it has no components with respect to coordinate charts on M. It is only expandable in spin bases on the tangent spaces. Gunn and Vetharaniam are motivated by this to declare that spin space is an internal symmetry space that has nothing to do with spacetime. I think this stance is problematic, as I shall now attempt to demonstrate.

7.2.

In thinking that the SL(2, C) 'complex representation' of spacetime points is necessitated by the existence of odd-indexed spinor fields, Gunn and Vetharaniam accuse Zangari of conflating spin space as an internal symmetry parameter space with spacetime as the arena governing the kinematics of special relativity:

> one regards quantum mechanical spin as an internal property of particles. The symmetries corresponding to the transformation properties of spinors are symmetries of an internal spinor space. [...] Internal spinor space must, therefore, not be confused with the external space of Minkowski space time (1995, p. 605).

Apparently, Gunn and Vetharaniam claim that the two degrees of additional freedom provided by spinors should be associated with an internal state of the field being described, in analogy with isotopic spin, electroweak charge, colour, and other internal gauge symmetries that appear in standard Yang–Mills theories. These additional two degrees of freedom should not, they argue, be associated with degrees of freedom in spacetime. In particular, spacetime coordinates need not transform in the same way as spinor field components.

Gunn and Vetharaniam are correct in noting that spacetime coordinates need not transform in the same way as spinor components. Simply put, as indicated above, clocks and measuring rods are not spinorial objects. However, it is misleading to view spin as an internal symmetry on a par with gauge symmetries. Spin is one of two fundamental properties of a relativistic spacetime field, the other being mass. Wigner's famous analysis (1939) has shown that a relativistic field is determined by its mass and spin, insofar as mass and spin are the only two Casimir invariants of the Poincaré group, the symmetry group of Minkowski spacetime.

Another way of making the distinction between gauge symmetries and spin can be seen in the fiber bundle formulation of gauge theories. In this formalism, for a given gauge theory with gauge symmetry group G, the internal symmetry spaces are fibers in a principle G-bundle over a spacetime manifold M. The associated vector bundle is a tangent bundle with sections identified as

[32] The curved-space treatment of the Dirac equation requires the existence of a global tetrad field; hence, it does not directly address the Spinor Realist's claim given above that such a global tetrad field provides the fact of the matter as to the temporal structure of spacetime.

spacetime fields that interact with the gauge field. If these fields are half-integer-spin fields, then the associated vector bundle must be a spinor bundle, and a third bundle over *M* arises, namely, an SL(2, *C*) dyad-bundle. This latter differs from the principle gauge bundle insofar as it is soldered to the base space *M* (technically, this means that a soldering form exists on the dyad bundle). The presence of soldering amounts to the following distinction: whereas the connection and curvature on the gauge bundle do not define structures on *M*, the connection and curvature on the dyad bundle do project down to structures on *M*. In this sense, spin is intimately connected with the spacetime manifold in a way that typical Yang–Mills gauge symmetries are not.

7.3.

Third, Zangari claims that the SL(2, *C*) ‘complex representation’ of spacetime points is to be preferred for its topological simplicity. Gunn and Vetharaniam contend, however, that this simplicity is offset by the inability within the SL(2, *C*) representation to account for parity violations. They claim that the O(1, 3) ‘real representation’ of spacetime points is needed in order to be able to represent parity operations, which have physical significance, in the light of confirmed parity violation in electroweak theory.[34] Gunn and Vetharaniam thus argue not only that Zangari’s empirical argument for the SL(2, *C*) representation is wrong, but, moreover, that there is an empirical argument against the SL(2, *C*) representation.

[33] The existence of a soldering form on frame bundles is the main distinction between fiber bundle formulations of Yang–Mills gauge theories and a fiber bundle formulation of general relativity as a theory of a metric and connection over a principle frame bundle (see e.g. Trautman, 1980).

Before leaving this subject, I should note that Karakostas (1997, pp. 261–263) describes spinor fields (conceived as sections of a spinor bundle) as ‘Lorentz covariant and coordinate invariant’. This seems to be at odds with the standard notions of covariance and invariance (see e.g. Earman (1974), Friedman (1983), Norton (1993)). Under these notions, covariance refers to the form an equation takes with respect to a coordinate chart. A generally covariant equation retains its form in arbitrary general linear coordinates. A Lorentz covariant equation retains its form only in Minkowski inertial coordinates. A test for general covariance is whether the equation can be given an expression purely in terms of tensors. (Similarly, a test for Lorentz covariance is whether the equation can be given an expression purely in terms of world-tensors.) Invariance, according to the standard notion, refers to the symmetries of a given mathematical object; in particular, how the object behaves under active transformations. Tensors are generally (or, one might say, ‘coordinate’) invariant: they are invariant under diffeomorphisms (viz, active general linear transformations). World-tensors and even-indexed 2-spinors are Lorentz invariant. Note that odd-indexed 2-spinors are not: they can change sign under an active Lorentz transformation. Odd-indexed spinors are invariant only under actions of SL(2, *C*).

Hence an equation involving only even-indexed 2-spinors is Lorentz invariant. It can be put in a Lorentz covariant form as well as a generally covariant form; in the same way that an equation involving only world-tensors can be put in a generally covariant form. A equation involving only odd-indexed 2-spinors, on the other hand, is not even Lorentz invariant; it is SL(2, *C*)-invariant. Furthermore, it can be put into a generally covariant form only up to a sign.

[34] From footnote 6, parity operations may be represented by elements of the $O^{\uparrow}_{-}(1, 3)$ component (determinant $= -1$) of O(1, 3). SL(2, *C*) is connected to the identity, hence none of its elements can represent a reflection about a spatial axis.

As noted above, the issue should not be parsed in terms of an SL(2, *C*) representation of points versus an O(1, 3) representation; rather, the issue concerns the choice between a 2-component spinor formalism and a tensor formalism. In this context, one might thus claim: the 2-component spinor formalism does not admit parity transformation, whereas the tensor formalism does. (}I do not claim that this is what Gunn and Vetharaniam hold; I grant that they are simply responding to Zangari's distinction between O(1, 3) representations of points vs SL(2, *C*) representations of points, which, I do claim, is technically incorrect.)

However, this claim is patently false. A homogeneous Lorentz transformation $\Lambda_a{}^b \in \mathrm{O}(1, 3)$ can be decomposed in the following manner in the 2-component spinor formalism (Penrose and Rindler, 1984, pp. 167–175):

$$\Lambda_{AA'}{}^{BB'} = \pm\,\theta^B{}_{A'}\bar{\theta}_A{}^{B'} \quad \text{or} \quad \Lambda_{AA'}{}^{BB'} = \pm\,\phi_A{}^B\bar{\phi}_{A'}{}^{B'}, \tag{28}$$

where $\det(\theta^B{}_{A'}) = \det(\phi_A{}^B) = 1$, with $\theta^B{}_{A'}$ and $\phi_A{}^B$ determined uniquely up to a sign. Specifically, it can be shown that

$$\begin{aligned} &+\theta^B{}_{A'}\bar{\theta}_A{}^{B'} \in \mathrm{O}_-^{\uparrow}(1, 3), \quad &&+\phi_A{}^B\bar{\phi}_{A'}{}^{B'} \in \mathrm{O}_+^{\uparrow}(1, 3), \\ &+\theta^B{}_{A'}\bar{\theta}_A{}^{B'} \in \mathrm{O}_-^{\downarrow}(1, 3), \quad &&-\phi_A{}^B\bar{\phi}_{A'}{}^{B'} \in \mathrm{O}_+^{\downarrow}(1, 3). \end{aligned} \tag{29}$$

The improper (parity) transformations are of the form $\pm\,\theta^B{}_{A'}\,\bar{\theta}_A{}^{B'}$. The point then is that, within the 2-component spinor formalism, such operators do occur. It is thus incorrect to say that the tensor formalism is superior to the 2-component formalism with regard to representing parity operations.

8. Conclusion

The existence of half-integer-spin fields, in and of itself, does not commit us to the standard simultaneity relation. For, according to current theory, such fields have nothing to do with determinations of the in-principle unobservable one-way speed of light that generates the skepticism that underlies the conventionalist's anti-realism. Simply put, as indicated above, clocks and measuring rods are not spinorial objects. Hence, while half-integer-spin fields may best be described by odd-indexed 2-spinor fields, and the components of the latter cannot take values in non-standard coordinate charts, this does not prevent the conventionalist from using non-standard coordinate charts in descriptions of clock synchrony experiments. In this article, I have argued that these observations become clear when one considers how half-integer-spin fields are described in the coordinate-independent 2-spinor formalism.

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References

Aharonov, Y. and Susskind, L. (1967) 'Observability of the Sign Change of Spinors under 2π Rotations', *Physical Review* **158,** 1237–1238.

Anderson, R., Vetharaniam, I. and Stedman, G. E. (1998) 'Conventionality of Synchronisation, Gauge Dependence and Test Theories of Gravity', *Physics Reports* **295,** 93–180.

Cartan, E. (1966) *The Theory of Spinors* (Dover Edition, New York: Dover, 1981).

Earman, J. (1974) 'Covariance, Invariance, and the Equivalence of Frames', *Foundations of Physics* **4,** 267–289.

Friedman, M. (1983) *Foundations of Spacetime Theories* (Princeton: Princeton University Press).

Gel'fand, I., Minlos, R. and Shapiro, Z. (1963) *Representations of the Rotation and Lorentz Groups and their Applications*, translated by G. Cummins and T. Boddington (New York: Macmillan).

Geroch, R. (1968) 'Spinor Structure of Spacetimes in General Relativity. I', *Journal of Mathematical Physics* **9,** 1739–1744.

Glöckeler, M. and Schücker, T. (1987) *Differential Geometry, Gauge Theories, and Gravity* (Cambridge: Cambridge University Press).

Gunn, D. and Vetharaniam, I. (1995) 'Relativistic Quantum Mechanics and the Conventionality of Simultaneity', *Philosophy of Science* **62,** 599–608.

Janis, A. (1998) 'Conventionality of Simultaneity', in E. Zalta (ed.), *Stanford Encyclopedia of Philosophy*, plato.stanford.edu/entries/spacetime-convensimul/.

Kaku, M. (1993) *Quantum Field Theory* (Oxford: Oxford University Press).

Karakostas, V. (1997) 'The Conventionality of Simultaneity in the Light of the Spinor Representation of the Lorentz Group', *Studies in History and Philosophy of Modern Physics* **28,** 249–276.

Malament, D. (1977) 'Causal Theories of Time and the Conventionality of Simultaneity', *Noûs* **11,** 293–300.

Naber, G. (1992) *The Geometry of Minkowski Spacetime* (New York: Springer-Verlag).

Ne'eman, Y. and Sijacki, D. (1987) '$\overline{\mathrm{GL}}(4, R)$ Group-Topology, Covariance and Curved-Space Spinors', *International Journal of Modern Physics* **A2,** 1655–1668.

Ne'eman, Y. and Sijacki, D. (1997) 'World Spinors—Construction and Some Applications', *Foundations of Physics* **27,** 1105–1122; Los Alamos e-print archive gr-qc/9804037.

Norton, J. (1992) 'Philosophy of Space and Time', in M. H. Salmon *et al*., *Introduction to the Philosophy of Science* (New Jersey: Prentice Hall), pp. 179–231.

Norton, J. (1993) 'General Covariance and the Foundations of General Relativity: Eight Decades of Dispute', *Reports on Progress in Physics* **56,** 791–858.

Penrose, R. and Rindler, W. (1984) *Spinors and Spacetime, Vol. 1: Two-Spinor Calculus and Relativistic Fields* (Cambridge: Cambridge University Press).

Sarkar, S. and Stachel, J. (1999) 'Did Malament Prove the Non-Conventionality of Simultaneity in the Special Theory of Relativity?', *Philosophy of Science* **66,** 208–220.

Sijacki, D. (1998) 'World Spinors Revisited', *Acta Physica Polonica* **B29,** 1089–1097; Los Alamos e-print archive gr-qc/9804038.

Stachel, J. (1986). What a Physicist Can Learn from the Discovery of General Relativity', in R. Ruffini (ed.) *Proceedings from the Fourth Marcel Grossmann Meeting on General Relativity* (Amsterdam: Elsevier Science Publishers), pp. 1857–1862.

Stachel, J. (1992) 'How Einstein Discovered General Relativity: A Historical Tale with some Contemporary Morals', in M. MacCallum (ed.), *General Relativity and Gravitation* (Cambridge: Cambridge University Press), pp. 200–208.

Torretti, R. (1983) *Relativity and Geometry* (New York: Pergamon Press).

Trautman, A. (1965) 'Foundations and Current Problems in General Relativity', in A. Trautman, F. A. E. Pirani and H. Bondi (eds.), *Lectures on General Relativity. Brandeis Summer Institute in Theoretical Physics*, *1964* (New York: Englewood Cliffs), pp. 1–248.

Trautman A. (1980). 'Fiber Bundles, Gauge Fields, and Gravitation', in A. Held (ed.), *General Relativity and Gravitation, One Hundred Years after the Birth of Albert Einstein* (New York: Plenum), pp. 287–308.

Wald, R. (1984) *General Relativity* (Chicago: University of Chicago Press).

Weingard, R. and Smith, G. (1982) 'Spin and Space', *Synthese* **50,** 213–231.

Werner, S., Colella, R., Overhauser, A. and Eagen, C. (1975) 'Observation of the Phase Shift of a Neutron Due to Precession in a Magnetic Field', *Physical Review Letters* **35,** 1053–1055.

Wigner, E. (1939) 'On Unitary Representations of the Inhomogeneous Lorentz Group', *Annals of Mathematics* **40,** 149–204.

Zangari, M. (1994) 'A New Twist in the Conventionality of Simultaneity Debate', *Philosophy of Science* **61,** 267–275.