RANDOM VARIABLES
A quote from the Scheinerman textbook:

“A random variable is neither random nor variable”
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\[ X[(5,5)] = 10 \]
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\[ \text{define random variable } Y : \text{parity of two dice rolls.} \]

\[ Y[(1,2)] = 1 \]
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2 dice \< \text{sum} \< \text{parity}
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\[ P(X < 3) = \frac{1}{36} \]
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We can also eliminate absurd events, e.g., \( P(X = 13) = 0 \).
EXPECTATION : the very basics
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EXPERIMENT: the very basics

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over all samples that define \( X \).
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\[ = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \]

\[ = \frac{1}{6} \left( 1 + 2 + 3 + 4 + 5 + 6 \right) \]

\[ = \frac{1}{6} \cdot 21 \]

\[ = \frac{21}{6} \]

\[ = \boxed{3.5} \]
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Possible values of \( X \rightarrow 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \)
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possible values of \( X \) \( \rightarrow \) 0, 1, 2, 3, 4, 5

# outcomes supporting value \( \rightarrow \) 6, 5, 2, 4, 2, 3, 2, 2

(for probability, divide by 36)
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example: roll 2 dice, \( X = |\text{difference between the 2}| \)

possible values of \( X \) → 0 1 2 3 4 5

# outcomes supporting value → 6 5·2 4·2 3·2 2·2 1·2

(for probability, divide by 36)

\[ E(x) = \frac{0 + 10 + 16 + 18 + 16 + 10}{36} \]

\( \approx 1.944 \)
EXPECTATION : PROPERTIES
Expectation: Properties

\[ E(X + Y) = E(X) + E(Y) \]
$c_1, c_2 \in \mathbb{R}$

$$E(c_1 X + c_2 Y) = c_1 \cdot E(X) + c_2 \cdot E(Y)$$
LINEARITY OF EXPECTATION (important)

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Generally, \[ \mathbb{E}(c_1X_1 + c_2X_2 + \cdots + c_nX_n) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2) + \cdots + c_n\mathbb{E}(X_n) \]
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**Independence:** \( P(X=a \ & \ Y=b) = P(X=a) \cdot P(Y=b) \)

for all \( a,b \ldots \)
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for all \( a, b \) ...

---

2 dice, A, B. \( X = \) result of A. \( Y = \) result of B. \( Z = X + Y \)

\[
E(Z) = E(X + Y) = E(X) + E(Y) = 2 \cdot 3.5 = 7
\]
EXPECTATION: PROPERTIES

\[ E(X+Y) = E(X) + E(Y) \]

Linearity of expectation doesn’t assume independence
\textbf{EXPECTATION : PROPERTIES}

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but \[ E(X \cdot Y) \neq E(X) \cdot E(Y) \] in general.
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If \( X \) & \( Y \) are independent, then \( E(X \cdot Y) = E(X) \cdot E(Y) \)

However, \( E(X \cdot Y) = E(X) \cdot E(Y) \) does **NOT** imply
\( X \) & \( Y \) are independent.

(see example 34.15) (Scheinerman)
- We are skipping the proofs of most statements in this section.

- You are not required to study these, but it would probably be beneficial.

- We are also skipping variance, which is an important concept to learn independently.