MAP COLORING
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\[ G = (V, \varepsilon) \]

COLORING \( G \) \( \rightarrow \) no adjacent vertices get same color
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$G$ is $k$-colorable if we can use $\leq k$ colors
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$\chi(G) : \min \# \text{colors we can use to color } G$

Chromatic number

$\chi \rho \mu \alpha = \text{color}$
**MAP COLORING**

\[ G = (V, E) \]

**COLORING** \( G \) → no adjacent vertices get same color

\( G \) is \( k \)-colorable if we can use \( \leq k \) colors

\( \chi(G) : \min \# \text{colors we can use to color } G \)

**chromatic number**

\( \chi \equiv \text{color} \)

Our map is 4-colorable \( \chi \leq 4 \)

etc
COLORING $G$ $\rightarrow$ no adjacent vertices get same color

$G$ is $k$-colorable if we can use $\leq k$ colors

$\chi(G) : \min \# \text{colors we can use to color } G$

chromatic number $\chiρ\nu\mu\alpha = \text{color}$

Our map is 4-colorable $\Rightarrow \chi \leq 4$

...but not 3-colorable

subgraph $K_4$

so $\chi \geq 4$
Exam scheduling

Students: \( s_1, s_2, s_3, s_4, s_5 \)

Classes: \( c_1, c_2, c_3, c_4, c_5 \)
EXAM SCHEDULING

students:  $s_1$, $s_2$, $s_3$, $s_4$, $s_5$

classes:  $c_1$, $c_2$, $c_3$, $c_4$, $c_5$
EXAM SCHEDULING

students: $S_1, S_2, S_3, S_4, S_5$

classes: $C_1, C_2, C_3, C_4, C_5$

Can't schedule exam simultaneously for classes taken by $S_i$
Want to minimize exam slots.
EXAM SCHEDULING

students: $S_1$ $S_2$ $S_3$ $S_4$ $S_5$
classes: $C_1$ $C_2$ $C_3$ $C_4$ $C_5$

Can't schedule exam simultaneously for classes taken by $S_i$.
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Make $G$: $V$ = classes $E$ = conflicts
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Make \( G: \ V = \text{classes} \quad E = \text{conflicts} \)
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Can't schedule exam simultaneously for classes taken by $S_i$

Want to minimize exam slots.

Make $G$: $V =$ classes $E =$ conflicts
EXAM SCHEDULING

students:  S₁  S₂  S₃  S₄  S₅  
           C₁  C₂  C₃  C₁  C₄  C₅  
           C₂  C₃  C₁  C₅  C₁  C₂  C₄  
classes:  C₁  C₂  C₃  C₄  C₅

Can’t schedule exam simultaneously for classes taken by Si
Want to minimize exam slots.

Make G:  V = classes   E = conflicts

Colors = slots (minimize colors)

If no edge has same color at endpoints, 
then no 2 classes are in same slot
EXAM SCHEDULING

students: $S_1, S_2, S_3, S_4, S_5$

classes: $C_1, C_2, C_3, C_4, C_5$

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Make $G$: $V = $ classes  $E = $ conflicts

Colors = slots (minimize colors)

If no edge has same color at endpoints, 
then no 2 classes are in same slot.
What is $\chi$ for cycles?
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\[
\chi = \begin{cases} 
2 & \text{if } V \text{ even} \\
3 & \text{if } V \text{ odd}
\end{cases}
\]
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even
$= 3$ if $V$ odd

For trees?
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even
$= 3$ if $V$ odd

For trees?
Remove a leaf, $v$.
2-color the rest...

...
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even
$\chi = 3$ if $V$ odd

For trees?
Remove a leaf, $v$.
2-color the rest.
Color $v$ opposite of $p(v)$

$\chi = 2$
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even

$\chi = 3$ if $V$ odd

For bipartite graphs?

For trees?

Remove a leaf, $v$.

2-color the rest.

Color $v$ opposite of $p(v)$.

$\chi = 2$
What is $\chi$ for cycles?

$\chi = 2$ if $|V|$ even
$\chi = 3$ if $|V|$ odd

For bipartite graphs?

$\chi = 2$

For trees?

Remove a leaf, $v$. 2-color the rest. Color $v$ opposite of $p(v)$

$\chi = 2$
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even

$\chi = 3$ if $V$ odd

For bipartite graphs?

$\chi = 2$

In fact if $\chi(G) = 2$ then $G$ is bipartite by definition

For trees?

Remove a leaf, $v$.

2-color the rest.

Color $v$ opposite of $p(v)$

$\chi = 2$

(trees are bipartite)
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even

$\chi = 3$ if $V$ odd

Claim: $G$ is bipartite if and only if $G$ contains no odd cycle.

For bipartite graphs?

$\chi = 2$

In fact, if $\chi(G) = 2$, then $G$ is bipartite by definition.

For trees?

Remove a leaf, $v$. 2-color the rest. Color $v$ opposite of $p(v)$.

$\chi = 2$

(trees are bipartite)
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ (for $K_n$)
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ $(K_n)$

What can $\chi$ be if max degree of $G = \Delta$?
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What can $\chi$ be if max degree of $G = \Delta$?

\[ K_n \rightarrow \]
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ (Kn).

What can $\chi$ be if max degree of $G = \Delta$?

- $K_n \rightarrow \Delta = n-1 : \chi = \Delta + 1$
- $\Delta = n-1 : \chi = 2$
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ ($K_n$)

What can $\chi$ be if max degree of $G = \Delta$?

$K_n \rightarrow \Delta = n-1 : \chi = \Delta + 1$

$\Delta = n-1 : \chi = 2$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ ($K_n$)

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$K_n \rightarrow \Delta=n-1 : \chi = \Delta+1$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)

Claim $\chi \leq \Delta+1$

$\Delta=4$

$\Delta=n-1 : \chi=2$
If \( G \) has \( n > 1 \) vertices, trivial bounds: \( 2 \leq \chi \leq n \) \((K_n)\).

What can \( \chi \) be if max degree of \( G = \Delta \)?

\[ K_n \rightarrow \Delta = n - 1 : \chi = \Delta + 1 \]

Can we have \( \chi \gg \Delta \)? (need \( n \gg \Delta \))

Claim \( \chi \leq \Delta + 1 \)

- Remove any vertex \( v \).

\( \Delta = 4 \)
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ \hspace{1cm} (K_n)

What can $\chi$ be if max degree of $G = \Delta$?

- $K_n$ \hspace{0.5cm} $\Delta = n - 1$ : $\chi = \Delta + 1$

Can we have $\chi \gg \Delta$? \hspace{1cm} (need $n \gg \Delta$)

Claim $\chi \leq \Delta + 1$

- Remove any vertex $v$.
- Color $G - v$ by induction.

$\Delta = 4$
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ (K_n)

What can $\chi$ be if max degree of $G = \Delta$?

K_n \rightarrow \Delta = n-1 : \chi = \Delta + 1

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- Remove any vertex $v$.
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- Re-insert $v$.
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ (K_n)

What can $\chi$ be if max degree of $G = \Delta$?

$K_n \rightarrow \Delta = n-1 \implies \chi = \Delta + 1$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)

Claim $\chi \leq \Delta + 1$

- Remove any vertex $v$.
- Color $G-v$ by induction.
- Re-insert $v$.
- $v$ has $\leq \Delta$ neighbors.
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ \hfill (K\(_n\))

What can $\chi$ be if max degree of $G = \Delta$?

$K_n \rightarrow \Delta = n-1 : \chi = \Delta + 1$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)

Claim $\chi \leq \Delta + 1$

- Remove any vertex $v$.
- Color $G - v$ by induction.
- Re-insert $v$.
- $v$ has $\leq \Delta$ neighbors.
- Use color $\Delta + 1$ for $v$. 
If $G$ has $n > 1$ vertices, trivial bounds: $2 \leq \chi \leq n$ (Kn).

What can $\chi$ be if max degree of $G = \Delta$?

$K_n \rightarrow \Delta = n-1 : \chi = \Delta + 1$

Can we have $\chi \gg \Delta$? (need $n \gg \Delta$)

Claim $\chi \leq \Delta + 1$

- Incrementally "add" vertices.
- When adding vertex $v$,
  look at all neighboring colors.
- Always have $> 1$ color available.

- Remove any vertex $v$.
- Color $G - v$ by induction.
- Re-insert $v$.
- $v$ has $< \Delta$ neighbors.
- Use color $\Delta + 1$ for $v$. 
COLORING PLANAR GRAPHS (like map duals)
COLORING PLANAR GRAPHS  

Claim: $\chi \leq 6$  

... trivial if $V \leq 6$
COLORING PLANAR GRAPHS  (like map duals)

Claim: $\chi \leq 6$  ... trivial if $V \leq 6$

We know planar graphs have a vertex w/ degree $\leq 5$  (Euler)
COLORING PLANAR GRAPHS  (like map duals)

Claim: $\chi \leq 6$ ... trivial if $V \leq 6$

We know planar graphs have a vertex w/ degree $\leq 5$

Given planar $G$ s.t. $V > 6$ & $u \in G$, $d(u) \leq 5$
COLORING PLANAR GRAPHS  (like map duals)

Claim: $\chi \leq 6$ ... trivial if $V \leq 6$

We know planar graphs have a vertex w/ degree $\leq 5$

Given planar $G$ s.t. $V > 6$ & $u \in G$, $d(u) \leq 5$ : look at $G-u$

still planar
COLORING PLANAR GRAPHS  (like map duals)

Claim: $\chi \leq 6$  ... trivial if $V \leq 6$

We know planar graphs have a vertex w/ degree $\leq 5$

Given planar G s.t. $V > 6$  &  $u \in G$, $d(u) \leq 5$ : look at $G-u$

Assume by induction that $G-u$ is 6-colorable
COLORING PLANAR GRAPHS  (like map duals)

Claim: $\chi \leq 6$ ... trivial if $V \leq 6$

We know planar graphs have a vertex w/ degree $\leq 5$

Given planar $G$ s.t. $V > 6$ & $u \in G$, $d(u) \leq 5$ : look at $G - u$

Assume by induction that $G - u$ is 6-colorable

re-insert $u$: give it a color not used by neighbors
Claim: $x \leq 5$ ... trivial if ???
Claim: $\chi \leq 5$ ... trivial if $V \leq 5$

Also trivial if neighbors use $< 5$ colors
Claim: $\chi \leq 5$ ... trivial if $V \leq 5$

Use induction & $d(u) \leq 5$ again

Also trivial if neighbors use $< 5$ colors

Consider any embedding of $G$

We need a neighbor of $u$ to change color
Try to change $\times$ from $\bullet$ to $\bullet$

[specifically skipping 2 over in $\text{adj}(u)$]
Try to change $\times$ from $\circ$ to $\circ$

[specifically skipping 2 over in $\text{adj}(u)$]

This works if $\times$ has no $\circ$ neighbors
Try to change $\times$ from $\bullet$ to $\bullet$ 

[specifically skipping 2 over in $\text{adj}(u)$]

$\Rightarrow$ This works if $\times$ has no $\bullet$ neighbors

$\Rightarrow$ else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\bullet \bullet$ and $\times$. 
Try to change $x$ from $\circ$ to $\circ$ \\
[specifically skipping 2 over in $\text{adj}(u)$] \\
$\Rightarrow$ This works if $x$ has no $\circ$ neighbors \\
$\Rightarrow$ else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\circ \circ$ and $x$. 
Try to change $x$ from $\bullet$ to $\circ$.

- Specifically skipping 2 over in $\text{adj}(u)$.
- This works if $x$ has no $\circ$ neighbors.
- Else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\circ \circ$ and $x$.

ONE PROBLEM?
Try to change $\times$ from $\bullet$ to $\bullet$.

[Specifically skipping 2 over in $\text{adj}(u)$]

→ This works if $\times$ has no $\bullet$ neighbors.

→ Else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\bullet$ and $\times$.

One problem...
The only bad case involves a path from $x$ to $y$ that alternates $x \cdots y$. 
The only bad case involves a path from $x$ to $y$ that alternates $x \_ \_ \_ \_ \_ y$

Together with $\odot$, the path forms a cycle surrounding the $\bullet^t$-neighbor of $u$.

$\text{So \ldots ?}$
The only bad case involves a path from $x$ to $y$ that alternates $x$$0$$0$$0$$...$$y$

Together with $u$ the path forms a cycle surrounding the $\bullet$ neighbor of $u$.

Restart the entire procedure using $s$ & $t$ instead of $x$ & $y$. 
The only bad case involves a path from \( x \) to \( y \) that alternates \( x \circ \circ \circ \circ \circ \circ y \).

Together with \( u \) the path forms a cycle surrounding the \( \circ \) neighbor of \( u \).

Restart the entire procedure using \( s \& t \) instead of \( x \& y \).

The only way to fail is if there is a path \( s \circ \circ \circ \circ \circ \circ t \)
The only bad case involves a path from \( x \) to \( y \) that alternates \( x \ldots y \).

Together with \( u \) the path forms a cycle surrounding the \( \bullet \) neighbor of \( u \).

Restart the entire procedure using \( s \) & \( t \) instead of \( x \) & \( y \).

The only way to fail is if there is a path \( s \ldots t \) but this would have to cross \( x \ldots y \).
The only bad case involves a path from $x$ to $y$ that alternates $\cdot \cdot \cdot \cdot \cdot \cdot \cdot y$

Together with $u$ the path forms a cycle surrounding the $\bullet$ neighbor of $u$.

Restart the entire procedure using $s \& t$ instead of $x \& y$.

The only way to fail is if there is a path $s \cdot \cdot \cdot t$

but this would have to cross $\cdot \cdot \cdot \cdot \cdot \cdot \cdot y$

<Impossible: This is a plane drawing>
Planar graphs:

6-coloring: ~ trivial
5-coloring: short elegant proof
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4-coloring: • unsolved from ≤1850 until 1977
• proof involved ~2000 cases solved by computer
Planar graphs:

6-coloring: ~ trivial
5-coloring: short elegant proof

4-coloring: • unsolved from ≤1850 until 1977
  • proof involved ~2000 cases solved by computer

3-coloring: • clearly not always possible
  • if triangle-free then 3-colorable
    (in fact if ≤3 triangles)