A bet: I randomly select half of the class...
(I was targeting a group of around 70/2 people when I wrote this)

If any 2 people in that group have the same birthday you give me a dollar.

If no birthday match is found I give you a dollar
Another bet:

I randomly select 10 people born in the same month

Same deal as before

My chances of winning: >80%
One last bet?

I randomly select 7 people.

If any 2 people in that group have birthdays within a week of each other...

I win 60% of the time. (52% if within 6 days)
DISCRETE PROBABILITY

Roll a die ... Possible outcomes: \{1, 2, 3, 4, 5, 6\}

Each has a probability: \(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\)

Sample space \(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\)

Sum to 1

Roll 2 dice ... Sample space \(\{(a,1), (a,2), (a,3), (a,4), (a,5), (a,6), (b,1), (b,2), \ldots\}\)

(a,b)

\[\text{prob. } \Rightarrow \frac{1}{36}\]

\((6,1), (6,2), \ldots\) ... (6,6)
Roll 2 indistinguishable dice...

Sample space $\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$ $(2,2), (2,3), (2,4), (2,5), (2,6),$ $(3,3), (3,4), (3,5), (3,6),$ $(4,4), (4,5), (4,6),$ $(5,5), (5,6),$ $(6,6) \}$

- $\Pr((a,a)) = \frac{1}{36}$
- $\Pr((a,b)) = \frac{2}{36}$ if $a \neq b$

$6 \cdot \frac{1}{36} = \frac{6}{36} \quad \text{and} \quad 15 \cdot \frac{2}{36} = \frac{30}{36}$

$\frac{6}{36} + \frac{30}{36} = 1$
Roll a die

\[ P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6} \]

\[ P(\text{roll even}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{2} \]

Roll 2 dice...

\[ P(\text{sum} = 7) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) \]

\[ = P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) \]

\[ = 6 \cdot \frac{1}{36} = \frac{1}{6} \]

\[ P(\text{sum} = 7) = P(\{(1,6), (2,5), (3,4)\}) = 3 \cdot \frac{2}{36} = \frac{1}{6} \]
Roll 10 dice (or 1 die 10 times)

Sample space size: \(6^{10} > 60\) million

\[P(\text{observe no 1's}) \Rightarrow \left(\frac{5}{6}\right)^{10}\]

How many outcomes have no 1's? \(\Rightarrow 5^{10}\)

\(\text{Or, say that each roll/die is independent}\)

\(\text{so for each roll, } P(\text{no 1}) = \frac{5}{6} \Rightarrow \left(\frac{5}{6}\right)^{10}\)
Poker: 52 cards (4x13 types); select 5.

\[ P(4 \text{ of a kind}) = P(4 \text{ of 5 are of same type}) \]

\[ \text{e.g. } 3,3,3,3,7 \quad \text{or} \quad 8,8,8,J,8 \]

\[ \text{ans: } \frac{\# 4\text{-of-a-kinds}}{\# \text{possible outcomes}} \rightarrow \frac{13 \cdot 48}{(\binom{52}{5})} = \frac{1}{4165} \sim 0.00024 \]
**Probability Visualization**

Sample space $\rightarrow$ area = 1

$P(\text{outcome } i) = \text{area}(i)$

Example: roll one die $\rightarrow$ all areas equal $= \frac{1}{6}$

$P(\text{event}) = \text{sum of appropriate areas}$

$e.g. \quad P(\text{roll prime\# OR even\#}) = \left\{ \begin{array}{c} 2, 3, 5 \quad \text{P(prime)} = \frac{3}{6} \\ 2, 4, 6 \quad \text{P(even)} = \frac{3}{6} \end{array} \right\} \frac{5}{6}$

Avoid double counting

NOT $\frac{3}{6} + \frac{3}{6}$
\[
P(\text{roll prime number or even number}) \rightarrow P(A \cup B)
\]

- All probability space

- "OR"

- "AND"
\[ P(A) + P(B) = P(A \cup B) + P(A \cap B) \]
\[ P(\text{someone in class was born on Feb. 29}) \]
\[ = P(\text{student 1 born on Feb. 29}) \cup \ldots \cup (\text{student } k \text{ born on Feb. 29}) \]
\[ \quad \vdots \]
\[ \cup (\text{student } k \text{ born on Feb. 29}) \]
\[ \leq \text{ awful, but we could say it is } \sum P(i) \approx 80 \cdot 0.07\% \approx 5.6\% \]
\[ \approx \frac{1}{365 \cdot 4 + 1} \approx 0.07\% \]
\[ = 1 - P(\text{nobody in class was born on Feb. 29}) \]
\[ = 1 - P(\text{student 1 NOT born on Feb. 29}) \cap \ldots \cap (\text{student } k \text{ NOT born on Feb. 29}) \]
\[ = 1 - \alpha^k \]
\[ \alpha = P(\text{student } i \text{ NOT born on Feb. 29}) \]
\[ P(\text{someone in class was born on Feb. 29}) \quad \text{(suppose } k = 80 \text{ students}) \]

\[ = P(\text{student 1 born on Feb. 29}) \cup (\text{student 2 born on Feb. 29}) \quad \ldots \]

\[ \ldots \cup (\text{student 3 born on Feb. 29}) \quad \ldots \cup \quad \ldots \quad \cup (\text{student } k \text{ born on Feb. 29}) \]

\[ \leftarrow \text{awful but we could say it is } \sum P(i) \approx 80 \cdot 0.07\% \]

\[ \quad \text{(approximation)} \quad \approx 5.6\% \quad \text{assuming all days equally likely & 1 leap year every 4 years.} \]

\[ = 1 - P(\text{nobody in class was born on Feb. 29}) \]

\[ = 1 - P(\text{student 1 not born on Feb. 29}) \cap (\text{student 2 not born on Feb. 29}) \quad \ldots \quad \cap (\text{student } k \text{ not born on Feb. 29}) \]

\[ = 1 - \alpha^k = 1 - \left(\frac{365 \cdot 4}{365 \cdot 4 + 1}\right)^k \quad \text{exactly} \]

\[ 80 \text{ students } \approx 5\% \]
\[ P_i + \overline{P}_i = 1 \} \text{ area in circle } i \]
+ area outside circle \( i \) = 1

\[
\bigcup P_i = 1 - \bigcap \overline{P}_i
\]

inside any circle

outside every circle
P( > 2 people in a group of k have same birthday)\[\text{no Feb. 29 allowed}\]

\[P[ (1,2) U (1,3) U (1,4) \ldots U (1,k) U (2,3) U (2,4) \ldots U (2,k) \ldots \ldots U (k-1,k)]\]

awful

\[\frac{1}{365}\]

if \(k > 365\) use pigeonhole

\[= 1 - P(\text{all } k \text{ have distinct birthdays})\]

\[P(\text{2nd person has different bday than 1st}) = \frac{364}{365} = P(A)\]

\[P(3rd \ldots 1st \& 2nd) = \frac{363}{365} = P(B)\]

assuming 1st & 2nd differ

"conditional probability"

I’m abusing notation a bit, it should be \(P(B \mid A)\), meaning \(P(B \text{ given } A)\).
\[ P( \text{2 people in a group of } k \text{ have same birthday}) \quad \text{no Feb.29 allowed} \]

\[ \Pr[(1,2) \cup (1,3) \cup (1,4) \ldots \cup (1,k) \cup (2,3) \cup (2,4) \ldots \cup (2,k) \ldots \ldots \cup (k-1,k)] \approx \frac{1}{365} \]

\[ \text{awful} \]

\[ = 1 - \Pr(\text{all } k \text{ have distinct birthdays}) \]

\[ \Pr(\text{2nd person has different bday than 1st}) = \frac{364}{365} = \Pr(A) \]

\[ \Pr(\text{3rd} \ldots \ldots \ldots \text{1st} & \text{2nd}) = \frac{363}{365} = \Pr(B) \]

\[ \Pr(\text{4th} \ldots \ldots \ldots (1-3)) = \frac{362}{365} = \Pr(C) \]

\[ \text{etc} \]

\[ = 1 - \left[ \Pr(A) \cap \Pr(B) \cap \Pr(C) \ldots \right] = 1 - \frac{365!}{(365-k)!} \cdot \frac{1}{365^k} = 1 - \frac{365 \cdot 364 \cdot 363 \ldots (365-k+1)}{365^k} \]
\[ P(\text{2 or more people in a group of } k \text{ have same birthday}) \]

\[ = 1 - \frac{365!}{(365-k)!} \cdot \frac{1}{365^k} = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365-k+1)}{365^k} = 1 - \frac{(365)_k}{365^k} \]

\[ k = 2 \quad \rightarrow \quad P \sim 0.27\% \quad \left( \frac{1}{365} \right) \]

\[ k = 4 \quad \rightarrow \quad P \sim 1.64\% \]

\[ k = 23 \quad \rightarrow \quad P \sim 50.73\% \]

\[ k = 30 \quad \rightarrow \quad P \sim 70.6\% \]

\[ k = 70 \quad \rightarrow \quad P \sim 99.9\% \]

\[ k \sim 116 \quad \rightarrow \quad P \sim 1 - \frac{1}{10^9} \]

\[ k = 300 \quad \rightarrow \quad P \sim 1 - \frac{1}{10^{80}} \quad \left( 10^{80} \sim \# \text{atoms in universe} \right) \]

\[ (k > 365 \quad \rightarrow \quad P = 1) \]
\[ P(\geq 2 \text{ people in a group of } k \text{ have same birthday}) \]

\[ 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365-k+1)}{365^k} \]

For the bet involving \( k \) people born in a month w/ 30 days substitute \( 365 \rightarrow 30 \)

\[ (k=10) \quad 1 - \frac{30 \cdot 29 \cdot 28 \cdots \cdot 23 \cdot 22 \cdot 21}{30^{10}} \approx 0.815 \]