Knowledge assumed in this document:

Algebraic definition of even, odd, irrational.

IF-THEN, IFF, →, ↔

∃

∀

prime & prime factor

every integer is a product of primes  (see document on induction)
if Greece wins the world cup, I will be happy (forever)
if Greece wins the world cup, I will be happy (forever)

if I'm not happy, Greece has not won the world cup
if Greece wins the world cup, I will be happy (forever)

\[\text{equivalent}\]

if I'm not happy, Greece has not won the world cup
CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

if I’m not happy, Greece has not won the world cup

if you are a square, you have corners
CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

if I'm not happy, Greece has not won the world cup

if you are a square, you have corners

??
CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

\[ \uparrow \text{equivalent} \uparrow \]

if I'm not happy, Greece has not won the world cup

if you are a square, you have corners

\[ \uparrow \text{equivalent} \uparrow \]

if you don't have corners, you are not a square
**CONTRAPOSITIVE**

if Greece wins the world cup, I will be happy *(forever)*

\[ \Leftrightarrow \]

if I'm *not* happy, Greece has *not* won the world cup

if you are a square, you have corners

\[ \Leftrightarrow \]

if you *don't* have corners, you are *not* a square

if A then B \[ \iff \] 
CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

Equivalent

if I'm not happy, Greece has not won the world cup

if you are a square, you have corners

Equivalent

if you don't have corners, you are not a square

if A then B ⇔ if not B, then not A
CONTRAPOSITIVE: \[ \text{if } A \text{ then } B \quad = \quad \text{if not } B, \text{ then not } A \]
CONTRAPOSITIVE: \hspace{1cm} \text{if } A \text{ then } B \quad = \quad \text{if not } B, \text{ then not } A

\hspace{1cm} A \rightarrow B \quad = \quad \neg B \rightarrow \neg A

\hspace{1cm} \text{not } B \rightarrow \text{ not } A
CONTRAPOSITIVE: \[ \text{if } A \text{ then } B \quad = \quad \text{if not } B, \text{ then not } A \]
\[ A \rightarrow B \quad = \quad \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but \( B \) is still true?

Greece hasn't won, but I'm still happy.

This shape isn't a square, but it has corners.
CONTRAPOSITIVE: \[ \text{if } A \text{ then } B = \text{if not } B, \text{ then not } A \]
\[ A \rightarrow B = \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but \( B \) is still true?

\[ \leftrightarrow \text{ That's OK; no contradiction. It's not } B \text{ IFF } A \]
CONTRAPPOSITIVE: \( \text{if } A \text{ then } B = \text{if not } B, \text{ then not } A \)

\[ A \rightarrow B = \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but \( B \) is still true?

\[ \Leftarrow \text{That's OK; no contradiction. It's not } B \text{ IFF } A \]

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CONTRAPOSITIVE: if A then B = if not B, then not A

A → B = ¬B → ¬A

What if ¬A holds, but B is still true?

¬A → ¬B

That's OK; no contradiction. It's not B IFF A

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CONTRAPOSITIVE: \[ \text{if A then B} = \text{if not B, then not A} \]
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What if \( \neg A \) holds, but B is still true?
\[ \iff \text{That's OK; no contradiction. It's not B IFF A} \]

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What if \( \neg A \) holds, but \( B \) is still true?

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| F & F & \checkmark \\
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don't contradict
CONTRAPOSITIVE: \[ \text{if } A \text{ then } B = \text{if not } B, \text{ then not } A \]

\[ A \rightarrow B = \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but \( B \) is still true?

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\{\text{don't contradict}\}:

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CONTRAPOSITIVE: \[ \text{if A then B} = \text{if not B, then not A} \]
\[ A \to B = \neg B \to \neg A \]

What if \( \neg A \) holds, but B is still true?

\[ \text{\( \neg A \)} \text{ holds, but B is still true?} \]

\[ \text{That's OK; no contradiction. It's not B IFF A} \]

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CONTRAPOSITIVE: if $A$ then $B$ = if not $B$, then not $A$

$A \rightarrow B = \neg B \rightarrow \neg A$

What if $\neg A$ holds, but $B$ is still true?

$\neg A \rightarrow \neg B$ That's OK; no contradiction. It's not $B$ IFF $A$

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don't contradict

| F | T | ✓ |
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**ContraPositive:** \( \text{if } A \text{ then } B = \text{ if not } B, \text{ then not } A \) 

\[ A \rightarrow B = \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but B is still true?

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*don't contradict*
**CONTRAPOSITIVE:** \[ \text{if } A \text{ then } B = \text{if not } B, \text{ then not } A \]
\[ A \rightarrow B = \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but \( B \) is still true?

\[ \downarrow \text{ That's OK; no contradiction. It's not } B \iff A \]

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don't contradict
**Contrapositive:** \( \text{if } A \text{ then } B = \text{ if not } B, \text{ then not } A \)

\[ A \rightarrow B = \neg B \rightarrow \neg A \]

What if \( \neg A \) holds, but \( B \) is still true?

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context so far: we know $A \rightarrow B$, so if we observe $\neg B$
then we can conclude $\neg A$

$\neg$ corners $\rightarrow$ $\neg$ square
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \) then we can conclude \( \neg A \)

**PROOF BY CONTRAPOSITIVE**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.
context so far: we know $A \rightarrow B$, so if we observe $\neg B$
then we can conclude $\neg A$

**Proof by Contrapositive**

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x + 9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)
context so far: we know $A\rightarrow B$, so if we observe $\neg B$
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PROOF BY CONTRAPOSCITIVE

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start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)

$7x + 9 = 2a$ \hspace{1cm} // a: integer $\rightarrow 7x + 9$ : even
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \) then we can conclude \( \neg A \).

**Proof by Contraposition**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( 7x + 9 \) is even, then \( x \) is odd (for \( x \in \mathbb{Z} \))

\[
\begin{align*}
7x + 9 &= 2a \quad \text{// } a: \text{integer} \rightarrow 7x + 9: \text{even} \\
x &= 2a - 6x - 9
\end{align*}
\]
Context so far: we know $A \rightarrow B$, so if we observe $\neg B$, then we can conclude $\neg A$.

**Proof by Contrapositive**

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x + 9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)

\[
7x + 9 = 2a \quad // a: \text{integer} \rightarrow 7x + 9: \text{even}
\]

\[
x = 2a - 6x - 9
\]

\[
x = 2a - 6x - 10 + 1
\]
context so far: we know $A \rightarrow B$, so if we observe $\neg B$
then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don’t know how to prove $A \rightarrow B$ (easily), so we try to
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Prove: if $7x + 9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)

\[
7x + 9 = 2a \quad \text{// $a$ : integer} \rightarrow 7x + 9 : \text{even}
\]

\[
x = 2a - 6x - 9
\]

\[
x = 2a - 6x - 10 + 1
\]

\[
x = 2(a - 3x - 5) + 1
\]
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \) then we can conclude \( \neg A \)

**PROOF BY CONTRAPOSITIVE**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( 7x + 9 \) is even, then \( x \) is odd \((\text{for } x \in \mathbb{Z})\)

\[
7x + 9 = 2a \quad // a: \text{integer} \rightarrow 7x + 9: \text{even} \\
x = 2a - 6x - 9 \\
x = 2a - 6x - 10 + 1 \\
x = 2(a - 3x - 5) + 1 \\
x = 2b + 1 \quad (b = a - 3x - 5)
\]
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \) then we can conclude \( \neg A \)

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**PROOF BY CONTRAPOSITIVE**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( 7x+9 \) is even, then \( x \) is odd (for \( x \in \mathbb{Z} \))

\[
\begin{align*}
7x + 9 &= 2a \quad \text{[}\text{for} \ a \in \text{integer}\text{]} \rightarrow 7x+9 \text{ is even} \\
\frac{x}{a} &= 2a - 6x - 9 \\
\frac{x}{a} &= 2(a - 3x - 5) + 1 \\
x &= 2b + 1 \text{ (odd)} (b = a - 3x - 5) \quad \square
\end{align*}
\]
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \) then we can conclude \( \neg A \)

---

**Proof by Contrapositive**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

---

**Prove:** if \( 7x+9 \) is even, then \( x \) is odd (for \( x \in \mathbb{Z} \))

\[
\begin{align*}
7x + 9 &= 2a & \text{direct} \\
x &= 2a - 6x - 9 \\
x &= 2a - 6x - 10 + 1 \\
x &= 2(a - 3x - 5) + 1 \\
x &= 2b + 1 \text{ (odd)} (b = a - 3x - 5)
\end{align*}
\]
Proof: if \(7x+9\) is even, then \(x\) is odd (for \(x \in \mathbb{Z}\))

\[
7x + 9 = 2a \quad \text{// } a: \text{integer } \Rightarrow 7x + 9: \text{even}
\]

\[
x = 2a - 6x - 9
\]

\[
x = 2a - 6x - 10 + 1
\]

\[
x = 2(a - 3x - 5) + 1
\]

\[
x = 2b + 1 \quad \text{(odd)} \quad (b = a - 3x - 5)
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Proof by Contrapositive

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( 7x + 9 \) is even, then \( x \) is odd \quad \text{for } x \in \mathbb{Z} \)

7x + 9 = 2a \quad // \text{a: integer } \rightarrow 7x + 9 : \text{even} \quad \text{direct}

\begin{align*}
x &= 2a - 6x - 9 \\
x &= 2a - 6x - 10 + 1 \\
x &= 2(a - 3x - 5) + 1 \\
x &= 2b + 1 \quad \text{(odd)} \quad (b = a - 3x - 5) \\
\end{align*}

\begin{align*}
\text{contrapositive} \\
\text{Suppose } x \text{ is not odd: } x = ? \\
\end{align*}
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \)
then we can conclude \( \neg A \)

---

**Proof by Contrapositive**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

---

Prove: if \( 7x + 9 \) is even, then \( x \) is odd  

\[
\begin{align*}
7x + 9 &= 2a \quad \text{even} \\
x &= 2a - 6x - 9 \\
x &= 2a - 6x - 10 + 1 \\
x &= 2(a - 3x - 5) + 1 \\
x &= 2b + 1 \quad \text{(odd)} \quad (b = a - 3x - 5)
\end{align*}
\]

Suppose \( x \) is not odd: \( x = 2c \)  

... then?
context so far: we know $A \rightarrow B$, so if we observe $\neg B$ then we can conclude $\neg A$

---

**PROOF BY CONTRAPOSITIVE**

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

**Prove:** if $7x + 9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)

\[
\begin{align*}
7x + 9 & = 2a \quad \text{// $a$: integer $\Rightarrow 7x + 9$: even} \\
x & = 2a - 6x - 9 \\
x & = 2a - 6x - 10 + 1 \\
x & = 2(a - 3x - 5) + 1 \\
x & = 2b + 1 \quad \text{(odd)} \quad (b = a - 3x - 5)
\end{align*}
\]

contrapositive

Suppose $x$ is not odd: $x = 2c$

\[
7x + 9 = 7 \cdot 2c + 9
\]
context so far: we know \( A \rightarrow B \), so if we observe \( \neg B \) then we can conclude \( \neg A \)

---

**PROOF BY CONTRAPOSITIVE**

We don’t know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

---

Prove: if \( 7x+9 \) is even, then \( x \) is odd (for \( x \in \mathbb{Z} \))

**Direct**

\[
\begin{align*}
7x+9 &= 2a \quad \text{// } a: \text{integer } \rightarrow 7x+9: \text{even} \\
x &= 2a - 6x - 9 \\
x &= 2a - 6x - 10 + 1 \\
x &= 2(a - 3x - 5) + 1 \\
x &= 2b + 1 \quad \text{(odd)} (b = a - 3x - 5)
\end{align*}
\]

**Contrapositive**

Suppose \( x \) is not odd: \( x = 2c \)

\[
\begin{align*}
7x + 9 &= 7 \cdot 2c + 9 \\
&= 14c + 8 + 1
\end{align*}
\]
context so far: we know $A \rightarrow B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSIITIVE

We don’t know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x + 9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)

\[
\begin{align*}
7x + 9 &= 2a & \text{// } & a: \text{integer } \rightarrow 7x + 9: \text{even} \\
x &= 2a - 6x - 9 \\
x &= 2a - 6x - 10 + 1 \\
x &= 2(a - 3x - 5) + 1 \\
x &= 2b + 1 \text{ (odd)} \quad (b = a - 3x - 5)
\end{align*}
\]

contrapositive

Suppose $x$ is not odd: $x = 2c$

\[
\begin{align*}
7x + 9 &= 7 \cdot 2c + 9 \\
&= 14c + 8 + 1 \\
&= 2 \cdot (7c + 4) + 1
\end{align*}
\]
context so far: we know $A \rightarrow B$, so if we observe $\neg B$
then we can conclude $\neg A$

**PROOF BY CONTRAPOSITIVE**

We don't know how to prove $A \rightarrow B$ (easily), so we try to
start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x + 9$ is even, then $x$ is odd

**Notation:** for $x \in \mathbb{Z}$

\[
7x + 9 = 2a \quad // \quad a: \text{integer} \rightarrow 7x+9: \text{even}
\]

\[
x = 2a - 6x - 9
\]

\[
x = 2a - 6x - 10 + 1
\]

\[
x = 2(a - 3x - 5) + 1
\]

\[
x = 2b + 1 \quad (\text{odd}) \quad (b = a - 3x - 5)
\]

---

**Contrapositive**

Suppose $x$ is not odd: $x = 2c$

\[
7x + 9 = 7 \cdot 2c + 9
\]

\[
= 14c + 8 + 1
\]

\[
= 2 \cdot (7c + 4) + 1
\]

\[
= 2 \cdot d + 1 \quad (d = 7c + 4)
\]

\[
\]
context so far: we know $A \rightarrow B$, so if we observe $\neg B$
then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x + 9$ is even, then $x$ is odd (for $x \in \mathbb{Z}$)

**Direct**

$7x + 9 = 2a \quad // a: integer \rightarrow 7x + 9: even$

$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

$x = 2b + 1$ (odd) ($b = a - 3x - 5$)

**Contrapositive**

Suppose $x$ is not odd: $x = 2c$

$7x + 9 = 7 \cdot 2c + 9$

$= 14c + 8 + 1$

$= 2 \cdot (7c + 4) + 1$

$= 2 \cdot d + 1$ ($d = 7c + 4$)

$7x + 9 = odd$

$\square$
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \Rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd.
PROOF BY CONTRAPOSITIVE

We don’t know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd

phrase mathematically?
PROOF BY CONTRAPOSITIVE

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x^2 - 6x + 5 \) is even, then \( x \) is odd.

phrase mathematically?

\[
(x^2 - 6x + 5 = 2a) \quad \rightarrow \quad (x = 2b + 1)
\]

& \( x, a, b \) are integers
PROOF BY CONTRAPOSITIVE

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x^2 - 6x + 5 \) is even, then \( x \) is odd

\[
x^2 - 6x + 5 = 2a \quad \text{direct}
\]
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd

$x^2 - 6x + 5 = 2a$ \hspace{1cm} \text{direct}

\text{?}

\text{?}

\Rightarrow

x = 2b + 1
PROOF BY CONTRAPOSITIVE

We don't know how to prove \( A \Rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x^2 - 6x + 5 \) is even, then \( x \) is odd

\[
x^2 - 6x + 5 = 2a
\]

\[
x^2 - 6x + (5 - 2a) = 0
\]

direct

\[
x = \frac{6 \pm \sqrt{36 + 8a - 20}}{2}
\]

\[
x = \frac{6 \pm \sqrt{4 + 2a}}{2}
\]

\[
\therefore \quad x = 2b + 1
\]
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \Rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd

$x^2 - 6x + 5 = 2a$  \hspace{1cm} \text{direct}  \hspace{1cm} \text{contrapositive} \hspace{1cm} \ldots ?$

\[ \therefore \text{?} \]

\[ x = 2b + 1 \]
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd

$x^2 - 6x + 5 = 2a$  

direct  

contrapositive  

Suppose $x$ is not odd: $x = 2c$

$\therefore$  

$\therefore$  

$x = 2b + 1$
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

**Prove:** if $x^2 - 6x + 5$ is even, then $x$ is odd

$x^2 - 6x + 5 = 2a$  
**direct**

contrapositive

Suppose $x$ is not odd: $x = 2c$

$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$

:: $\neg$ $\neg$

$x = 2b + 1$
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd.

$$x^2 - 6x + 5 = 2a$$

**direct**

**contrapositive**

Suppose $x$ is not odd: $x = 2c$

$$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$$

$$= 4c^2 - 12c + 5$$

$: ?$

$: ?$

$x = 2b + 1$
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd.

$x^2 - 6x + 5 = 2a$  

**direct**

contrapositive

Suppose $x$ is not odd: $x = 2c$

$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$

$= 4c^2 - 12c + 5$

$= 4c^2 - 12c + 4 + 1$

because we want to get something odd

$x = 2b + 1$
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \Rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd

$$x^2 - 6x + 5 = 2a$$  \hspace{1cm} \text{direct} \hspace{1cm} \text{contrapositive}

Suppose $x$ is not odd: $x = 2c$

$$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$$
$$= 4c^2 - 12c + 5$$
$$= 4c^2 - 12c + 4 + 1$$
$$= 2 \cdot (2c^2 - 6c + 2) + 1$$

\[ \therefore \]
PROOF BY CONTRAPOSITIVE

We don’t know how to prove \( A \Rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x^2 - 6x + 5 \) is even, then \( x \) is odd

\[
x^2 - 6x + 5 = 2a
\]

\begin{align*}
\text{Direct} & \quad \text{Contrapositive} \\
\text{Suppose } x \text{ is not odd: } x = 2c & \quad \text{Suppose } x = 2c \\
\Rightarrow x^2 - 6x + 5 & = (2c)^2 - 6 \cdot 2c + 5 \\
& = 4c^2 - 12c + 5 \\
& = 4c^2 - 12c + 4 + 1 \\
& = 2 \cdot (2c^2 - 6c + 2) + 1 \\
& = 2 \cdot d + 1 \quad (d = 2c^2 - 6c + 2)
\end{align*}

\[
x = 2b + 1
\]
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then $x$ is odd

$x^2 - 6x + 5 = 2a$

**direct**

**contrapositive**

Suppose $x$ is not odd: $x = 2c$

$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$

$= 4c^2 - 12c + 5$

$= 4c^2 - 12c + 4 + 1$

$= 2 \cdot (2c^2 - 6c + 2) + 1$

$= 2 \cdot d + 1$ (since $d = 2c^2 - 6c + 2$)

$= \text{not even}$

$x = 2b + 1$
PROOF BY CONTRAPOSITIVE

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x \) is irrational then \( \sqrt{x} \) is irrational.
PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x$ is irrational then $\sqrt{x}$ is irrational

direct

???
We don't know how to prove $A \rightarrow B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x$ is irrational then $\sqrt{x}$ is irrational

direct

contrapositive
PROOF BY CONTRAPOSITIVE

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x \) is irrational then \( \sqrt{x} \) is irrational

direct
???

contrapositive

Suppose \( \sqrt{x} \) is not irrational
PROOF BY CONTRAPOSITIVE

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

Prove: if \( x \) is irrational then \( \sqrt{x} \) is irrational

direct

contrapositive

Suppose \( \sqrt{x} \) is not irrational

\[
\sqrt{x} = \frac{a}{b} \quad a, b \in \mathbb{Z}
\]
**Proof by Contrapositive**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

**Prove:** if \( x \) is irrational then \( \sqrt{x} \) is irrational

<table>
<thead>
<tr>
<th>Direct</th>
<th>Contrapositive</th>
</tr>
</thead>
<tbody>
<tr>
<td>???</td>
<td>Suppose ( \sqrt{x} ) is not irrational</td>
</tr>
</tbody>
</table>

\[
\sqrt{x} = \frac{a}{b} \quad a, b \in \mathbb{Z}
\]

\[
x = \frac{a^2}{b^2}
\]
**Proof by Contrapositive**

We don't know how to prove \( A \rightarrow B \) (easily), so we try to start by assuming \( \neg B \). If we conclude \( \neg A \), we are done.

### Prove:

If \( x \) is irrational then \( \sqrt{x} \) is irrational

<table>
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<td>Suppose ( \sqrt{x} ) is not irrational</td>
</tr>
</tbody>
</table>

Suppose \( \sqrt{x} \) is not irrational

\[
\sqrt{x} = \frac{a}{b} \quad a, b \in \mathbb{Z}
\]

\[
x = \frac{a^2}{b^2} \quad \text{not irrational } \square
\]
PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving \( \text{if } A \text{ then } B \) for now
PROOF BY CONTRAPOSITION

A slight generalization of proof by contrapositive

still proving if A then B for now

\[(a+b)(a-b) \rightarrow a^2 - ab + ba - b^2 \rightarrow a^2 - b^2\]

You can prove something directly (in one direction)
PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving \( \text{if } A \text{ then } B \) for now

\[(a+b) \cdot (a-b) \Rightarrow a^2 - ab + ba - b^2 \Leftrightarrow a^2 - b^2\]

You can prove something directly (in one direction)
or work in both directions
PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving \( \text{if } A \text{ then } B \) for now

\[(a+b)(a-b) \iff a^2 - ab + ba - b^2 \iff a^2 - b^2\]

You can prove something directly (in one direction) or work in both directions

contrapositive \( \iff \) starting w/ \( \neg B \) & leading to \( \neg A \)

contradicts \( A \rightarrow \neg B \)
PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving \( \text{if } A \text{ then } B \) for now

\[(a+b)(a-b) \iff a^2-ab+ba-b^2 \iff a^2-b^2\]

You can prove something directly (in one direction) or work in both directions

instead of starting w/ \( \neg B \) & leading to \( \neg A \)

(which contradicts \( A \rightarrow \neg B \))
PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving if A then B for now

\[(a+b)(a-b) \iff a^2 - ab + ba - b^2 \iff a^2 - b^2\]

You can prove something directly (in one direction)
or work in both directions

instead of starting w/ \(\neg B\) & leading to \(\neg A\)
(which contradicts \(A \rightarrow \neg B\))

assume both A and \(\neg B\) are true
& arrive at some contradicting statement
Proof by Contradiction

If $x$ is even then $x$ is not odd
PROOF BY CONTRADICTION

If $x$ is even, then $x$ is not odd.

A

B
Proof by Contradiction

If $x$ is even then $x$ is not odd

$A$  $B$

Assume $A \land \neg B$: ...

and
PROOF BY CONTRADICTION

If $x$ is even, then $x$ is not odd

\[ A \quad B \]

Assume $A \land \neg B$: $x$ is even & $x$ is odd
PROOF BY CONTRADICTION

If $x$ is even then $x$ is not odd

$A$ $B$

Assume $A \land \neg B$: $x$ is even & $x$ is odd

? ?
If \( x \) is even, then \( x \) is not odd

\[ A \quad \text{and} \quad B \]

Assume \( A \land \neg B \): \( x \) is even & \( x \) is odd

\( (a: \text{int.}) \quad x = 2a \)

\( (b: \text{int.}) \quad x = 2b + 1 \)
PROOF BY CONTRADICTION

If $x$ is even, then $x$ is not odd

$A$ \hspace{2cm} $B$

Assume $A \land \neg B$: $x$ is even $\land$ $x$ is odd

$(a: \text{int.}) \quad x = 2a \quad \quad x = 2b+1 \quad (b: \text{int.})$

$2a = 2b+1$
**Proof by Contradiction**

If $x$ is even, then $x$ is not odd

A

B

Assume $A \land \neg B$: $x$ is even & $x$ is odd

$(a: \text{int.})$

$x = 2a$

$x = 2b + 1$  $(b: \text{int.})$

$2a = 2b + 1$

$a = b + \frac{1}{2}$
PROOF BY CONTRADICTION

If $x$ is even, then $x$ is not odd

A \hspace{1cm} B

Assume $A \land \neg B$: $x$ is even \& $x$ is odd

(a: int.) \hspace{1cm} x = 2a \hspace{1cm} x = 2b + 1 \hspace{1cm} (b: int.)

\[ 2a = 2b + 1 \]

\[ a = b + \frac{1}{2} \]

impossible/absurd/contradiction □
PROOF BY CONTRADICTION

If \( x \) is even, then \( x \) is not odd

\[ \text{A} \quad \text{B} \]

Assume \( A \land \neg B \): \( x \) is even \& \( x \) is odd

\[ a : \text{int.} \quad x = 2a \quad x = 2b + 1 \quad (b : \text{int.}) \]

\[ 2a = 2b + 1 \]

\[ a = b + \frac{1}{2} \]

Notice we met halfway at an incorrect statement.

impossible/absurd/contradiction \( \Box \)
Proof by Contradiction

If $x$ is even then $x$ is not odd

\[ A \quad B \]

Assume $A \land \neg B$: $x$ is even $\land$ $x$ is odd

\[ x = 2a \quad x = 2b+1 \]

(a: int.) \quad (b: int.)

Notice we met halfway at an incorrect statement.

impossible / absurd / contradiction $\square$

Could also plug $b + \frac{1}{2}$ into $x = 2a$ & conclude $x$ is odd.
PROOF BY CONTRADICTION

For integers $a \neq 0$ & $b$, there is only one number $x$ such that $ax + b = 0$.

State this in IF-THEN form.
For integers $a \neq 0$ & $b$, there is only one number s.t. $ax + b = 0$.

(if $ax + b = 0$ then for $y \neq x$, $ay + b \neq 0$)
Proof by Contradiction

For integers $a \neq 0$ & $b$, there is only one number $s.t.$ $ax + b = 0$.

(If $ax + b = 0$ then for $y \neq x$, $ay + b \neq 0$)

$A$ $B$
PROOF BY CONTRADICTION

For integers $a \neq 0$ & $b$, there is only one number s.t. $ax+b=0$.

$(\text{if } ax+b=0 \text{ then for } y \neq x, ay+b \neq 0)$

Assume $A \land \neg B$: $ax+b=0$ & $ay+b=0$
For integers $a \neq 0$ & $b$, there is only one number s.t. $ax+b=0$.

(If $ax+b=0$ then for $y \neq x$, $ay+b \neq 0$)

Assume $A \land \neg B$:

$ax+b=0$ & $ay+b=0$

$ax+b = ay+b$
For integers $a \neq 0$ & $b$, there is only one number s.t. $ax + b = 0$.

(If $ax + b = 0$ then for $y \neq x$, $ay + b \neq 0$)

Assume $A \land \neg B$:

$ax + b = 0$ & $ay + b = 0$

$ax + b = ay + b$

$ax = ay$
Proof by Contradiction

For integers $a \neq 0$ & $b$, there is only one number s.t. $ax + b = 0$. 

*(if $ax + b = 0$ then for $y \neq x$, $ay + b \neq 0$)*

$A \quad B$

Assume $A \land \neg B$:

$ax + b = 0 \quad \& \quad ay + b = 0$

$ax + b = ay + b$

$ax = ay$

$x = y$
PROOF BY CONTRADICTION

For integers $a \neq 0$ & $b$, there is only one number s.t. $az + b = 0$.

(If $ax + b = 0$ then for $y \neq x$, $ay + b \neq 0$)

Assume $A \land \neg B$:

$ax + b = 0$ & $ay + b = 0$

$ax + b = ay + b$

$ax = ay$

$x = y$ —— contradicts

$\square$
Prove: if $A$ then $B$

Assume $A \land \neg B$, get contradiction. ✓
Prove: if $A$ then $B$

Assume $A \land \neg B$, get contradiction. $\checkmark$

Does it work if we assume $\neg A \land B$ and get a contradiction?
Prove: if $A$ then $B$

Assume $A \land \neg B$, get contradiction. ✓

Does it work if we assume $\neg A \land B$ and get a contradiction?

NO

$A \rightarrow B$ tells us nothing about what happens when $\neg A$. 
Prove: if A then B

Assume $A \land \neg B$, get contradiction. √

Does it work if we assume $\neg A \land B$ and get a contradiction?

NO

$A \rightarrow B$ tells us nothing about what happens when $\neg A$.

It would work if we were proving $A \leftrightarrow B$. 
Let's prove something not in IF-THEN format
$\sqrt{2}$ is irrational - proof by contradiction
$\sqrt{2}$ is IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?
\[ \sqrt{2} \text{ is irrational} \] - Proof by Contradiction

1) What does the claim mean?

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
\[ \sqrt{2} \text{ is irrational} \] - Proof by contradiction

1) what does the claim mean?
2) assume the contrary is true

1) \( \exists \text{ integers } \{a,b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \)
2) \( \exists \text{ integers } \{a,b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \)

\[ \Rightarrow \sqrt{2} \text{ is rational} \]
\[
\sqrt{2} \text{ is irrational} - \text{ Proof by contradiction}
\]

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)

2) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
\[ \sqrt{2} \text{ is irrational} - \text{ proof by contradiction} \]

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)

2) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)

3) if (2) is true, then choose \( \{a, b\} \) with no common divisor

(simplify)
$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong

1) $\exists$ integers \(\{a, b\}\) s.t. $\sqrt{2} = \frac{a}{b}$
2) $\exists$ integers \(\{a, b\}\) s.t. $\sqrt{2} = \frac{a}{b}$
3) if (2) is true, then choose \(\{a, b\}\) w/ no common divisor

By (2), $2 = \frac{a^2}{b^2}$
\[ \sqrt{2} \text{ is IRRATIONAL - PROOF BY CONTRADICTION} \]

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
2) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
3) if (2) is true, then choose \( \{a, b\} \) w/ no common divisor

By (2), \( 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \)
$\sqrt{2}$ is Irrational - Proof by Contradiction

1) What does the claim mean?

2) Assume the contrary is true

3) Use this to establish something that you know is wrong

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)

2) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)

3) If (2) is true, then choose \( \{a, b\} \) with no common divisor

By (2), \( 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2: \text{even} \)
\[ \sqrt{2} \text{ is irrational} \text{ - Proof by contradiction} \]

1) What does the claim mean?
2) Assume the contrary is true
3) Use this to establish something that you know is wrong

1) \( \exists \text{ integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \)
2) \( \exists \text{ integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \)
3) If (2) is true, then choose \( \{a, b\} \) with no common divisor.

By (2), \( 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow \frac{a^2}{2} \) (a: even)

\[ \Rightarrow \text{ why?} \]
\[ \sqrt{2} \text{ is IRRATIONAL - PROOF BY CONTRADICTION} \]

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
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3) if (2) is true, then choose \( \{a, b\} \) w/ no common divisor
   By (2), \( 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2 : \text{even} \)
   \( (2x+1) \cdot (2x+1) = 4x^2 + 4x + 1 \)
   \( = 2 \cdot (2x^2 + 2x) + 1 \)
   \( a : \text{odd} \rightarrow a^2 : \text{odd} \)
\[ \sqrt{2} \text{ is IRRATIONAL - PROOF BY CONTRADICTION} \]

1) what does the claim mean?

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3) use this to establish something that you know is wrong

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By (2), \( 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2 : \text{even} \) with \( (a: \text{even}) \)

\[ a = 2c \text{ w/ } c: \text{int.} \]
\[ \sqrt{2} \text{ is IRRATIONAL} \quad - \quad \text{Proof by Contradiction} \]

1) What does the claim mean?

2) Assume the contrary is true

3) Use this to establish something that you know is wrong

\begin{align*}
1) \quad & \exists \text{ integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \\
2) \quad & \exists \text{ integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \\
3) \quad & \text{if (2) is true, then choose } \{a, b\} \text{ with no common divisor} \\
& \text{By (2), } 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2 : \text{ even} \\
& \quad (a : \text{ even}) \\
& \Rightarrow a = 2c \land \{c: \text{int.}\} \Rightarrow 2b^2 = 4c^2
\end{align*}
\[ \sqrt{2} \text{ is IRRATIONAL} \] - Proof by Contradiction

1) What does the claim mean?
2) Assume the contrary is true
3) Use this to establish something that you know is wrong

\[ \begin{align*}
1) & \quad \text{Integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \\
2) & \quad \exists \text{ integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b} \\
3) & \quad \text{If (2) is true, then choose } \{a, b\} \text{ with no common divisor} \\
\text{By (2), } & \quad 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2: \text{even} \\
& \quad (a: \text{even}) \\
& \quad \Rightarrow a = 2c \quad \{c: \text{int.}\} \Rightarrow 2b^2 = 4c^2 \Rightarrow b: \text{even} \\
& \quad (b^2 = 2c^2 \Rightarrow b^2: \text{even})
\end{align*} \]
$\sqrt{2}$ is IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong

1) \exists \text{ integers } \{a, b\} \text{ s.t. } \sqrt{2} = \frac{a}{b}
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3) if (2) is true, then choose \{a, b\} w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2: \text{ even (}a: \text{ even)}$

\[ a = 2c \quad \exists c: \text{ int.} \Rightarrow 2b^2 = 4c^2 \Rightarrow b: \text{ even} \quad (b = 2d) \]

\[ \Rightarrow \sqrt{2} = \frac{a}{b} = \frac{2c}{2d} \]
$\sqrt{2}$ is IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong

1) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$
2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$
3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ even (a: even)

$\Rightarrow a = 2c \ \forall c \in \mathbb{N}$ \Rightarrow $2b^2 = 4c^2 \Rightarrow b$ even

$\Rightarrow \sqrt{2} = \frac{a}{b} = \frac{2c}{2d}$ contradiction
\[ \sqrt{2} \text{ is IRRATIONAL} \] - Proof by Contradiction

1) what does the claim mean?
2) assume the contrary is true
3) use this to establish something that you know is wrong
4) conclude that (2) is false thus the initial claim is true

1) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
2) \( \exists \) integers \( \{a, b\} \) s.t. \( \sqrt{2} = \frac{a}{b} \)
3) if (2) is true, then choose \( \{a, b\} \) with no common divisor

\[ \text{By (2), } 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2: \text{even} \]
\[ \Rightarrow a = 2c \; \text{\( \exists \) c : int} \] \Rightarrow \( 2b^2 = 4c^2 \Rightarrow b: \text{even} \)
\[ \therefore \sqrt{2} = \frac{a}{b} = \frac{2c}{2d} \] contradiction
THERE ARE AN INFINITE NUMBER OF PRIMES
(proof by contradiction)
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• Assume that #primes is finite: \( p_1, p_2, \ldots, p_n \)
There are an infinite number of primes
(proof by contradiction)

• Assume that \#primes is finite: \(p_1, p_2, \ldots, p_n\)

• Let \(t = 1 + \prod_{i=1}^{n} p_i\) (i.e., \(1 + p_1 \times p_2 \times \ldots \times p_n\))
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• Notice \( t > p_i \) for all \( i \).
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• If \( t \) is not prime then \( \exists \) prime factor \( q \neq t \) of \( t \)
THERE ARE AN INFINITE NUMBER OF PRIMES
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• Assume that \#primes is finite: \( p_1, p_2, \ldots, p_n \)

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• Notice \( t > p_i \) for all \( i \). So if \( t \) is prime, contradiction.

• If \( t \) is not prime then \( \exists \) prime factor \( q \neq t \) of \( t \)
  
  - if \( q \neq p_i \) (for all \( i \)) \( t \) is composite
THERE ARE AN INFINITE NUMBER OF PRIMES
(proof by contradiction)

• Assume that \#primes is finite: \( p_1, p_2, \ldots, p_n \)

• Let \( t = 1 + \prod_{i=1}^{n} p_i \) (i.e., \( 1 + p_1 \times p_2 \times \ldots \times p_n \))

• Notice \( t > p_i \) for all \( i \). So if \( t \) is prime, contradiction.

• If \( t \) is not prime then \( \exists \) prime factor \( q \neq t \) of \( t \)
  - if \( q \neq p_i \) (for all \( i \)), contradiction.

\( t \) is composite
There are an infinite number of primes (proof by contradiction)

- Assume that there are finitely many primes: \( p_1, p_2, \ldots, p_n \)
- Let \( t = 1 + \prod_{i=1}^{n} p_i \) (i.e., \( 1 + p_1 \times p_2 \times \cdots \times p_n \))
- Notice that \( t > p_i \) for all \( i \). So if \( t \) is prime, contradiction.
- If \( t \) is not prime then there exists a prime factor \( q \neq t \) of \( t \)
  - if \( q \neq p_i \) (for all \( i \)), contradiction.
  - if \( q = p_j \)
THERE ARE AN INFINITE NUMBER OF PRIMES
(proof by contradiction)

• Assume that \#primes is finite: \(p_1, p_2, \ldots, p_n\)

• Let \(t = 1 + \prod_{i=1}^{n} p_i\) (i.e., \(1 + p_1 \times p_2 \times \cdots \times p_n\))

• Notice \(t > p_i\) for all \(i\). So if \(t\) is prime, contradiction.

• If \(t\) is not prime then \(\exists\) prime factor \(q \neq t\) of \(t\)
  - if \(q \neq p_i\) (for all \(i\)), contradiction.
  - if \(q = p_j\), we know \(q\) divides \(\prod_{i=1}^{n} p_i\)
THERE ARE AN INFINITE NUMBER OF PRIMES
(proof by contradiction)

• Assume that \#primes is finite: \( p_1, p_2, \ldots, p_n \)

• Let \( t = 1 + \prod_{i=1}^{n} p_i \) (i.e., \( 1 + p_1 \times p_2 \times \cdots \times p_n \))

• Notice \( t > p_i \) for all \( i \). So if \( t \) is prime, contradiction.

• If \( t \) is not prime then \( \exists \) prime factor \( q \neq t \) of \( t \)
  - If \( q \neq p_i \) (for all \( i \)), contradiction.
  - If \( q = p_j \), we know \( q \) divides \( \prod_{i=1}^{n} p_i \)
    but then ?
THERE ARE AN INFINITE NUMBER OF PRIMES
(proof by contradiction)

• Assume that \#primes is finite: \( p_1, p_2, \ldots, p_n \)

• Let \( t = 1 + \prod_{i=1}^{n} p_i \) (i.e., \( 1 + p_1 \times p_2 \times \cdots \times p_n \))

• Notice \( t > p_i \) for all \( i \). So if \( t \) is prime, contradiction.

• If \( t \) is not prime then \( \exists \) prime factor \( q \neq t \) of \( t \)
  - if \( q \neq p_i \) (for all \( i \)), contradiction.
  - if \( q = p_j \), we know \( q \) divides \( \prod_{i=1}^{n} p_i \)
    but then it can't also divide \( 1 + \prod_{i=1}^{n} p_i \) (contr.)
Next:

A variant of proof by contradiction
Prove that the first $n$ odd natural numbers sum to $n^2$. 
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + ? \quad ? \]
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]
Prove that the first $n$ odd natural numbers sum to $n^2$.

$$i = 1 \quad 2 \quad 3 \quad 4 \ldots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: 1 4 9 16 ... 
so far so good
Prove that the first $n$ odd natural numbers sum to $n^2$.

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: $1 \quad 4 \quad 9 \quad 16 \quad \cdots$

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$. 
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[
\begin{align*}
\sum_{i=1}^n & 1 \quad 3 \quad 5 \quad 7 \quad \cdots \quad (2n-3) \quad (2n-1) \\
1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) & = n^2
\end{align*}
\]

Sum: $1 \quad 4 \quad 9 \quad 16 \quad \ldots$

Suppose not. Then $\sum_{i=1}^n 2i-1 \neq n^2$.

We saw the claim is true for small $n$. 
Prove that the first $n$ odd natural numbers sum to $n^2$.

\[ i = 1 \ 2 \ 3 \ 4 \ \cdots \ \ (n-1) \ n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

Sum: 1 4 9 16 ...

Suppose not. Then \[ \sum_{i=1}^{n} 2i-1 \neq n^2 \].

We saw the claim is true for small $n$.

If the claim is false, there must be some smallest number $x \ (\leq n)$ for which the claim is false.
Prove that the first \( n \) odd natural numbers sum to \( n^2 \).

\[
i = 1 \ 2 \ 3 \ 4 \ \cdots \ (n-1) \ n
1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2
\]

Sum: \( 1 \ 4 \ 9 \ 16 \ \cdots \)

Suppose not. Then \( \sum_{i=1}^{n} 2i-1 \neq n^2 \).

We saw the claim is true for small \( n \).

If the claim is false, there must be some smallest number \( x \) (\( \leq n \)) for which \( \sum_{i=1}^{x} 2i-1 \neq x^2 \).
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \ldots + (2n-3) + (2n-1) = n^2 \]
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true → in fact for all \( x \).
\( \sum_{i=1}^{n} 2i - 2 = n^2 \)

If false, then there exists \( x \) for which it is false & there exists \( x \) for which it is true.

\( \sum_{i=x-1}^{2x-3} \)
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false \& \( x-1 \) for which it is true \( \Rightarrow \) in fact for all \( < x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) \]

\[ = (x-1)^2 \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]
\[ 1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true → in fact for all \( x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) = (x-1)^2 \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x-1 \neq x^2 \]
\[
\begin{align*}
\text{i = 1, 2, 3, 4, \ldots, (n-1), n} \\
1 + 3 + 5 + \cdots + (2n-3) + (2n-1) &= n^2 \\
\text{if false, then } \exists x \text{ for which it is false & } x-1 \text{ for which it is true} \\
\text{in fact for all } x \\
1 + 3 + 5 + \cdots + (2x-3) + (2x-1) &= (x-1)^2 \\
1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \\
(x-1)^2 + 2x-1 &\neq x^2 \\
x^2 - 2x + 1 + 2x-1 &\neq x^2
\end{align*}
\]
\[ i = 1 \ 2 \ 3 \ 4 \ \cdots \ (n-1) \ n \]
\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true

\[ 1 + 3 + 5 + \cdots + (2x-3) \]
\[ = (x-1)^2 \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x - 1 \neq x^2 \]

\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]

\[ 1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2 \quad ? \]

If false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true. 
\[ \Rightarrow \text{in fact for all } x \]

\[ 1 + 3 + 5 + \cdots + (2x-3) \]
\[ \quad i = x-1 \]

\[ \quad \quad \quad i = x \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x-1 \neq x^2 \]

\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]

\text{contradiction}
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]

\[ 1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false & \( x-1 \) for which it is true in fact for all \( x \)

\[ 1 + 3 + 5 + \cdots + (2x-3) \]

\[ 1 + 3 + 5 + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x-1 \neq x^2 \]

\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]

\[ \text{contradiction} \]
\[ i = 1, 2, 3, 4, \ldots, (n-1), n \]

\[ 1 + 3 + S + 7 + \cdots + (2n-3) + (2n-1) = n^2 \]

If false, then \( \exists x \) for which it is false \& \( x - 1 \) for which it is true \( \Rightarrow \) in fact for all \( x \)

\[ 1 + 3 + S + \cdots + (2x-3) \]

\[ 1 + 3 + S + \cdots + (2x-3) + (2x-1) \neq x^2 \]

\[ (x-1)^2 + 2x-1 \neq x^2 \]

\[ x^2 - 2x + 1 + 2x - 1 \neq x^2 \]

\[ \therefore \text{contradiction} \]
\[ i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n \]
\[ 1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2 ? \]

If false, then \( \exists x \) for which it is false & \( x \leq x \) for which it is true

\[
\begin{align*}
1 + 3 + S + \cdots + (2x-3) & = (x-1)^2 \\
1 + 3 + S + \cdots + (2x-3) + (2x-1) & \neq x^2
\end{align*}
\]

\[
(x-1)^2 + 2x-1 \neq x^2
\]

\[
x^2 - 2x + 1 + 2x - 1 \neq x^2
\]

Contradiction

\[ \leftarrow \text{either this should have been } \neq \]
\[ \leftarrow \text{or this should have been } = \]

\[ \ldots \text{which contradicts the smallest counterexample assumption, i.e.,} \]

\[ \text{THERE IS NO (SMALLEST) COUNTEREXAMPLE} \]

\[ \leftarrow \text{CLAIM IS TRUE} \]
SMALLEST COUNTEREXAMPLE
SMALLEST COUNTEREXAMPLE

- be able to "count" & "order" instances of the claim

(case/example)
SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim
- prove the claim for smallest instance (case/example)
  (& prove a smallest instance exists)
SMALLEST COUNTEREXAMPLE

- Be able to "count" & "order" instances of the claim
- Prove the claim for smallest instance (case/example)
- Assume the claim is false
SMALLEST COUNTEREXAMPLE recap

• be able to “count” & “order” instances of the claim

• prove the claim for smallest instance (case/example)

• assume the claim is false: then there is a smallest instance, $E_i$, for which it is false (smallest counterexample)
SMALLEST COUNTEREXAMPLE recap

• be able to "count" & "order" instances of the claim

• prove the claim for smallest instance (case/example)

• assume the claim is false: then there is a smallest instance, \( E_i \), for which it is false (smallest counterexample)

• this implies the claim is true for the next smallest instance, \( E_{i-1} \).
- be able to “count” & “order” instances of the claim
- prove the claim for smallest instance (case/example)
- assume the claim is false: then there is a smallest instance, \( E_i \), for which it is false (smallest counterexample)
- this implies the claim is true for the next smallest instance, \( E_{i-1} \).
- use \( E_i \) & \( E_{i-1} \) to get a contradiction (to the existence of any counterexample)
Claim: For \( n \in \mathbb{Z}, \ n \geq 5, \ 2^n > n^2 \)
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

Notice:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^n$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
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<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

- Use smallest counterexample

$n^2$: $0, 1, 4, 9, 16, 25$

$2^n$: $1, 2, 4, 8, 16, 32$

$\Rightarrow$ which is ...?
Claim: For \( n \in \mathbb{Z}, \ n \geq 5, \ 2^n > n^2 \)

- use smallest counterexample

\[
\begin{array}{c|c|c|c|c|c|c}
 n & 0 & 1 & 2 & 3 & 4 & 5 \\
 \hline
 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\
 2n^2 & 0 & 1 & 4 & 9 & 16 & 25 \\
\end{array}
\]

\[\Rightarrow\] which is some unknown hypothetical \( x \).
Claim: For \( n \in \mathbb{Z}, \ n > 5, \ 2^n > n^2 \) 

- use smallest counterexample

\( n = 2, 3, 4 \) are not counterexamples

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
n & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
2^n & 1 & 2 & 4 & 8 & 16 & 32 \\
\hline
n^2 & 0 & 1 & 4 & 9 & 16 & 25 \\
\hline
\end{array}
\]
Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

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</tr>
<tr>
<td>$n^2$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

- Use smallest counterexample
  
  ($n=2,3,4$ are not counterexamples)

Why can we? \rightarrow Claim is true for smallest instance ($n=5$)
Claim: For $n \in \mathbb{Z}$, $n > 5$, $2^n > n^2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^n$</th>
<th>$n^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>1</td>
<td>2</td>
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<td>3</td>
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<tr>
<td>4</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>25</td>
</tr>
</tbody>
</table>

Notice:

- **use smallest counterexample**
- $(n=2, 3, 4$ are not counterexamples)

Why can we? $\implies$ Claim is true for smallest instance ($n=5$)

- **assume \exists smallest counterexample $x$. $\implies ?$**
Claim: For $n \in \mathbb{Z}, n \geq 5$, $2^n > n^2$

Notice:

<table>
<thead>
<tr>
<th>$n$</th>
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<th>1</th>
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<tbody>
<tr>
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<td>1</td>
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<td>4</td>
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<td>25</td>
</tr>
</tbody>
</table>

- Use smallest counterexample
  
  ($n = 2, 3, 4$ are not counterexamples)

  Why can we? $\rightarrow$ Claim is true for smallest instance ($n = 5$)

- Assume $\exists$ smallest counterexample $x$. $\exists 2^x \leq x^2$ ($x > 5$)

  (What other condition?)
Claim: For \( n \in \mathbb{Z}, n > 5, 2^n > n^2 \) (notice \( \begin{array}{c} 2^n \\ \ 1 \ 2 \ 4 \ 8 \ 16 \ 32 \\ n^2 \ \ \ \ \ \ \ \ \ 0 \ 1 \ 4 \ \ \ 9 \ 16 \ 25 \end{array} \))

- use smallest counterexample
  \( n = 2, 3, 4 \) are not counterexamples

  \( \Rightarrow \) why can we? \( \rightarrow \) Claim is true for smallest instance \( (n = 5) \)

- assume \( \exists \) smallest counterexample \( x \). \( 2^x \leq x^2 \) \( (x > 5) \)
  (& for \( y > 5 \), if \( y < x \) then \( 2^y > y^2 \))
Claim: For \( n \in \mathbb{Z}, n > 5 \), \( 2^n > n^2 \)

- Use smallest counterexample
  
  \( n = 2, 3, 4 \) are not counterexamples

- Why can we? → Claim is true for smallest instance (\( n = 5 \))

- Assume \( \exists \) smallest counterexample \( x \).
  
  \[ 2^x \leq x^2 \quad (x > 5) \]

  (\& for \( y > 5 \), if \( y < x \) then \( 2^y > y^2 \))

- Focus on \( x - 1 \)
Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

Notice:

\[
\begin{align*}
&n & 0 & 1 & 2 & 3 & 4 & 5 \\
2^n & 1 & 2 & 4 & 8 & 16 & 32 \\
n^2 & 0 & 1 & 4 & 9 & 16 & 25
\end{align*}
\]

- Use smallest counterexample
  \((n=2, 3, 4\) are not counterexamples)

- Why can we? → Claim is true for smallest instance \((n=5)\)

- Assume \(\exists\) smallest counterexample \(x\).
  \[2^x \leq x^2\] \((x \geq 5)\)
  \((\&\) for \(y \geq 5\), if \(y < x\) then \(2^y > y^2)\)

- Focus on \(x-1\):
  \[2^{x-1} > (x-1)^2\]
Claim: For \( n \in \mathbb{Z}, \ n > 5, \ 2^n > n^2 \) (notice \( \begin{array}{c|c|c|c|c|c|c} n & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ \hline n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \) )

- Use smallest counterexample

\( (n=2, 3, 4 \text{ are not counterexamples}) \)

\rightarrow \text{why can we?} \rightarrow \text{Claim is true for smallest instance (n=5)}

- Assume \( \exists \) smallest counterexample \( x \).

\( 2^x \leq x^2 \) \( (x > 5) \)

\( (\& \text{ for } y > 5, \text{ if } y < x \text{ then } 2^y > y^2) \)

- Focus on \( x-1 : \)

\[ 2^{x-1} > (x-1)^2 \]

Combine to get contradiction
$2^x \leq x^2$

because $x$ is a counterexample
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because ?
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

next?
\[ \begin{align*} 2^x & \leq x^2 \\
\text{because } x \text{ is a counterexample} \end{align*} \]

\[ \begin{align*} 2^{x-1} & > (x-1)^2 \\
\text{because } x \text{ is the smallest counterexample} \text{ and not the smallest case} \\
2^{x-1} & > x^2 - 2x + 1 \]
Because $x$ is a counterexample, the inequality $2^{x-1} > (x-1)^2$ holds for all $x > 0$. This is because $x$ is the smallest counterexample and not the smallest case. Therefore, we have:

\[
2^{x-1} > x^2 - 2x + 1
\]

And

\[
2^{x-1} \cdot 2 > 2x^2 - 4x + 2
\]
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$

if $x^2 - 4x + 2 \geq 0$ then?
$2^x \leq x^2$

because $x$ is a counterexample

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$

if $x^2 - 4x + 2 > 0$ we will get a contradiction
because $x$ is a counterexample

$2^x \leq x^2$

$2^{x-1} > (x-1)^2$

because $x$ is the smallest counterexample and not the smallest case

$2^{x-1} > x^2 - 2x + 1$

$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$

$2^x > 2x^2 - 4x + 2$

$2^x > x^2 + (x^2 - 4x + 2)$

if $x^2 - 4x + 2 > 0$ we will get a contradiction

$(x-2)(x-2) > 2$

true for $x > 4$
Because \( x \) is a counterexample and not the smallest case, we have:

\[
2^{x-1} > (x-1)^2
\]

because \( x \) is the smallest counterexample and not the smallest case.

\[
2^{x-1} > x^2 - 2x + 1
\]

\[
2^{x-1} \cdot 2 > 2x^2 - 4x + 2
\]

\[
2^x > 2x^2 - 4x + 2
\]

\[
2^x > x^2 + (x^2 - 4x + 2)
\]

If \( x^2 - 4x + 2 \geq 0 \), we will get a contradiction:

\[
(x-2)(x-2) \geq 2
\]

true for \( x \geq 4 \)

We have assumed \( x \geq 5 \)

**Conclusion**

For \( n \in \mathbb{Z}, n \geq 5 \), \( 2^n > n^2 \)
FIBONACCI NUMBERS
FIBONACCI NUMBERS

$F_0 = 1 \quad F_1 = 1$
Fibonacci Numbers

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1 \quad F_1 = 1$
Fibonacci Numbers

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1$

$F_1 = 1$

$F_2 = 2$
FIBONACCI NUMBERS

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)
FIBONACCI NUMBERS

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1$
$F_1 = 1$

$F_2 = 2$
$F_3 = 3$
$F_4 = 5$
FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1$

$F_1 = 1$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$
FIBONACCI NUMBERS

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1$
$F_1 = 1$

$F_2 = 2$
$F_3 = 3$
$F_4 = 5$
$F_5 = 8$
$F_6 = 13$

etc
Fibonacci Numbers

\[ F_0 = 1 \quad F_1 = 1 \]

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, \ n \geq 0 \), \( F_n \leq 1.7^n \)
**Fibonacci Numbers**

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

Suppose smallest counterexample is $n = x$.

$F_x > 1.7^x$
FIBONACCI NUMBERS

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geq 0$, $F_n \leq 1.7^n$

Suppose smallest counterexample is $n = x$

$F_X > 1.7^x$

We want a contradiction, so most likely this will involve $F_{x-1}$
Fibonacci Numbers

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, n > 0 \), \( F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n = x \)

\( F_x > 1.7^x \)

We want a contradiction, so most likely this will involve \( F_{x-1} \)

Slight hiccup?
FIBONACCI NUMBERS

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n > 0$, $F_n \leq 1.7^n$

Suppose smallest counterexample is $n = x$

$\Rightarrow F_x > 1.7^x$

We want a contradiction, so most likely this will involve $F_{x-1}$

It will be hard to use only $F_x$ & $F_{x-1}$
Fibonacci Numbers

For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n > 0$, $F_n \leq 1.7^n$

Suppose smallest counterexample is $n = x$

$\Rightarrow F_x > 1.7^x$

We want a contradiction, so most likely this will involve $F_{x-1}$

It will be hard to use only $F_x$ & $F_{x-1}$

So why not use $F_{x-2}$ also?

(Why not $F_{x+1}$?)
FIBONACCI NUMBERS

for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z}, n \geq 0 \), \( F_n \leq 1.7^n \)

suppose smallest counterexample is \( n = x \)

\[ F_x > 1.7^x \]

we want a contradiction, so most likely this will involve \( F_{x-1} \)

it will be hard to use only \( F_x \) & \( F_{x-1} \), so why not use \( F_{x-2} \) also: assume \( x \geq 2 \)
**Fibonacci Numbers**

\[F_0 = 1\]
\[F_1 = 1\]
\[F_2 = 2\]
\[F_3 = 3\]
\[F_4 = 5\]
\[F_5 = 8\]
\[F_6 = 13\]

\[\text{for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2}\]

Claim: for \(n \in \mathbb{Z}, n > 0, \quad F_n \leq 1.7^n\)

Suppose smallest counterexample is \(n = x\)

\[\Rightarrow F_x > 1.7^x\]

We want a contradiction, so most likely this will involve \(F_{x-1}\)

It will be hard to use only \(F_x\) & \(F_{x-1}\), so why not use \(F_{x-2}\) also? assume \(x \geq 2\)

\[\Rightarrow \text{is } F_0 \leq 1.7^0?\]
Fibonacci Numbers

For \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)

Claim: for \( n \in \mathbb{Z} \), \( n \geq 0 \), \( F_n \leq 1.7^n \)

Suppose smallest counterexample is \( n=x \)

\( \iff F_x > 1.7^x \)

We want a contradiction, so most likely this will involve \( F_{x-1} \)

It will be hard to use only \( F_x \) & \( F_{x-1} \), so why not use \( F_{x-2} \) also: assume \( x \geq 2 \)

\( \iff is F_0 \leq 1.7^0 ? \text{ yes. Is } F_1 \leq 1.7^1 ? \)
FIBONACCI NUMBERS

\[ F_0 = 1 \]
\[ F_1 = 1 \]
\[ F_2 = 2 \]
\[ F_3 = 3 \]
\[ F_4 = 5 \]
\[ F_5 = 8 \]
\[ F_6 = 13 \]

\[ \text{for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

\[ \text{Claim: for } n \in \mathbb{Z}, \quad n \geq 0, \quad F_n \leq 1.7^n \]

Suppose smallest counterexample is \( n = x \)

\[ F_x > 1.7^x \]

We want a contradiction, so most likely this will involve \( F_{x-1} \)

It will be hard to use only \( F_x \) & \( F_{x-1} \), so why not use \( F_{x-2} \) also: assume \( x \geq 2 \)

\[ \text{is } F_0 \leq 1.7^0? \quad \text{yes. Is } F_1 \leq 1.7^1? \quad \text{yes. OK!} \]
\[ F_0 = F_1 = 1 \quad \text{// for } n > 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n > 0, \quad F_n \leq 1.7^n \]
F_0 = F_1 = 1  // for n \geq 2, \quad F_n = F_{n-1} + F_{n-2}

Claim: for n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n

smallest counterexample: \quad F_x > 1.7^x  \quad & \quad we \ can \ safely \ assume \\
(x \geq 2)  \quad F_y \leq 1.7^y  \quad for \ y < x
$F_0 = F_1 = 1 \quad \forall \ n \geq 2, \ F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}, n \geq 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x \quad \& \text{we can safely assume} \quad F_y \leq 1.7^y \text{ for } y < x$

next?
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) & we can safely assume
\((x \geq 2)\)

\( F_y \leq 1.7^y \) for \( y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \quad \ldots \)
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \; n > 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) \& we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

\((x \geq 2)\)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)
\[
F_0 = F_1 = 1 \quad \forall \quad \text{for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2}
\]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n

smallest counterexample: \( F_x > 1.7^x \) \quad \& \quad \text{we can safely assume} \quad F_y \leq 1.7^y \quad \text{for} \ y < x

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} = 1.7^{x-2} \cdot (1.7 + 1) \)
\[ F_0 = F_1 = 1 \]  \hspace{1em} // \hspace{1em} \text{for } n \geq 2, \hspace{0.5em} F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \ F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) \hspace{1em} (\( x \geq 2 \)) \hspace{1em} \& \hspace{1em} \text{we can safely assume} \ F_y \leq 1.7^y \hspace{1em} \text{for } y < x 

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)

\[ = 1.7^{x-2} \cdot (1.7 + 1) \]

\[ = 1.7^{x-2} \cdot 2.7 \]
$F_0 = F_1 = 1$ \hspace{1cm} // \hspace{1cm} \text{for } n \gg 2, \quad F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}, \ n > 0$, $F_n \leq 1.7^n$

smallest counterexample: $F_x > 1.7^x$ \hspace{1cm} \& we can safely assume $F_y \leq 1.7^y$ for $y < x$

$(x \gg 2)$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$\leq 1.7^{x-2}(1.7 + 1)$

$= 1.7^{x-2} \cdot 2.7$

$< 1.7^{x-2}(1.7)^2$ \hspace{1cm} [1.7^2 = 2.89]
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n > 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) \( (x \geq 2) \)

\& we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)
\[ = 1.7^{x-2} \cdot (1.7 + 1) \]
\[ = 1.7^{x-2} \cdot 2.7 \]
\[ < 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89] \]
\[ = 1.7^x \quad \text{so?} \]
\[ F_0 = F_1 = 1 \quad \text{// for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2} \]

Claim: for \( n \in \mathbb{Z}, \ n \geq 0, \quad F_n \leq 1.7^n \)

smallest counterexample: \( F_x > 1.7^x \) \( (x \geq 2) \)

& we can safely assume \( F_y \leq 1.7^y \) for \( y < x \)

we can now say: \( F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2} \)

\[ = 1.7^{x-2} \cdot (1.7 + 1) \]

\[ = 1.7^{x-2} \cdot 2.7 \]

\[ < 1.7^{x-2} \cdot (1.7)^2 \]

\[ = 1.7^x \]

so \( F_x < 1.7^x \)

CONTRACTION

\[ [1.7^2 = 2.89] \]
geometry time
6 points in **convex** position.

This is in 2D, aka "the plane".

x, y coordinates are real numbers, so our point set is in \( \mathbb{R}^2 \)
still 6 points in convex position.
Theorem: in $\mathbb{R}^2$, every set of $\geq 17$ points with no 3 on a line has 6 points in convex position.
Theorem: in $\mathbb{R}^2$, every set of $\geq 17$ points with no 3 on a line has 6 points in convex position.

"has a hexagon" (not necessarily regular)
Claim: In $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon...
Claim: in $\mathbb{R}^2$, given a set of points $P$ w/ no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.
Claim: in $\mathbb{R}^2$, given a set of points $P$ with no 3 on a line, if $P$ has 6 points forming a hexagon then $P$ has 5 points forming an empty pentagon.

Stronger claim:
Every hexagon contains an empty pentagon.
Claim: Every hexagon $H$ contains an empty pentagon

Trivial example:

- If $H$ is empty, DONE.
Claim: Every hexagon $H$ contains an empty pentagon

Trivial examples:

- if $H$ is empty, DONE.
- if $H$ contains exactly 1 point,
  "split" $H"
Claim: Every hexagon $H$ contains an empty pentagon

Trivial examples:

- If $H$ is empty, DONE.
- If $H$ contains exactly 1 point, "split" $H$ and then we are DONE.
Claim: Every hexagon $H$ contains an empty pentagon

Trivial examples:

- If $H$ is empty, DONE.
- If $H$ contains exactly 1 point, "split" $H$ and then we are DONE.

• We can order hexagons by #points inside.
Claim: Every hexagon $H$ contains an empty pentagon

Trivial examples:
- if $H$ is empty, DONE.
- if $H$ contains exactly 1 point, "split" $H$ and then we are DONE.

- We can order hexagons by #points inside.
- If claim is false there must be a smallest counterexample
Claim: Every hexagon $H$ contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon $H$ containing min #pts, for which claim is false.

- Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.
Claim: Every hexagon $H$ contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon $H$ containing min #pts, for which claim is false.

Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.

Hypothetical smallest counterexample
Claim: Every hexagon $H$ contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon $H$ containing min #pts, for which claim is false.
- Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.
- If any "extreme segment" of interior points "isolates" 3 points of $H$...
Claim: Every hexagon $H$ contains an empty pentagon

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- if any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.

This wasn't a counterexample, CONTRADICTION
Claim: Every hexagon \( H \) contains an empty pentagon

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- if any "extreme segment" of interior points "isolates" 3 points of \( H \), DONE.

$\Rightarrow$ so every such segment isolates 1 or 2 points.
Claim: Every hexagon $H$ contains an empty pentagon

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-Shown: if $H$ contains $\leq 1$ points, DONE $\rightarrow$ so assume $\geq 2$ pts inside.
-If any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.
  $\rightarrow$ so every such segment isolates 1 or 2 points.
  $\rightarrow$ use one segment
  $\&$ form a hexagon $H'$
  (why?)
Claim: Every hexagon $H$ contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon $H$ containing min #pts, for which claim is false.
- Shown: if $H$ contains $\leq 1$ points, DONE $\implies$ so assume $\geq 2$ pts inside.
- If any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.
  $\implies$ so every such segment isolates 1 or 2 points.

$\implies$ use one segment & form a hexagon $H'$ containing fewer points than $H$. (So?)
Claim: Every hexagon $H$ contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon $H$ containing minimum number of points, for which the claim is false.

- Shown: if $H$ contains $\leq 1$ points, DONE $\implies$ so assume $\geq 2$ pts inside.

- if any "extreme segment" of interior points "isolates" 3 points of $H$, DONE.

$\implies$ so every such segment isolates 1 or 2 points.

$\implies$ use one segment

& form a hexagon $H'$ containing fewer points than $H$.

If $H$ is smallest counterexample, claim is true for $H'$ AND $H$!
The “smallest counterexample” method is useful and elegant, and essentially the same as another extremely useful method: **Induction**
proof by **INDUCTION**

explained quickly

via conversion from smallest counterexample

To learn Induction from scratch, see other set of notes.
proof by INDUCTION

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)
proof by INDUCTION

like proof by smallest counterexample,
(1) prove your claim for a base case
(should be ~easy)

like proof by smallest counterexample,
(2) focus on two "neighboring" cases [call them n-1 & n]
proof by **INDUCTION**

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~ easy)

like proof by smallest counterexample,

(2) focus on two "neighboring" cases [call them n-1 & n]

"unlike" proof by smallest counterexample,

(3) show that if the claim is true for n-1 \( \Rightarrow \) then it is true for n \( A \rightarrow B \)
proof by INDUCTION

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)

like proof by smallest counterexample,

(2) focus on two “neighboring” cases [call them n-1 & n]

“unlike” proof by smallest counterexample, ... which proves (A ∧ ¬B) = F
which is the same

(3) show that if the claim is true for n-1 then it is true for n

\[ A \rightarrow B \]