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The RT formula and its discontents: spacetime and entanglement

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Abstract
This essay is concerned with a number of related proposals that claim there is a link between spacetime topology and quantum entanglement. I indicate the extent to which these proposals can be understood as stating a duality, and then consider two general approaches to articulating such a duality: a "state-based" approach, under which one attempts to identify relevant topological states as dual to quantum entangled states; and an "observable-based" approach, under which one attempts to identify relevant topological observables as dual to quantum entangement observables. Both approaches are faced with issues, essentially due to the ambiguous nature of quantum entanglement, that remain to be addressed.

Keywords Entanglement · Topology · Duality · AdS/CFT correspondence · Non-locality

Contents

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1 Introduction

According to Van Raamsdonk (2010), the connectivity of a bulk spacetime is encoded in the quantum entanglement of physical systems on its boundary; according to Maldacena and Susskind (2013), two physical systems that exhibit quantum entanglement are connected by an Einstein-Rosen wormhole; and according to Levin and Wen (2006), and Kitaev and Preskill (2006), the quantum entanglement between degrees of freedom of a physical system in regions on either side of a boundary depends on both the geometry of the boundary and the topology of the regions. What these proposals all have in common is the claim that quantum entanglement is, in some sense, encoded in topology. This essay is concerned with some of the implications of this claim. First, are we to understand it in terms of a "duality"; i.e., a 1-1 correspondence between topological features of spacetime on the one hand, and quantum entanglement, on the other? If so, should it be understood as a duality between topological *states* of a physical system, and quantum entangled states; or as a duality between topological and quantum entanglement *observables*? Moreover, how should we understand the notion of quantum entanglement it refers to?

With respect to the first question, dualities have become a much discussed topic in the recent philosophy of physics literature. In that literature, the term "duality" refers to an isomorphism between two theories under which states and observables of one theory are bijectively mapped onto those of the other in a way that preserves the dynamics.[1] Various philosophical topics have been discussed in the context of this concept, including how best to define a physical theory, and the relation of theoretical equivalence between theories (De Haro et al. 2017), realist interpretative options in the light of empirically equivalent dual theories (Le Bihan and Read 2018), the relation between duality and symmetry (De Haro and Butterfield 2019), and the relation between duality and notions of emergence (De Haro 2017; Teh 2013; Vistarini 2017), to name a few. In the physics literature, the primary example of duality comes from an approach to reconciling general relativity and quantum theory under which dualities are sought between a "bulk" classical theory of gravity and a "boundary" quantum field theory in one less dimension. The AdS/CFT correspondence is one example of this approach (see, e.g., Teh 2013; De Haro 2017; Vistarini 2017). Determining the extent to which the claim that quantum entanglement is encoded in spacetime topology is a duality is important in order to situate it in both the philosophical and physics literatures. For instance, as we will see, one motivation for the proposals of van Raamsdonk (2010) and Maldacena and Susskind (2013) is the AdS/CFT correspondence. Thus, if these proposals can be understood as stating a duality, they suggest a way of reconciling general relativity with quantum theory at a fairly fundamental level.

If the alleged relation between topology and quantum entanglement is intended as a duality, a further question is whether it should be understood as a duality between states, or as a duality between observables. This distinction may appear to be trivial in the context of non-relativistic quantum mechanics: the intuition might be that a formulation of the latter in terms of states (i.e., the "Schrödinger picture") can be shown to

[1] See, e.g., de Haro (2017, p. 110); de Haro and Butterfield (2019, p. 20). My use of the term in this essay is restricted to the entries (states and/or observables) in the "dictionary" that this mapping sets up.

be equivalent to a formulation in terms of observables, or operators (i.e., the "Heisenberg picture"). However, the distinction becomes less trivial for relativistic quantum field theories (RQFTs). In that context, the physical systems of interest possess infinite degrees of freedom, and this problematizes theoretical descriptions in terms of states. For instance, states that carry unitarily inequivalent representations of the canonical commutation relations belong to different Hilbert spaces, even though they may only differ in the boundary conditions (at infinity) they satisfy (see, e.g., Wallace 2006, pp. 56–58). Moreover, as Swanson (2018, p. 9) indicates, the vacuum state of an RQFT is highly entangled, and this makes it difficult to assign local Hilbert spaces of states to relativistic spatiotemporal regions. These issues with states can be addressed by adopting a formalism in which observables are fundamental and states are derivative (e.g., the algebraic formalism). Alternatively, it might be argued that most of these issues involve assumptions about the high-energy behavior of RQFTs, and can be adequately addressed by treating the latter as effective field theories (Wallace 2006). In any event, the point to make is that an equivalence between state-based approaches and observable-based approaches should not be taken for granted for RQFTs; thus the question of whether a given duality claim, understood in the context of RQFTs, is intended to be about states or observables has non-trivial implications.[2]

Finally, as Earman (2015) has indicated, the notion of quantum entanglement, a fundamental concept in quantum theory, is, surprisingly, still little understood (appart from the simple bipartite case). Proposals that seek a relation between topology and quantum entanglement thus offer an intriging tool to probe the nature of the latter, particularly if the relation is intended as a duality. On the other hand, as we will see, care needs to be taken to keep distinct notions of quantum entanglement separate in assessing such proposals.

To address these concerns, I will initially focus on van Raamsdonk's proposal, which is based on an interpretation of the Ryu-Takayanagi (RT) formula that appears in the AdS/CFT correspondence. I first motivate the RT formula by comparing it in Sect. 2.1 with a formula due to Srednicki (1993), according to which the entanglement entropy of a massless scalar field with respect to a spherical region of spacetime is a function of the area of the region's bounding surface. Srednicki's formula was applied by Levin and Wen (2006), and, independently, Kitaev and Preskill (2006), to the case of a 2-dim condensed matter system with an energy gap between its ground and excited states. According to these authors, the entanglement entropy of such a system with respect to a bounded region of space depends on both the length of the boundary and the topology of the region, with the additional topological contribution referred to as "topological entanglement entropy". Section 2.2 then considers the RT formula as an extension of Srednicki's formula to the AdS/CFT correspondence. As originally expressed by Ryu and Takayanagi (2006), the RT formula only contained an area term. A "quantum correction" to the formula was suggested by Faulkner et al.

[2] Swanson (2018) indicates that one route to establishing an equivalence between state-based and observable-based approaches in RQFT requires nothing less than a complete reformulation of algebraic quantum field theory (AQFT), in which the basic object of the latter, a net of local $\mathcal{C}^*$-algebras (the elements of which represent local observables), is replaced with a presheaf of convex oriented sets (the elements of which represent local states). To establish a formal equivalence between these objects requires, in addition, replacing the axioms of AQFT with a set of new axioms that have yet to be fully specified.

(2013), and in Sect. 2.3, I explain Susskind's (2016) topological interpretation of this correction, based on the "ER = EPR" hypothesis of Maldacena and Susskind (2013). According to the latter, two physical systems in a quantum entangled state ("EPR") are connected by an Einstein-Rosen wormhole ("ER"). I point out that Susskind's topological interpretation of the quantum correction to the RT formula is analogous to the topological entanglement entropy correction to Srednicki's formula, but whereas the (ER = EPR)-modified RT formula admits an interpretation under which it states a duality between spacetime topology and quantum entanglement, Srednicki's formula with topological correction does not.

Sections 3 and 4 are concerned with how to understand a duality between spacetime topology and quantum entanglement in terms of states and in terms of observables. Under a "state-based" approach, one seeks to identify appropriate topological states as the dual to quantum entangled states; while under an "observable-based" approach, one seeks to identify appropriate topological observables as the dual to observables associated with quantum entanglement. An example of the former approach is an analogy suggested by Aravind (1997) between topologically entangled n-links and quantum entangled n-partite states. Section 3 considers the extent to which this analogy can be turned into a 1-1 correspondence that might underwrite a relation of duality, and concludes that this is not possible, insofar as (i) a topologically entangled n-link can correspond to many distinct quantum entangled n-partite states, and (ii) a quantum entangled n-partite state can correspond to many distinct topologically entangled n-links.

Section 4 then considers an example of an observable-based approach, motivated in part by Aravind's analogy, that takes the form of a program initiated by Kaufmann and Lomonaco (2002). This program seeks to identify a correspondence between quantum entangling operators, viewed as unitary representations of the braid group, on the one hand, and link invariants, on the other. Limitations of this program raise the question of how a quantum entanglement observable should be characterized. According to some authors, quantum entanglement manifests itself in two distinct ways, one associated with a violation of an entropic inequality, and another associated with a violation of a Bell inequality (e.g., Horodecki et al. 2009). This distinction raises the following concerns, addressed in Sect. 5:

(a) *Non-linearity* To the extent that quantum entanglement is characterized by a violation of an entropic inequality, it is non-linear, in the sense that it cannot be represented by a linear operator; but typical examples of topological observables are linear in this sense.
(b) *Non-locality* To the extent that quantum entanglement is characterized by a violation of a Bell inequality, it is linear, insofar as it can be characterized by a linear "Bell" operator; however, it is also characterized by a notion of non-locality which is distinct from the notion of non-locality associated with topological observables.
(c) *Correlations* The non-locality associated with the violation of a Bell inequality is exhibited by a particular type of correlation between observables. Correlations can be characterized in four distinct ways, depending on the type of observable (local *vs*. non-local), and the strength of the correlation (short-range *vs*. long-range). If topological duals to quantum entanglement observables are required to exhibit

Bell inequality-violating correlations, care must be taken to keep these different types of correlation distinct.

These concerns, combined with those raised in Sects. 3 and 4, suggest that caution is advised in evaluating any proposed duality between spacetime topology and quantum entanglement.

2 The RT formula

To set the stage for van Raamsdonk's interpretation of the RT formula, I will first consider another example of a relation between quantum entanglement and spacetime topology; namely, a topological correction, due to Levin and Wen (2006), and Kitaev and Preskill (2006), to Srednicki's (1993) formula that relates the entanglement entropy of a massless scalar field to the area of a bounding surface.

2.1 Srednicki's formula and topological entanglement entropy

Both Srednicki's formula and the RT formula are expressions for the entanglement entropy of a physical system. For a bipartite system AB characterized by a density operator ρ_{AB}, the entanglement entropy S_A of subsystem A is defined as the von Neumann entropy S_{vN} of the reduced density operator ρ_A; thus, $S_A \equiv S_{vN}(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$, where $\rho_A = \mathrm{Tr}_B \rho_{AB}$. One can show that if ρ_{AB} is a pure state, then it is decomposable if and only if $S_A \leq S_{vN}(\rho_{AB})$.[3] Thus, if a bipartite system is in a pure state, and if an entangled state is defined as an indecomposable state, then the entanglement entropy of one of its subsystems is a measure of the extent to which that subsystem is entangled with the other subsystem.

One might initially be concerned that entanglement entropy, so-defined, is rather limited in its applicability, constrained to bipartite systems in indecomposable pure states. To be fair, the establishment of a link between spacetime topology and quantum entanglement restricted to bipartite systems in pure states would be a significant achievement in its own right. On the other hand, one might be a bit more concerned with the definition of an entangled state as an indecomposable state. On the surface, this fails to appreciate an important aspect of entanglement; namely, the type of non-locality associated with the violation of a Bell inequality.[4] I will return to this concern in Sects. 4 and 5 below. For the moment, I will bracket it off, in order to continue with the exposition.

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[3] A bipartite state ρ_{AB} is *decomposable* if and only if it can be written as a convex combination of product states, $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$, where $\sum_i p_i = 1$, and $0 \leq p_i \leq 1$. (Here I follow Earman 2015, p. 311, in using the term "decomposable" as opposed to "separable".) For pure states, decomposability reduces to being a product state, and one can show that the above entropic inequality holds if and only if the bipartite state is a product state (Nielson and Chuang 2010, p. 514).

[4] Indecomposability is, arguably, the standard way of defining quantum entanglement in the physics literature (see, e.g., Horodecki et al. 2009, p. 873). On the other hand, Earman (2015) considers four distinct notions of a quantum entangled state, the weakest being a non-product state and the strongest involving a violation of a Bell inequality, with indecomposability ranked in-between.

Srednicki (1993) considered a massless scalar field decomposed into degrees of freedom with support inside and outside a spherical region R of spacetime, and derived the following expression for the entanglement entropy,

$$S_R = \alpha|\partial R| \tag{1}$$

where ∂R is the boundary of R, $|\partial R|$ is its area, and α is a constant that depends on the short wavelength modes of the system.[5] Equation (1) states that the entanglement (as expressed by the indecomposability of the composite state) between the degrees of freedom of the scalar field with support on R and the degrees of freedom with support on its complement is a function of the area of the boundary of R.

Levin and Wen (2006), and independently Kitaev and Preskill (2006), considered a version of (1) for a 2-dim condensed matter system with an energy gap between its ground state and excited states.[6] The gap property entails that ground state correlations between local observables decay exponentially as a function of their separation distance; thus if the system is decomposed into degrees of freedom with support inside and outside a 2-dim region R, the entanglement entropy S_R only gets contributions from the degrees of freedom in the vicinity of the boundary ∂R.[7] This suggests that Srednicki's formula (1) should hold (with area replaced by boundary length). Moreover, Levin and Wen, and Kitaev and Preskill, suggested a topological correction Γ to the boundary term in (1), referred to as "topological entanglement entropy", that depends on the topology of R,[8]

$$S_R = \alpha|\partial R| + \Gamma \tag{2}$$

On first blush, Γ is supposed to be an indication of the presence of anyons (i.e., physical systems that exhibit fractional exchange statistics). More precisely, Γ scales as $\log \mathcal{D}$, where $\mathcal{D}$ is the total quantum dimension of the system, defined by $\mathcal{D} = \sqrt{\sum_a d_a^2}$, where d_a is the quantum dimension of an anyon of type a (anyons being distinguished by the type of fractional statistics they exhibit) (Kitaev and Preskill 2006, p. 1; Levin

[5] As Hartman (2015, p. 175) indicates, in a continuum quantum field theory in Minkowski spacetime, there are high-energy modes at small scales across any surface that divides the system into two regions, and this requires a regularization scheme to prevent the entanglement entropy defined with respect to the surface from becoming divergent. Srednicki (1993, p. 669) adopted a high-energy cutoff defined by $M = a^{-1}$, where a is the spacing between sites on a discrete lattice, and derived the relation $S_R = \kappa M^2(4\pi r^2)$, where r is the radius of R and κ is a constant. This relation holds specifically for a massless scalar field in flat Minkowski spacetime, and assumedly could be extended (in standard limited cases) to a relation in a curved spacetime using techniques from quantum field theory in curved spacetimes. Equation (1) is formally similar to Bekenstein's formula for the thermodynamic entropy of a black hole, $S_{BH} = Area(horizon)/4G$.

[6] These authors were primarily concerned with gapped systems that exhibit "topological order". The subsequent notion of "topological entanglement entropy" was proposed as a way of characterizing the latter.

[7] More precisely, the gap property entails that the system has a finite correlation length ξ, and this entails that for observables A, B with support on regions X, Y, $\langle AB\rangle - \langle A\rangle\langle B\rangle \sim e^{-dist(X,Y)/\xi}$, where $dist(X, Y)$ is the distance between X and Y, and expectation values are taken in the ground state. Thus contributions to S_R should come from a strip on either side of ∂R of width ξ (Pachos 2012, p. 179).

[8] Equation (2) is due to Kitaev and Preskill (2006), who use the phrase "topological entropy" in referring to Γ.

and Wen 2006, p. 1). The quantum dimension d_a is the "asymptotic dimension" of the Hilbert space $\mathcal{H}_a^{(n)}$ of n type-a anyons, in the sense that $\dim(\mathcal{H}_a^{(n)}) \to (d_a)^n$, as $n \to \infty$. It can be thought of as the number of degrees of freedom of a type-a anyon (see, e.g., Tong 2016, p. 133). Thus $\mathcal{D}$, and consequently Γ, can be thought of roughly as the number of degrees of freedom of a collection of anyons of different types. In 2-dim, the fractional exchange statistics of a collection of anyons encodes aspects of the topology of the space; thus knowing $\mathcal{D}$, and hence Γ, gives one information about the topology of the 2-dim space in which the physical system is localized.

To demonstrate how the topology of R contributes to the entanglement entropy of the system, Levin and Wen considered the following difference in entanglement entropies

$$(S_1 - S_2) - (S_3 - S_4) \tag{3}$$

of the four versions of R in Figure 1.[9] If this difference is calculated using Srednicki's formula (1), the result is zero (i.e., the difference in the boundary lengths of R_1 and R_2 is the same as the difference in the boundary lengths of R_3 and R_4). On the other hand, if one assumes that each connected boundary is associated with a topological correction Γ and uses expression (2) to calculate (3), then the result is 2Γ.[10] According to Levin and Wen (2006, p. 2), "a nonzero value for [Γ] signals the presence of non-local correlations and topological order". Intuitively, a nonzero difference $(S_1 - S_2)$ comes from observables that contribute to the entanglement associated with region R_1 and that do not contribute to the entanglement associated with region R_2. Such observables have support on R_1 but do not have support on R_2. One example is a "local" observable with support on a contractible subregion of the upper horizontal part of R_1 (contractible in the sense of deformable into a point). Another example is a "non-local" observable with support on a subregion that extends completely around R_1; for example, a loop operator with support on a non-contractible loop that wraps around the region R_1 (non-contractible in the sense of not being deformable into a point). Note that whereas there are two types of observable that can find support on R_1 but not R_2 (namely, "local" and "non-local"), there is only one type of observable that can find support on R_3 but not R_4; namely, "local" observables with support in the upper horizontal part of R_3. In particular, R_3 cannot support the sort of "non-local" loop observables that R_1 potentially can. Thus, a nonzero difference Γ would come from observables that could be supported on R_1 but not on R_3, and these are "non-local" (non-contractible loop) observables with support on subregions that extend completely around R_1. Evidently, a correlation involving such a non-local observable is what Levin and Wen refer to as a "non-local" correlation.

[9] Kitaev and Preskill (2006) reached the same conclusion using a different configuration of regions and a different linear combination of entanglement entropies.

[10] To show that $(S_1 - S_2) - (S_3 - S_4) = 2\Gamma$, one notes, for instance, that the region R_1 has two disconnected boundaries (∂R_{1a} and ∂R_{1b} in Fig. 1), and assumes each contributes a separate topological correction Γ to S_1. Thus S_1 and S_4 each contain two Γ's (R_4 also has two disconnected boundaries, ∂R_{4a} and ∂R_{4b}), whereas S_2 and S_3 each contain a single Γ. The difference $(S_1 - S_2) - (S_3 - S_4)$ thus contains the term 2Γ, as well as the sum of boundary lengths $\{(|\partial R_{1a}| + |\partial R_{1b}|) - |\partial R_2|\} - \{|\partial R_3| - (|\partial R_{4a}| + |\partial R_{4b}|)\}$, which is zero.

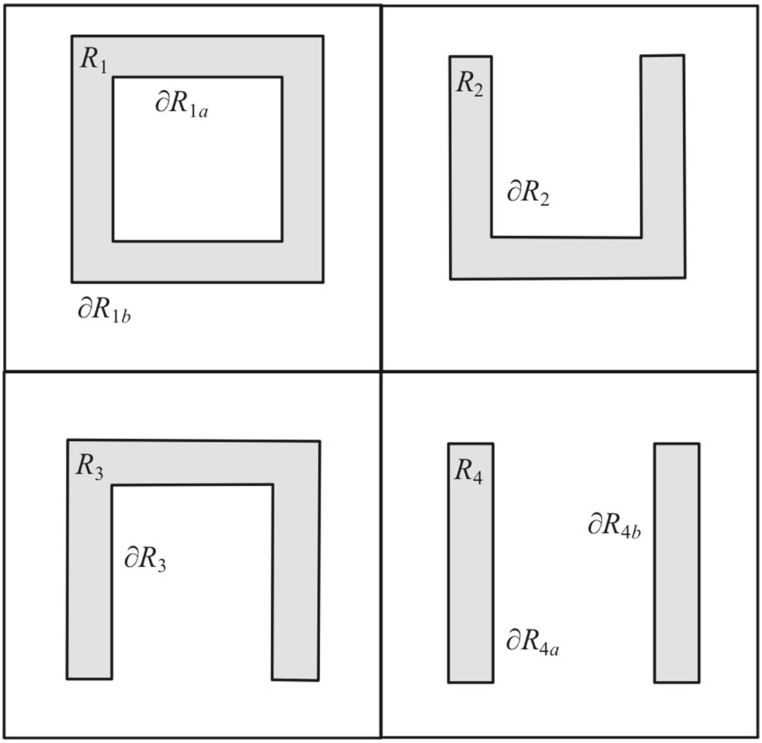


Fig. 1 Regions used to calculate topological entanglement entrophy

The considerations of Levin and Wen (and Kitaev and Preskill) suggest a link between quantum entanglement, on the one hand, and spacetime topology, on the other. In particular, according to the discussion above, the extent to which the subsystems of a bipartite system are quantum entangled depends on the topology of the regions of spacetime on which they have support, as well as the type of observables they are characterized by (local versus non-local), and the type of correlations these observables enter into (again, local versus non-local). Note that, without further ado, this link does not appear to be enough to establish a duality between topology and quantum entanglement. Such a duality should minimally establish a 1-1 relation between some aspect of spacetime topology and some aspect of quantum entanglement. In the context of formula (2), this would seem to require that a non-zero value of the entanglement entropy S_R is both necessary and sufficient for some aspect of the topology of R, and this is not the case. It is not sufficient, since S_R may be non-zero due solely to the first "geometric" term on the right-hand-side of (2) that depends on the length of the boundary ∂R. And it is not necessary, insofar as a non-zero value of topological entanglement entropy Γ depends not just on the topology of the region R, but on the presence of a non-local observable with support on R that, in some sense, exhibits a "non-local correlation". An attempt will be made to unpack these notions in Sect. 5.3 below, but first I'd like to consider the similarities between this initial example of a relation between spacetime topology and quantum entanglement, and van Raamsdonk's interpretation of the RT formula.

2.2 The RT formula and the connectedness/entanglement hypothesis

The RT formula was introduced by Ryu and Takayanagi (2006) in the context of the AdS/CFT correspondence. The latter is a dictionary that relates a $(d+1)$-dim "bulk" theory of gravity in anti-de Sitter (AdS) spacetime, to a d-dim "boundary" conformal field theory (CFT).[11] The RT formula can be thought of as an entry in this dictionary insofar as it is an expression for the entanglement entropy of a physical system on the boundary in terms of bulk quantities. Consider a boundary spatial region given by the intersection of the boundary with a bulk timeslice, and partitioned into a subregion R and its complement $\bar{R}$ (see Fig. 2). Let γ_R be the extremal surface in the bulk with the same boundary as R, and let H_R be the bulk region bounded by $\gamma_R \cup R$. The RT formula is then given by,

$$S_R = |\gamma_R|/4G + S_{H_R} \tag{4}$$

where S_R is the entanglement entropy of a subsystem of a boundary composite system with respect to a decomposition into degrees of freedom with support on R and its complement $\bar{R}$, $|\gamma_R|$ is the area of γ_R, G is the Newtonian gravitational constant, and S_{H_R} is the entanglement entropy of a subsystem of a bulk composite system with respect to a decomposition into degrees of freedom with support on H_R and its complement $\overline{H_R}$. According to Eq. (4), the entanglement between boundary degrees of freedom with support on R and $\bar{R}$, as measured by S_R, is given by the area of the extremal bulk surface γ_R, and by the entanglement (if any) between bulk systems with support on H_R and $\overline{H_R}$.[12]

The second term on the right of Eq. (4) does not appear in Ryu and Takayanagi's original formula, and was derived by Faulkner et al. (2013) as a "quantum correction". The original formula (with the second term missing) was motivated in part by an analogy with Bekenstein's formula for the entropy of a black hole (see Footnote 5). For a bulk (anti-de Sitter) Schwarzschild black hole with support on $\overline{H_R}$, in the limit in which R encompasses the entire boundary timeslice, the extremal bulk surface γ_R wraps around the black hole and becomes the event horizon, and the RT formula

[11] See, e.g., Teh (2013), De Haro (2017), Vistarini (2017) for reviews of aspects of the AdS/CFT correspondence. See Jaksland (2018) for a discussion of aspects of the RT formula and van Raamsdonk's interpretation of it.

[12] As Headricks (2019, p. 48) reports, the RT formula (4) without the second term on the right is limited in three ways: The bulk theory must be a (i) classical (ii) Einsteinian theory of gravity, and (iii) the bulk spacetime (in addition to being asymptotically AdS) must have a time-reflection symmetry under which the boundary subregion R is invariant (in order to pick out a bulk timeslice with the relevant properties). These restrictions can be relaxed in various ways: A covariant version of the RT formula can be derived for bulk spacetimes without assuming any symmetries (Hubeny et al. 2007). Higher derivative corrections to the bulk gravitational action can be included for non-Einsteinian theories of gravity, which subsequently requires including terms beyond the area term. Finally, corrections to the Newtonian gravitational constant can be included to move away from classical theories. One result of the latter is the second term on the right in Eq. (4) due to Faulkner et al. (2013).

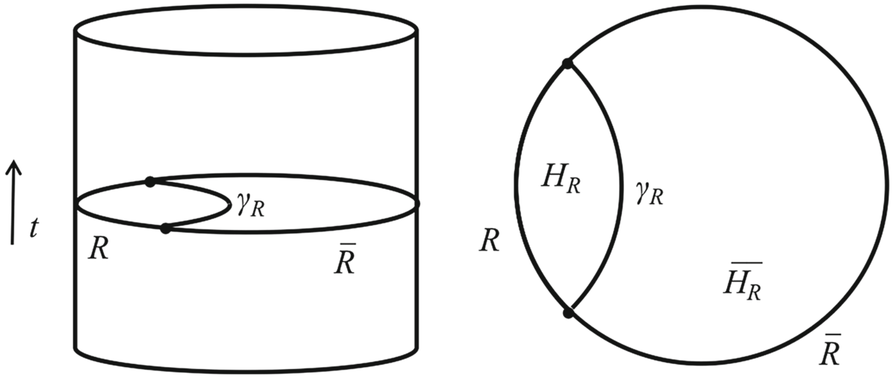


Fig. 2 Regions apperaring in the RT formula

(without the second term) becomes Bekenstein's formula (Ryu and Takayanagi 2006, p. 4).

Van Raamsdonk's (2010) interpretation of the RT formula is based on a bipartite decomposition of the boundary CFT Hilbert space $\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_{\bar{R}}$ with respect to degrees of freedom with support on R and $\bar{R}$.[13] A CFT boundary state entangled over these regions can be expressed as

$$|\Psi\rangle = \sum_{i,j} p_{ij} |\psi_i^R\rangle \otimes |\psi_j^{\bar{R}}\rangle \tag{5}$$

where $|\psi_i^R\rangle$ and $|\psi_j^{\bar{R}}\rangle$ are bases for $\mathcal{H}_R$ and $\mathcal{H}_{\bar{R}}$, and the p_{ij} are constants. Consider the connected bulk timeslice that has boundary $R \cup \bar{R}$ and that consists of the union of the region H_R with its complement $\overline{H_R}$, separated by the extremal surface γ_R. Now suppose we ignore the "quantum correction" term in the RT formula (4). Then the latter entails that as the entanglement entropy of the state (5) decreases, so does the area of γ_R, and in the limit as S_R goes to zero, γ_R becomes a point. In this limit, according to van Raamsdonk, the bulk spatial region $H_R \cup \overline{H_R}$ splits into two disconnected pieces, H_R and $\overline{H_R}$, and the initially entangled state (5) becomes a product state

$$|\Phi\rangle = \left(\sum_i c_i |\psi_i^R\rangle\right) \otimes \left(\sum_j d_j |\psi_j^{\bar{R}}\rangle\right) \tag{6}$$

[13] As Hartman (2015, p. 175) notes, "Quantum field theory is strictly speaking *not* bipartite". To address this, Hartman recommends inserting a UV cutoff to render the theory finite. On the other hand, according to Earman (2015, p. 309), the restriction of any discussion of quantum entanglement to finite tensor product Hilbert spaces "...is to be deplored because it neglects possibilities that need to be explored". Moreover, even granted this restriction, there still remain various ambiguities associated with quantum entanglement. While I won't attempt to address Earman's general concern in this essay, Sect. 5.3 attempts to address some of the latter concerns.

where c_i and d_j are constants.[14] In this way, the degree of entanglement of the boundary state (5), as encoded in its entanglement entropy, tracks the connectedness of the corresponding bulk region $H_R \cup \overline{H_R}$. Van Raamsdonk (2010, p. 2325) suggested viewing the reverse process that begins with (6) and ends with (5) as one in which "[c]lassical connectivity arises by entangling the degrees of freedom of the two components [of a CFT product state]." Thus, according to what I will refer to as van Raamsdonk's Connectedness/Entanglement hypothesis, bulk connectivity is dual to boundary quantum entanglement.

On the other hand, if Faulkner *et al.*'s (2013) quantum correction to the RT formula is not ignored, then the Connectedness/Entanglement hypothesis fails to state a 1-1 correspondence between (bulk) spacetime topology and (boundary) quantum entanglement. With the quantum correction, boundary entanglement is a necessary, but not sufficient, condition for bulk connectivity: According to (4), the boundary quantum entanglement associated with a particular partition $R \cup \bar{R}$ of a boundary spatial region is encoded in both the connectivity of the corresponding bulk timeslice, and in the bulk quantum entanglement associated with the partition $H_R \cup \overline{H_R}$ of the latter. The concern then is that van Raamsdonk's Connectedness/Entanglement hypothesis cannot be viewed as a 1-1 duality relation. This concern can be addressed by another proposal that links spacetime topology with quantum entanglement; namely, Maldacena and Susskind's (2013) ER = EPR hypothesis.

2.3 The (ER = EPR)-modified RT formula

Maldacena and Susskind's (2013) ER = EPR hypothesis states that two physical systems in a quantum entangled state (EPR) are connected by an Einstein-Rosen wormhole (ER). This proposal was initially motivated by an example due to Maldacena (2003) of an AdS/CFT duality between a bulk AdS-Schwarzschild spacetime and a boundary CFT entangled thermal state.

The general idea behind Maldacena's example is that a decomposition of a spacetime into regions induces a decomposition of the Hilbert space of a field theory defined on the spacetime; and for regions separated by a horizon (event horizon or particle horizon), the corresponding states are entangled. For instance, the Kruskal decomposition of a Schwarzschild spacetime decomposes the latter into four regions separated by event horizons, with the two asymptotically flat exterior regions connected by an

[14] van Raamsdonk (2010, p. 2326) describes the limit as a process in which "the two regions of space are pinching off from each other", but immediately qualifies this by cautioning "Here and below, we should keep in mind that the spacetime will likely cease to have a completely geometrical description before the entanglement is strictly zero". Elsewhere, he suggests that "without entanglement, we have a product state in two non-interacting systems, and the only possible interpretation would be two disconnected spacetimes" (van Raasmdonk 2016, p. 22). Further motivation for this interpretation of the limit comes from an argument that relates the mutual information $I(C, D)$ of subsystems localized in bulk subregions $C \subset H_R$ and $D \subset \overline{H_R}$ near the boundary, on the one hand, to the spatiotemporal distance $d(p, q)$ between bulk points $p \in C$ and $q \in D$ on the other hand: the argument shows that as $I(C, D)$ decreases to zero, $d(p, q)$ increases to infinity (van Raamsdonk 2010, p. 2327). Thus, insofar as $I(C, D)$ is a measure of the degree of entanglement of subsystems localized in C and D, as the entanglement near the boundary goes to zero, the distance between bulk points near the boundary increases. This suggests a process in which "the two regions of [bulk] spacetime pull apart and pinch off from each other" (van Raamsdonk 2010, p. 2327).

Einstein-Rosen wormhole (see, e.g., Harlow 2016, p. 4). This decomposition of the spacetime induces a decomposition of the Hilbert space of a scalar field defined on the spacetime. With respect to the Hilbert space decomposition, one can show that the degrees of freedom of the field localized in the exterior regions are in a bipartite entangled thermal state (Hartle and Hawking 1976; Israel 1976). Another, perhaps more familiar, example is the Rindler decomposition of Minkowksi spacetime, under which the latter decomposes into four regions separated by particle horizons. This induces a decomposition of the Hilbert space of a scalar field defined on Minkowski spacetime, with respect to which the degrees of freedom of the field localized in the left and right Rindler wedges (the analogs of the exterior regions in the Kruskal decomposition) are in a bipartite entangled thermal state (see Harlow 2016, pp. 9-10, 20, for a comparison of these examples.) In the AdS/CFT correspondence, a bulk AdS geometry corresponds to a boundary CFT state. Maldacena (2003) showed that the Kruskal decomposition of an AdS-Schwarzschild spacetime induces a decomposition of the boundary CFT Hilbert space with respect to which the degrees of freedom of the boundary CFT that correspond to the bulk exterior regions are in an entangled thermal state.

In Maldacena's example, the exterior regions of the bulk AdS-Schwarzschild spacetime are connected by a wormhole, as they are in the Kruskal decomposition of Schwarzschild spacetime; hence these exterior regions constitute a multiply connected space. One might be tempted to identify this topologically non-trivial bulk space as the dual to the corresponding boundary entangled CFT thermal state. However it would require a rather large leap of intuition to infer from this specific duality between a particular topologically non-trivial space and a particular entangled state, to a duality between spacetime topology and quantum entanglement in general, even if we restrict such a duality to the AdS/CFT correspondence. On the other hand, the RT formula would provide just this sort of general duality relation in the context of the AdS/CFT correspondence, if it could be interpreted in a way that made bulk connectivity both necessary and sufficient for boundary entanglement. Just such a way was proposed by Susskind (2016) who suggested that ER = EPR provides the basis for a topological interpretation of the quantum correction term in the RT formula.

According to both Srednicki's formula (1) and the original RT formula (without the quantum correction), entanglement entropy is encoded in the area of an extremal surface. In the original RT formula, boundary entanglement is encoded in the area of a bulk extremal surface, and one might consider what extremal surface should be associated with bulk entanglement. Susskind (2016, p. 73) suggested this extremal surface be identified with the cross-section σ of the wormhole that, according to ER = EPR, connects the two entangled bulk systems (Fig. 3). In particular, Susskind's suggestion is that the entanglement entropy S_{H_R} of a bulk subsystem with support on H_R with respect to a bulk subsystem with support on $\overline{H_R}$ is proportional to the area, $|\sigma|$, of σ. Thus the RT formula (4) can be re-written as

$$S_R = |\gamma_R|/4G + \alpha'|\sigma| \tag{7}$$

for some constant α'. Unlike Eq. (4), Eq. (7) expresses a 1-1 correspondence between bulk spacetime topology and boundary quantum entanglement. In particular, in Eq. (7),

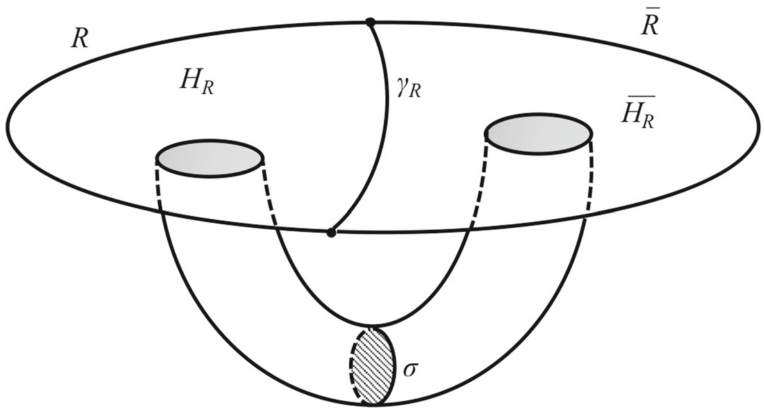


Fig. 3 Regions appearing in the (ER=)-modified RT formula

bulk connectivity is a necessary and sufficient condition for boundary quantum entanglement, as van Raamsdonk observes:

> We have argued that entanglement between fundamental degrees of freedom underlies the connectivity of spacetime. Maldacena and Susskind's suggestion in [2013] is that not only is this entanglement a necessary condition for connectedness, it is also sufficient. (van Raamsdonk 2016, footnote 25, p. 35.)

3 State-based approaches

As we've seen in Sect. 2 above, van Raamsdonk's Connectedness/Entangle-ment hypothesis can be viewed as a duality that relates a quantitative measure of quantum entanglement (entanglement entropy), on the one hand, to an aspect of spacetime topology (connectedness), on the other. In this section and the next, I'd like to consider two ways of understanding such a duality, one "state-based" and the other "observable-based".

In a state-based approach to a duality between spacetime topology and quantum entanglement, one attempts to identify appropriate topological states that are dual to quantum entangled states. For instance, Maldacena and Susskind's ER = EPR hypothesis suggests that the appropriate topological states are those that characterize Einstein-Rosen wormhole geometries. The hard work would then involve identifying the essential characteristics of such states that are dual to relevant aspects of quantum entangled states.[15] To get a sense of some of the issues that can arise with such a program, the remainder of this section focuses on a more simple example due to Aravind (1997) who proposed an analogy between quantum entangled n-partite states and topologically entangled n-links.

[15] One way of carrying this out is suggested by Bao et al. (2015a) who attempt to identify "no-go" theorems for Einstein-Rosen wormholes that are dual to no-go theorems associated with quantum entangled states.

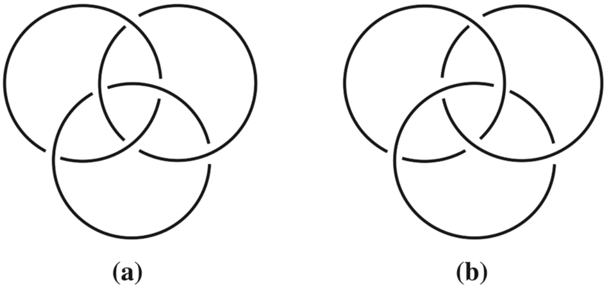


Fig. 4 **a** The Borromean rings 3-link; **b** the three-Hopf rings 3-link

An n-link is an embedding of n circles in the 2-dim plane up to ambient isotopy. The Borromean rings 3-link, for example, consists of three entangled circles with the feature that, if any one is cut, the other two become disentangled (Fig. 4a). Aravind (1997, p. 54) suggested that cutting one of the links of the Borromean rings is analogous to measuring one of the subsystems of a tripartite system in an entangled GHZ state:

$$\sqrt{\frac{1}{2}}\left\{|\uparrow_{z1}\rangle|\uparrow_{z2}\rangle|\uparrow_{z3}\rangle - |\downarrow_{z1}\rangle|\downarrow_{z2}\rangle|\downarrow_{z3}\rangle\right\} \tag{8}$$

where, e.g., $|\uparrow_{z_1}\rangle$ is an eigenstate of subsystem 1 characterized by spin-up along the z-axis. A measurement of spin along the z-axis of subsystem 1, for instance, disentangles subsystems 2 and 3, insofar as the post-measurement state (assuming the projection postulate) is a product state.

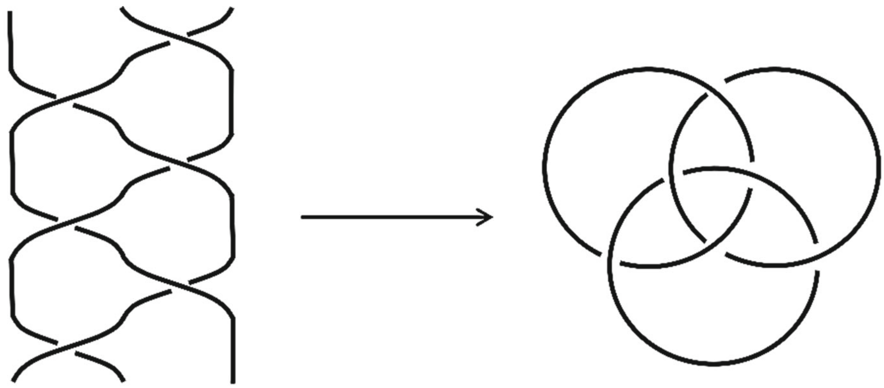
To make this analogy more precise, one would like a formal definition of a topologically entangled n-link. Towards this end, first recall that the n-strand braid group B_n is the group generated by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ that satisfy the braid relations (see, e.g., Alagic et al. 2016, p. 2):

(a) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j = 1, \ldots, n-1$, with $|i-j| \geq 2$
(b) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $i = 1, \ldots, n-2$

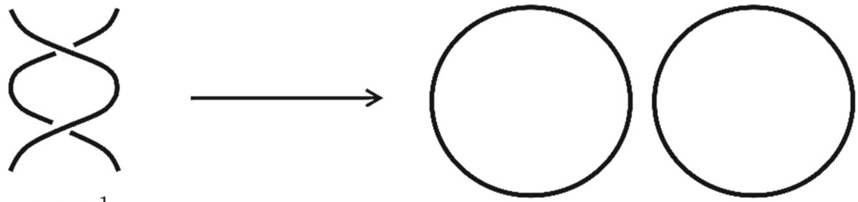
Under the intended interpretation, elements of B_n act on "n-strands" (n vertical line segments). In particular, the action of the ith element σ_i is to braid the ith strand counterclockwise about the $(i+1)$th strand, and the action of its inverse, σ_i^{-1}, is to braid the ith strand clockwise about the $(i+1)$th strand. An n-braid, expressed as a sequence $\sigma_i\sigma_j\sigma_k$... of elements of B_n, where $i, j, k \in 1, \ldots, n-1$, can be defined as an n-strand that carries a representation of B_n.

Every n-link can be represented by a closed n-braid.[16] The Borromean rings 3-link, for instance, is the closure of the 3-braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$, and the trivial 2-link is the closure of the 2-braid $\sigma_1\sigma_1^{-1}$ (Fig. 5). These examples suggest that an n-braid can be said to be topologically entangled just when it contains at least one pairwise set of terms $\sigma_i\sigma_j$ such that $\sigma_i\sigma_j \neq 1$. Thus the 3-braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$ is topologically entangled, whereas the 2-braid $\sigma_1\sigma_1^{-1}$ is not. This suggests the following definition of a topologically entangled n-link:

[16] See, e.g., Alagic et al. (2016, p. 2) for a discussion of this theorem due to Alexander.



$\sigma_2{}^{-1}\sigma_1\sigma_2{}^{-1}\sigma_1\sigma_2{}^{-1}\sigma_1$



$\sigma_1\sigma_1{}^{-1}$

Fig. 5 n-links as the closure of n-braids

> **Def. 1**. An ***n*-link** is **topologically entangled** just when it is the closure of a topologically entangled n-braid.

Under this definition, the Borromean rings 3-link is topologically entangled, whereas the trivial 2-link is not. Def. 1 might be compared with a typical definition of a quantum entangled n-partite state:

> **Def. 2**. A vector $|\psi\rangle \in \mathcal{H}$ is **quantum entangled** with respect to an n-partite decomposition $\mathcal{H} = V_1 \otimes ... \otimes V_n$ just when it cannot be expressed as a product of n terms $|\psi\rangle = |v_1\rangle \otimes ... \otimes |v_n\rangle$, where $|v_i\rangle \in V_i$.[17]

Given these definitions, one can now ask, is topological entanglement of n-links the dual of quantum entanglement of n-partite state vectors? Alas, no; to the extent that the duality requires a 1-1 correspondence. Aravind (1997) provided examples of the following claims:

[17] This definition of a quantum entangled state vector (i.e., a pure state) as a non-product state is sufficient for the purpose of an initial comparison with Def. 1. However, it does not fully capture the sense of non-locality associated with the correlations that subsystems in a quantum entangled state may exhibit, as Sect. 5 below indicates.

(i) A topologically entangled n-link can correspond to more than one quantum entangled n-partite state.
(ii) A quantum entangled n-partite state can correspond to more than one topologically entangled n-link.

Aravind's (1997, pp. 54-55) example of (i) is the three-Hopf rings 3-link, which has the feature that if any of its links is cut, the other two remain entangled (Fig. 4b). Under Aravind's analogy, this single topologically entangled 3-link corresponds to an infinite number of distinct quantum entangled tripartite states of the general form

$$\alpha|\uparrow_{a_1}\rangle\{|\uparrow_{b_2}\rangle|\uparrow_{b_3}\rangle - |\downarrow_{b_2}\rangle|\downarrow_{b_3}\rangle\} + \beta|\downarrow_{a_1}\rangle\{|\uparrow_{b_2}\rangle|\uparrow_{b_3}\rangle + |\downarrow_{b_2}\rangle|\downarrow_{b_3}\rangle\} \tag{9}$$

where a, b are arbitrary spin axes inclined at some angle θ with respect to each other, and α, β are constants determined by θ and constrained by $|\alpha|^2 + |\beta|^2 = 1$. These states have the feature in common with the three-Hopf rings that, if a measurement is performed on any one subsystem, the other subsystems remain in a quantum entangled state.

Aravind's (1997, p. 56) example of (ii) is the tripartite state

$$\sqrt{\frac{1}{3}}\{|\uparrow_{z_1}\rangle(|\uparrow_{z_2}\rangle|\downarrow_{z_3}\rangle + |\downarrow_{z_2}\rangle|\uparrow_{z_3}\rangle) + |\downarrow_{z_1}\rangle|\uparrow_{z_2}\rangle|\uparrow_{z_3}\rangle\} \tag{10}$$

In this state, if a measurement is performed on subsystem 1, there is a probability of 2/3 that the outcome will be spin-up, and a probability of 1/3 that it will be spin-down. Moreover, if the measurement outcome is spin-up, the other subsystems remain entangled, which corresponds, under Aravind's analogy, to the three-Hopf rings 3-link; whereas if the measurement outcome is spin-down, the other subsystems disentangle, which corresponds to the Borromean rings 3-link. Thus the quantum entangled tripartite state (10) corresponds to the three-Hopf rings with a probability of 2/3, and to the Borromean rings 3-link with a probability of 1/3.

Claims (i) and (ii) indicate that there is no 1-1 correspondence between topologically entangled n-links and quantum entangled n-partite states, according to Definitions 1 and 2; and this problematizes a state-based duality between topological entanglement and quantum entanglement. This does not rule out state-based approaches in general, but it does suggest additional constraints may be required to support a 1-1 duality relation based on states. In any event, the next section considers an alternative approach in which the focus is on observables, as opposed to states.

4 Observable-based approaches

Our sample proposal for a duality between spacetime topology and quantum entanglement was based on the RT formula, and it suggests a duality between a topological property (bulk connectivity), and a property associated with quantum entanglement (entanglement entropy). Thus perhaps a more direct approach to this type of duality should focus on properties (i.e., observables), as opposed to states.

An example of an observable-based approach is a research program initiated by Kauffman and Lomonaco (2002) that seeks correspondences between quantum entangling operators, identified with unitary representations of the braid group, and aspects of n-links. This program was motivated in part by Aravind's (1997) analogy, discussed in Sect. 3 above. Concerns with the limitations of this analogy prompted Kauffman and Lomonaco to switch the focus from states to operators (*viz*., observables):

> The main point for the exploration of the analogy is that, from the point of view of a braid representation, each braid is seen as an operator rather than a state (Kauffman and Lomonaco 2002, p. 5).

A unitary representation of the braid group B_n on the vector space $V^{\otimes n}$ is a map $\rho_n^{(R)}$ from elements of B_n to unitary operators on $V^{\otimes n}$ such that, for each $\sigma_k \in B_n$,

$$\rho_n^{(R)}(\sigma_k) = I^{\otimes k-1} \otimes R \otimes I^{\otimes n-k-1} \tag{11}$$

where I is the identity on V, and $R : V \otimes V \to V \otimes V$ is a unitary invertible bipartite operator that satisfies the Yang–Baxter equation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \tag{12}$$

Under a unitary representation of B_n, an n-strand is associated with two copies of $V^{\otimes n}$ (one for each set of its endpoints), and each elementary braid generator σ_k is associated with a unitary bipartite operator R (referred to as a Yang–Baxter operator).[18] Given a unitary representation of B_n, one can construct a representation of an n-braid b by replacing each occurance of σ_k in the expression for b with $\rho_n^{(R)}(\sigma_k)$. The result is an n-partite unitary operator $\rho_n^{(R)}(b)$ on $V^{\otimes n}$. Let an n-partite operator $\mathcal{O}$ be "quantum entangling" just when there is a vector $|\Phi\rangle$ in $V^{\otimes n}$ such that $|\Phi\rangle$ is a product state and $\mathcal{O}|\Phi\rangle$ is an entangled state. We can now pose the question, *Is* $\rho_n^{(R)}(b)$ *quantum entangling if and only if b is topologically entangled?*[19]

We might not expect this to be the case, since we know from Sect. 3 that there is no 1-1 correspondence between topologically entangled n-links and quantum entangled n-partite states. Indeed, Kauffman and Lomonaco (2002, p. 6) consider the case $V = \mathbf{C}^2$ and an explicit matrix form of R given by

$$R = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \tag{13}$$

One can show that (13) is unitary, invertible, and satisfies the Yang–Baxter equation (12), hence it induces a unitary representation of B_2 on $\mathbf{C}^2 \otimes \mathbf{C}^2$. Moreover, there are

[18] See, e.g., Alagic et al. (2016). Conditions (11) and (12) guarantee that R satisfies the braiding relations in the definition of B_n in Sect. 3 above.

[19] Recall from Sect. 3 that an n-braid is topologically entangled just when it contains at least one pairwise set of terms $\sigma_i \sigma_j$ such that $\sigma_i \sigma_j \neq 1$. Recall, too, that we are (still!) bracketing off the concern over defining an entangled state as a non-product state.

product states $|\Phi\rangle$ in $\mathbf{C}^2 \otimes \mathbf{C}^2$ such that the action of (13) on $|\Phi\rangle$ produces an entangled state, provided $\alpha\beta \neq \delta\gamma$.[20] But for $\alpha\beta = \delta\gamma$, R is not quantum entangling. In particular, we should expect that, for R given by (13) with $\alpha\beta = \delta\gamma$, a unitary representation of a topologically entangled 2-braid is not a quantum entangling operator on $\mathbf{C}^2 \otimes \mathbf{C}^2$.

Thus a 1-1 correspondence (and hence a duality) does not exist between quantum entangling operators and topologically entangled n-braids. What can be shown, however, is that, under limited circumstances, relations can be obtained between quantum entangling operators, on the one hand, and certain *link invariants*, on the other. A link invariant is a function that takes values on links, and that is the same for isotopically equivalent links. For instance, Kauffman and Lomonaco (2002, p. 11) showed that the following function on 2-links K is a link invariant:

$$Z_K = 2[1 + (\gamma^2/\alpha^2)^{lk(K)}] \tag{14}$$

where $\gamma, \alpha \in \mathbf{C}$, and $lk(K)$ is the linking number of the two components of K (i.e., the number of times each component winds around the other). Moreover, Z_K can be encoded in the expression (13) for a Yang–Baxter operator for the special case $\beta = \alpha, \delta = \gamma$. Recall that R is quantum non-entangling just when $\alpha\beta = \delta\gamma$, or $\alpha^2 = \gamma^2$ for the special case that encodes Z_K. In this quantum non-entangling case, we thus have $Z_K = 2(1 + 1^{lk(K)}) = 4$; i.e., Z_K is constant for all 2-links K, and hence is trivial. Thus, if Z_K is non-trivial, then the corresponding Yang–Baxter operator R is quantum entangling. According to Kaufman and Lomonaco (2002, p. 11), "...for this specialization of the R matrix, the operator R entangles quantum states exactly when it can detect linking numbers in the topological context."

This result was generalized by Alagic et al. (2016) in the following way: Let $\rho_n^{(R)}(b)$ be, as above, a representation of an n-braid b, under a unitary representation of B_n generated by a Yang–Baxter operator R. Then one can show that $\mathrm{Tr}[\rho_n^{(R)}(b) \cdot \mu^{\otimes n}]$ is a link invariant, where $\mu : V \to V$ is an endomorphism on V such that R commutes with $\mu \otimes \mu$, and $\mathrm{Tr}_2[R \cdot \mu \otimes \mu] = \mathrm{Tr}_2(R^{-1} \cdot \mu \otimes \mu) = \mu$ (where Tr_2 is the partial trace over the second tensor factor).[21] Alagic et al. (2016, p. 4) proved that if $\mathrm{Tr}[\rho_n^{(R)}(b) \cdot \mu^{\otimes n}]$ is non-trivial, then R is quantum entangling.

These results, while intriguing, face the following concerns, to the extent one might appeal to them to underwrite a duality between spacetime topology and quantum entanglement. First, both results indicate that quantum entanglement is necessary, but not sufficient, for a non-trivial link invariant.[22] Moreover, that a link possesses a non-trivial link invariant does not necessarily imply that it is topologically entangled. For instance, the 2-link invariant (14) cannot distinguish between the trivial 2-link and the Whitehead 2-link, insofar as both have linking number 0; but, by Def. 1 above, the

[20] Kauffman and Lomonaco (2002, p. 8). Let $\{|0\rangle, |1\rangle\}$ be a basis for $\mathbf{C}^2$, and let $|\Phi\rangle = \{|0\rangle + |1\rangle\}\{|0\rangle + |1\rangle\}$. Then $R|\Phi\rangle = \alpha|00\rangle + \gamma|10\rangle + \delta|01\rangle + \beta|11\rangle$. This is a product state just when $\alpha|00\rangle + \gamma|10\rangle + \delta|01\rangle + \beta|11\rangle = \{X|0\rangle + Y|1\rangle\}\{X'|0\rangle + Y'|1\rangle\}$, and this requires $\alpha = XX'$, $\gamma = X'Y$, $\delta = XY'$, $\beta = YY'$. This holds just when $\alpha\beta = \delta\gamma$.

[21] The appearance of μ in the trace ensures that the latter is invariant under the Markov moves, which are a set of transformations that leave the closure of a braid invariant.

[22] Alagic et al. (2016, p. 10) explicitly show this by counterexample.

trivial 2-link is not topologically entangled, whereas the Whitehead link is.[23] Of course the advocate of link invariants might respond, so much the worse for any attempt to use Def. 1 to underwrite the notion of a topological observable that is dual to a quantum entanglement observable. But even if we grant that link invariants are the appropriate type of topological observable to focus on, we might question the notion of a quantum entanglement observable represented simply by a quantum entangling operator. It's finally time to address the concern, raised initially in Sect. 2.1, about the nature of quantum entanglement.

4.1 Two manifestations of quantum entanglement

Quantum entanglement is typically characterized by a type of non-locality associated with a violation of a Bell inequality, and this is not entirely captured by the notion of a quantum entangling operator. Indeed, according to Kauffman and Lomonaco (2009, p. 103), "[t]he Bell inequality violation is an indication of quantum mechanical entanglement. One's intuition suggests that it is this sort of entanglement that should have a topological context." This suggests that the task of identifying appropriate dual notions of topological observable and quantum entanglement observable may be more complicated than the examples we've considered so far. Moreover, as Horodecki et al. (2009) observe, Bell inequality-violating manifestations of entanglement are, in general, distinct from entropic inequality-violating manifestations. Recall that one example of an entropic inequality is the relation $S_A \leq S_{vN}(\rho_{AB})$ between the von Neumann entropy of a bipartite system AB, and the entanglement entropy of one of its subsystems: this inequality is violated if and only if the composite pure state ρ_{AB} is indecomposable. More generally, for a bipartite state ρ_{AB}, an entropic inequality takes the form $S_\alpha(\rho_A) \leq S_\alpha(\rho_{AB})$, where $S_\alpha(\rho) = (1-\alpha)^{-1} \log \mathrm{Tr}\rho^\alpha$ is the α-Renyi entropy, and $\alpha \geq 0$ (Horodecki et al. 2009, p. 880). The von Neumann entropy is the limiting case for the limit as α goes to 1, $S_1 \equiv S_{vN}$. Werner (1989) showed that there are indecomposable mixed states (subsequently called "Werner states") that cannot be detected by a Bell inequality. In fact, one can show that there are Werner states whose indecomposablity can be detected by a Bell inequality but not by an entropic inequality, and vice-versa, there are Werner states whose indecomposablity can be detected by an entropic inequality but not by a Bell inequality.

To see this, consider the particular example of a 2-dim Werner state:

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)(I_2 \otimes I_2)/4 \tag{15}$$

where $|\psi^-\rangle$ is the maximally entangled spin-1/2 singlet state, I_2 is the identity on $\mathbf{C}^2$, and $0 \leq p \leq 1$. One can show that, for $p > 1/3$, ρ_W is indecomposable, for $p > 1/\sqrt{3} \approx 0.57735$, ρ_W violates the $\alpha = 2$ entropic inequality, and for $p > 1/\sqrt{2} \approx 0.707107$, ρ_W violates the CHSH Bell inequality (Horodecki et al. 1996, p. 380). Moreover, for $p > 0.7476$, ρ_W violates the von Neumann ($\alpha = 1$)

[23] See, e.g., Kauffman and Lomonaco (2002, p. 10). The trivial 2-link can be obtained from the closure of the 2-braid $\sigma_1\sigma_1^{-1}$, which is not topologically entangled. The Whitehead 2-link can be obtained from the closure of the 3-braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-2}$, which is topologically entangled.

entropic inequality (Krammer 2005, p. 24). Thus, if entanglement is defined in terms of indecomposability, then for $0.57735 < p < 0.707107$, the entanglement of ρ_W is detected by an entropic inequality but not a Bell inequality, and for $0.707107 < p < 0.7476$, the entanglement of ρ_W is detected by a Bell inequality but not the von Neumann entropic inequality.

This example raises the question, Where should we seek topological duals to quantum entanglement: in Bell inequality-violating manifestations of the latter, or in entropic inequality-violating manifestations? On the surface, this question may seem moot: Insofar as the RT formula and its relatives reviewed above in Sect. 2 all involve formal correspondences between entanglement entropy and some aspect of topology, entropic inequality-violating manifestations of quantum entanglement seem to be the relevant place to seek topological duals. Moreover, these examples involve bipartite systems in pure states, and for these simple cases (as opposed to, for instance, mixed states like (15)), the distinction between entropic inequality-violating manifestations of quantum entanglement and Bell inequality-violating manifestations breaks down, to the extent that, for these simple cases, a violation of an entropic inequality is both necessary and sufficient for a violation of a Bell inequality.[24]

On the other hand, a bipartite system in a pure non-product state will typically decompose into subsystems in mixed states; thus, as Earman (2015, p. 310) notes, "...even if we deal only with pure states at the composite system level, mixed states will out at the subsystem level". Similarly, the assumption that a composite system is in a pure state entails that it is not itself a subsystem of a larger composite system in a non-product state, and this seems unlikely if environmental interactions typically produce composite systems in non-product states. Thus the general case of a mixed composite state should not be ignored, and for this general case, the distinction between entropic and Bell inequality-violating measures of quantum entanglement is quantitatively non-trivial, as the example of Werner states indicates. Moreover, the distinction is also conceptually non-trivial: Entropic measures involve non-linear logarithmic functions of a quantum state, whereas Bell inequality-violating measures can be encoded in linear operators (see below Sect. 5.1); and the latter measures, but not the former, are explicitly associated with non-local correlations.

Motivated by these concerns, the next section considers three general issues that any observable-based approach to a spacetime topology/quantum entanglement duality should address; namely, non-linearity, non-locality, and the nature of a correlation.

5 Topological versus quantum entanglement observables

To summarize the discussion at the end of Sect. 4.1 above, the distinction between entropic inequality-violating manifestations of quantum entanglement and Bell inequality-violating manifestations raises three concerns for any observable-based

[24] For a pure bipartite state, one can show that indecomposability is both necessary and sufficient for a violation of a Bell inequality (Brunner et al. 2014, p. 437). On the other hand, a pure bipartite state is indecomposable *if and only if* it is a non-product state, and "non-productness" is both necessary and sufficient for a violation of an entropic inequality (Nielson and Chuang 2010, p. 514, as noted above in footnote 2).

approach to a duality between spacetime topology and quantum entanglement. These concerns are

(a) *Non-linearity* To the extent that quantum entanglement is characterized by an entropic inequality, it is non-linear, in the sense that it cannot be represented by a linear operator; whereas typical examples of topological observables are linear in this sense.
(b) *Non-locality* To the extent that quantum entanglement is characterized by a Bell inequality, it exhibits a type of non-locality that is distinct from the non-locality associated with topological observables.
(c) *Correlations* The non-locality that characterizes a violation of a Bell inequality is exhibited by a particular type of correlation between observables. If topological observables are required to exhibit this type, care must be taken in distinguishing local versus non-local correlations, on the one hand, and short-range versus long-range correlations, on the other.

The remainder of this section addresses each of these concerns in turn.

5.1 Non-linearity

Any observable-based approach to a duality between spacetime topology and quantum entanglement must explain how the typically linear nature of topological observables can be reconciled with the non-linear aspect of quantum entanglement. This non-linearity takes a concrete form in entropic inequality-violating manifestations of quantum entanglement, which, as mentioned above, involve nonlinear logarithmic functions of a quantum state.[25] Some authors identify this non-linearity with the claim that quantum entanglement cannot be directly represented by a linear operator. Bao et al. (2015b, p. 2), for instance, observe that the set of all entangled states of a Hilbert space is not closed under addition, and thus does not form a subspace. Hence there is no projector onto this set, and hence no corresponding linear observable.[26]

On the other hand, typical examples of topological observables are represented by linear operators. Consider the way such observables are represented in topological quantum field theories (see, e.g., Labastida and Lozano 1997, p. 5). In the functional integral formalism, a local quantum field theory consists of a smooth manifold M (i.e., spacetime), a Lorentzian metric $g_{\mu\nu}$ on M, a set of fields $\phi_i(x)$, and an action $S[\phi_i]$ that is a functional of the fields and their derivatives evaluated at the same point. The observables are vacuum expectation values of products of local linear operators $O[\phi_i]$ constructed as functionals of the fields. These are defined by $\langle O_1...O_n\rangle = \int D\phi_i O_1...O_n e^{-S[\phi_i]}$. In this approach, a topological quantum field theory is a local quantum field theory in which $\delta/\delta g_{\mu\nu}\langle O_1...O_n\rangle = 0$ for some set of local operators. In words: The vacuum expectation value of these operators is invariant under variations of

[25] Statements to this effect can be found in, e.g., Bovino et al. (2005, p. 1), Walborn et al. (2006, p. 1022), Mintet and Buchleitner (2007, p. 1).

[26] Recall that a linear operator O on a Hilbert space $\mathcal{H}$ is a map $O : \mathcal{H} \to \mathcal{H}$ such that $O(\alpha|\phi\rangle + \beta|\psi\rangle) = \alpha O|\phi\rangle + \beta O|\psi\rangle$, for any $\alpha, \beta \in \mathbf{C}$ and any $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. Thus if O is linear and responds "yes" to two entangled states separately, then it should respond "yes" to their sum; but the latter may not be an entangled state.

the metric and hence is a topological invariant of M. The point is that such topological observables are represented by (combinations of) *linear* operators.

An example of a linear topological observable is a Wilson loop operator $W[C]$, defined, for gauge field A, and closed spacelike loop C, by $W[C] = \mathrm{Tr}[P \exp(\int_C A)]$, where P is the path ordering operator. $W[C]$ can be interpreted as adding a loop of "electric" flux along C to a state. The vacuum expectation value $\langle W[C]\rangle$ is then the probability for a loop of electric flux $W[C]|0\rangle$ to annihilate the vacuum. Again, the point is that, so-defined, a Wilson loop operator is a linear topological observable.[27]

Of course the lesson drawn from the non-linear aspect of quantum entanglement might be just that we should seek the same sort of non-linearity in the choice of dual topological observable; and if this means typical topological observables have no quantum entanglement duals, then so be it. Alternatively, one might interpret the non-linear aspect of quantum entanglement as indicating its non-detectability (the underlying assumption perhaps being that to be detectable, an observable must be representable by a linear operator), and then attempt to identify non-detectable topological duals in the relevant circumstances. An example of this approach is Bao et al. (2015b): in the context of the ER = EPR hypothesis, they view the fact that there is no linear observable associated with quantum entanglement as dual to the non-detectability of an Einstein-Rosen wormhole geometry by an observer in the interior of a Schwarzschild black hole.

5.2 Non-locality

While entropic inequality-violating manifestations of quantum entanglement are non-linear, Bell inequality-violating manifestations are linear, at least to the extent that they can be encoded in a constraint imposed on a linear operator. For instance, the CHSH inequality is given by

$$|E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2)| \leq 2 \tag{16}$$

where $E(A_i, B_j)$ is the expectation value of the joint outcome $A_i B_j$ of two pairs of observables (A_1, A_2) and (B_1, B_2) associated with two spatially separated physical systems assumed to be conditionally statistically independent of each other. The latter condition requires that the joint probability of obtaining the values a and b of A_i and B_j, respectively, satisfies

$$p(a, b|A_i, B_j) = \int d\lambda q(\lambda) p(a|A_i, \lambda) p(b|B_j, \lambda) \tag{17}$$

where $q(\lambda)$ is a probability distribution over a random variable λ. One can show that (16) is violated for expectation values of the joint outcomes of some pairs of

[27] The notion of linearity here is that described in the previous footnote 26. Note that a Wilson loop operator is distinct from a loop state. The latter is a state with respect to which a loop operator has a non-zero expectation value. Issues concerning how to characterize an appropriate space of loop states (arising in "loop representations" of gauge theories, and in loop quantum gravity) are distinct from the linear nature of a loop operator.

spin-1/2 observables taken with respect to the indecomposable spin-1/2 singlet state $|\Psi\rangle = \sqrt{1/2}\{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\}$.[28] In this sense, a violation of the CHSH inequality (16) is an indication of quantum entanglement. Moreover, a linear Bell operator can be defined by

$$\mathcal{B} = A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2) \tag{18}$$

and (16) can then be encoded by the condition $|\mathrm{Tr}(\mathcal{B}\rho)| \leq 2$, for all states ρ that satisfy (17).

Thus, to the extent that quantum entanglement is characterized by the violation of a Bell inequality, it is linear. On the other hand, Bell inequality-violating manifestations of quantum entanglement are characterized by a type of non-locality that isn't explicit in entropic inequality-violating manifestations. The concern then is that this type of non-locality is very different from the non-locality that topological observables exhibit.[29]

To say an observable satisfies *localization* is to say it has support on a finite contractible region of spacetime. This notion of localization is motivated by two naive intuitions: First, the region of support should be sufficiently finite in order to distinguish a localizable observable from a global observable that takes non-zero values everywhere in the spacetime. Second, the region of support should be contractible (i.e., continuously deformable into a point) in order to avoid finite but topologically non-trivial regions of support; the intuition being that an observable with support in a finite disconnected region, or a finite region with one or more missing points, should not be considered localizable. A failure of localization in this second sense leads to what might be called topological non-locality:

> **Def. 3**. An observable exhibits **topological non-locality** just when it has support on a non-contractible region of spacetime.

Non-contractibility is a characteristic of a disconnected space (i.e., a space in which there are points that cannot be connected by a curve) and a multiply connected space (i.e., a space that contains a closed curve that cannot be continuously deformed into a point), and hence captures the topological aspects of the examples in Sect. 2. So, for instance, van Raamsdonk's Connectedness/Entanglement hypothesis states a duality between entropic inequality-violating manifestations of quantum entanglement on the boundary, and observables associated with a disconnected bulk region. And similarly, observables with support on multiply connected regions contribute to the topological entanglement entropy of Levin and Wen, and Kitaev and Preskill. In both of these examples, the relevant topological observables exhibit topological non-locality, according to Def. 3, and hence violate localization.

The type of non-locality associated with Bell inequality-violating manifestations of quantum entanglement is exhibited by a correlation between the observables of two

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[28] For instance, let $A_1 = \hat{e}_1 \cdot \vec{\sigma}$, $A_2 = \hat{e}_2 \cdot \vec{\sigma}$, $B_1 = -\sqrt{1/2}(\hat{e}_1 + \hat{e}_2) \cdot \vec{\sigma}$, and $B_2 = -\sqrt{1/2}(-\hat{e}_1 + \hat{e}_2) \cdot \vec{\sigma}$, where $\hat{e}_1, \hat{e}_2$ are any two choices of spin measurement axes, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ encodes the Pauli operators.

[29] Here I follow Bain's (2019, p. 25) distinction between two types of non-locality claimed to be present in intrinsic topologically ordered condensed matter systems.

spatially separated physical systems that violates (17). Note that (17) entails that λ screens A_i off from B_j, and *vice-versa*; hence, when (17) holds, λ acts as a common cause that can explain the correlation between A_i and B_j. Thus one might say that a Bell-inequality violating correlation cannot be explained by a common cause (see, e.g., Bub 2016, p. 74). Moreover, if the distance between the subsystems exceeds an appropriate bound on causal signal propagation, any correlation between their observables cannot be explained by a direct cause in the form of a causal signal that propagates from one subsystem to the other. When this is the case, call the correlation a "distant correlation". The upshot is that it is possible for the observables of two physical systems in an entangled state to exhibit a distant, Bell inequality-violating correlation; i.e., a correlation that cannot be explained by either a common cause or a direct cause. This amounts to a second notion of non-locality:[30]

> **Def. 4**. Two observables exhibit **quantum entanglement non-locality** just when they exhibit a distant, Bell inequality-violating correlation.

An observable that exhibits topological non-locality violates localization. On the other hand, quantum entanglement non-locality is compatible with localization: two observables can exhibit a distant, Bell inequality-violating correlation, and yet still have support in finite contractible regions of spacetime.

Thus, if one characterizes quantum entanglement in terms of a violation of a Bell inequality, then any attempt at articulating a duality between topology and quantum entanglement must provide an account of the circumstances in which topological non-locality entails, and is entailed by, quantum entanglement non-locality. In particular, such an account must explain the sense in which the topological non-locality exhibited by a topological observable entails, and is entailed by, a distant, Bell inequality-violating correlation between it and another observable.[31]

5.3 Correlations

In what sense can an observable that exhibits topological non-locality enter into a distant, Bell inequality-violating correlation? Recall that Levin and Wen (2006) refer to the notion of a "non-local" correlation in explaining the topological contribution to the formula (2) for the entanglement entropy S_R of a 2-dim gapped condensed matter system with respect to a bounded region R of space. Can this notion of a "non-local" correlation be understood in terms of a distant, Bell inequality-violating correlation involving a topological observable?

According to Levin and Wen (2006), the first boundary term in formula (2) comes from contributions to S_R from correlations between local observables: such correlations decay exponentially, due to the gap property; hence, any contribution they make to the entanglement between degrees of freedom inside R and degrees of freedom out-

[30] Indeed, the conditional statistical independence condition (17) is sometimes referred to as "Bell locality", and the requirement that causal signal propagation not exceed an appropriate bound is sometimes referred to as "Einstein locality". Quantum entanglement non-locality, Def. 4, can then be thought of as the denial of both Bell locality and Einstein locality.

[31] There is a limited sense in which topological non-locality entails quantum entanglement non-locality in condensed matter systems that exhibit topological order (see, e.g., Bain 2019).

side R must be a function of the size of the boundary. The second term (the "topological entanglement entropy" Γ) can arise in cases in which R has non-trivial topology, and it can then come from contributions to S_R from "non-local" observables with support in R, one example being an observable with support on a non-contractible loop within R. According to Levin and Wen (2006, p. 2), the presence of such a non-local observable indicates "non-local correlations". Evidently this establishes a link between a topological observable, in the form of a "non-local" observable with support on a topologically non-trivial region R, on the one hand; and quantum entanglement, as measured by the entanglement entropy S_R, on the other hand. But is this link enough to establish a duality? At the end of Sect. 2.1, I argued that it was not, but it may help to run through the argument again, this time making precise the distinctions between a local versus a non-local observable, and a local versus a non-local correlation. To this end, consider the following three questions:

1. *In what sense do correlations between observables with support on R and its complement $\bar{R}$ contribute to S_R?* To say observables A and B are correlated is to say there is a state $|\psi\rangle$ with respect to which

$$\langle\psi|AB|\psi\rangle \neq \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle \tag{19}$$

   This is to say that, with respect to probabilities prescribed by the Born rule, A and B are statistically dependent. On the other hand, S_R is a measure of the extent to which the bipartite pure state of the system is indecomposable, and one can show that statistical dependence holds for any A and B with respect to any pure state, *if and only if* that state is indecomposable. Thus, S_R is non-zero *if and only if* there are observables with support on R and $\bar{R}$, and a pure state $|\psi\rangle$ with respect to which the observables exhibit a correlation.
2. *What is the distinction between a "local" observable and a "non-local" observable?* Levin and Wen's account suggests cashing out this distinction in terms of the notion of localization in Sect. 5.2; namely, a local observable is an observable with support on a finite contractible region of spacetime, and a non-local observable is an observable that is not local (i.e., an observable that does not have support on a finite contractible region of spacetime).
3. Finally, *In what sense can a non-local observable contribute to the entanglement entropy S_R of a 2-dim gapped condensed matter system with respect to a bounded region R of space?* Given the answer to question #1, observables contribute to S_R to the extent that they are correlated. So in what sense can a non-local observable be correlated with other observables? Assumedly, just in the sense of (19). This suggests the following distinctions:

   (i) A *local correlation* is a correlation between local observables.
   (ii) A *non-local correlation* is a correlation involving at least one non-local observable.

   The above senses of correlation should be kept distinct from two more senses:

   (iii) A *short-range correlation* is a correlation that goes to zero at sufficiently large separation distances.

(iv) A *long-range correlation* is a correlation that does not go to zero at sufficiently large separation distances.

The upshot seems to be that, according to Levin and Wen, for a gapped 2-dim condensed matter system decomposed into a bipartite system with respect to the interior and exterior of a bounded region R, a non-zero value of the entanglement entropy S_R indicates *either* local, and hence (due to the gap property) short-range (relative to the correlation length) correlations along the boundary ∂R; *or* non-local, long-range (relative to the correlation length), correlations far from ∂R, *or both*.

Arguably, this is (still) not enough to establish a duality between topologically non-local observables with support in R, and quantum entanglement on either side of the boundary ∂R. Such a duality would seem to require that the presence of a topological observable inside R be both necessary and sufficient for the presence of quantum entanglement across the boundary ∂R, and this has not been established. In fact, the presence of a non-local observable with support on R is neither necessary nor sufficient for a non-zero S_R: It is not necessary, since S_R may be non-zero due solely to local correlations along the boundary; and it is not sufficient since, evidently, what contributes to the topological entanglement entropy Γ isn't just the presence of a non-local observable; rather, it is the presence of a non-local correlation. Moreover, the presence of a non-local correlation is necessary and sufficient for a non-zero Γ, and hence is sufficient, but not necessary for a non-zero S_R.

Finally, recall that for mixed states, the distinction between entropic inequality-violating manifestations of quantum entanglement and Bell inequality-violating manifestations is non-trivial. In particular, for a mixed state, a non-local correlation is not sufficient for a Bell inequality-violating manifestation of quantum entanglement. For mixed states, then, it seems reasonable to return to the question that began this subsection and request partisans who characterize quantum entanglement in terms of Bell inequality violations to give an account of the conditions under which a non-local correlation that involves a topological observable is a distant, Bell inequality-violating correlation.

6 Conclusion

Is nontrivial spacetime topology the dual of quantum entanglement? More precisely, are there topological states that are dual to quantum entangled states? Alternatively, are there topological observables that are dual to quantum entanglement observables? Despite a number of recent proposals that suggest the answer to these questions, in certain circumstances, is "yes", issues associated with the ambiguous nature of quantum entanglement suggest a fair amount of caution. For a duality between states, even for a simple pure quantum state, it's not apparent what the corresponding dual topological state should be, as Aravind's analogy indicates. For a duality between observables, if the duality is meant to capture entropic inequality-violating manifestations of quantum entanglement, then its topological side should reflect the non-linear aspects of its quantum entanglement side, and it is not entirely clear what the nature of this non-linearity is with respect to topological observables. On the other hand, if the duality is meant to capture Bell inequality-violating manifestations of quantum entanglement, then non-linearity is not so much of an issue as is non-locality; in particular, it needs to be shown how the distinct notions of non-locality associated with topological observables and with distant, Bell inequality-violating correlations can be made compatible. Finally, exactly how a non-local topological observable might enter into a distant, Bell inequality-violating correlation remains to be explained.

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