PLANE GRAPH

no crossings
PLANE GRAPH
no crossings

PLANAR GRAPH
can redraw without crossings
**Plane Graph**

*no crossings*

**Planar Graph**

*can redraw without crossings*

**Non-planar graphs**

*(can't redraw)*
**Plane Graph**

- No crossings.

**Planar Graph**

- Can redraw without crossings.

**Non-planar graphs**

- (Can't redraw)

- $K_{3,3}$
- $K_5$
PLANE GRAPH
no crossings

PLANAR GRAPH
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Non-planar graphs
(can't redraw)

$K_{3,3}$
$K_5$
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Non-planar graphs
(can't redraw)

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Non-planar graphs
(can't redraw)

\( K_{3,3} \)

\( K_5 \)
PLANE GRAPH
no crossings

PLANAR GRAPH
can redraw without crossings

Non-planar graphs (can't redraw)

$K_{3,3}$  $K_5$
Planes Graph: no crossings

Planar Graph: can redraw without crossings

Non-planar graphs:
- (can't redraw)
- $K_{3,3}$
- $K_5$
PLANE GRAPH
no crossings

PLANAR GRAPH
can redraw
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Non-planar graphs
(can't redraw)

$K_{3,3}$

$K_5$
**Plane Graph**

No crossings

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**Non-planar graphs**

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Non-planar graphs
(can't redraw)

$K_{3,3}$

$K_5$
**Plane Graph**

No crossings

**Planar Graph**

Can redraw without crossings

**Non-planar graphs**

(Can't redraw)

$K_{3,3}$

$K_5$
PLANE GRAPH
no crossings

PLANAR GRAPH
can redraw without crossings

Non-planar graphs
(can't redraw)

K_{3,3}  
K_{5}
**Plane Graph**
- no crossings

**Planar Graph**
- can redraw without crossings

**Non-planar graphs**
- (can't redraw)

- obtained by successive contractions

A graph is non-planar if and only if it "contains" a $K_{3,3}$ or $K_5$
FYI

Every planar graph can be drawn without crossings. In fact the edges can be drawn straight as well.
\[ G = (V, E) \]

\[ V = 6 \]
\[ E = 8 \]

\[ F = \# \text{faces} = 4 \]

Disjoint regions, one of which is unbounded
Euler Formula for planar connected graphs: $V - E + F = 2$
EULER FORMULA for planar connected graphs: $V - E + F = 2$

Proof by induction on number of faces:

Base case $\rightarrow F = 1 \rightarrow ?$
Euler formula for planar connected graphs: \( V - E + F = 2 \)

Proof by induction on number of faces:

Base case \( F = 1 \) \( \rightarrow \) \( G \) is a tree \( \rightarrow ? \)
EULER FORMULA for planar connected graphs: \[ V - E + F = 2 \]

Proof by induction on number of faces:

Base case \( F = 1 \) \( \rightarrow \) \( G \) is a tree \( \rightarrow \) \( V = E + 1 \)
Euler Formula for planar connected graphs: \( V - E + F = 2 \)

Proof by induction on number of faces:

Base case \( F = 1 \) → G is a tree → \( V = E + 1 \)

so \( (E+1) - E + 1 = 2 \) \( \checkmark \)
EULER FORMULA for planar connected graphs: \[ V - E + F = 2 \]

Proof by induction on number of faces:

Base case \( F = 1 \) \( \rightarrow \) \( G \) is a tree \( \rightarrow \) \( V = E + 1 \)

so \( (E + 1) - E + 1 = 2 \) \( \checkmark \)

Given \( G = (V, E) \) w/ \( F > 1 \) faces, remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).
EULER FORMULA for planar connected graphs: \( V - E + F = 2 \)

Proof by induction on number of faces:

Base case → \( F = 1 \) → \( G \) is a tree → \( V = E + 1 \)

so \( (E+1)-E+1 = 2 \) ✓

Given \( G=(V,E) \) w/ \( F > 1 \) faces,
remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).

Either \( f_1 \) or \( f_2 \) is a bounded face
EULER FORMULA for planar connected graphs: \( V - E + F = 2 \)

Proof by induction on number of faces:

Base case \( F = 1 \) → G is a tree → \( V = E + 1 \)

so \( (E+1) - E + 1 = 2 \) ✓

Given \( G = (V, E) \) w/ \( F > 1 \) faces,
remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).
Either \( f_1 \) or \( f_2 \) is a bounded face, so \( e \) is on a cycle (\( e \) is not a cut edge).
EULER FORMULA for planar connected graphs: \( V - E + F = 2 \)

Proof by induction on number of faces:

Base case \( F = 1 \) \( \rightarrow \) \( G \) is a tree \( \rightarrow \) \( V = E + 1 \)

so \( (E+1) - E + 1 = 2 \) \( \checkmark \)

Given \( G = (V,E) \) \( \omega \) \( F > 1 \) faces,
remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).

Either \( f_1 \) or \( f_2 \) is a bounded face,
so \( e \) is on a cycle (\( e \) is not a cut edge)
\( \downarrow \) \( G - e \) is connected \& \( f_1, f_2 \) merge
EULER FORMULA for planar connected graphs: \[ V - E + F = 2 \]

Proof by induction on number of faces:

Base case \( F = 1 \) \( \rightarrow \) \( G \) is a tree \( \rightarrow \) \( V = E + 1 \)

so \( (E + 1) - E + 1 = 2 \)

Given \( G = (V, E) \) w/ \( F > 1 \) faces,
remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).
Either \( f_1 \) or \( f_2 \) is a bounded face,
so \( e \) is on a cycle (\( e \) is not a cut edge)
\( \Rightarrow \) \( G - e \) is connected & \( f_1, f_2 \) merge:

\[
\begin{align*}
V - (E - 1) - (F - 1) & \geq 2 \\
G - e & \end{align*}
\]
EULER FORMULA for planar connected graphs: \[ V - E + F = 2 \]

Proof by induction on number of faces:

Base case \( F = 1 \) \( \Rightarrow \) \( G \) is a tree \( \Rightarrow V = E + 1 \)

so \( (E+1) - E + 1 = 2 \) \( \checkmark \)

Given \( G = (V, E) \) w/ \( F > 1 \) faces, remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).

Either \( f_1 \) or \( f_2 \) is a bounded face, so \( e \) is on a cycle (\( e \) is not a cut edge)

\( \xrightarrow{\text{hypothesis}} (V - (E - 1) + (F - 1) = 2) \)
EULER FORMULA for planar connected graphs: \( V - E + F = 2 \)

Proof by induction on number of faces:

Base case \( F = 1 \) → G is a tree → \( V = E + 1 \)

so \((E+1) - E + 1 = 2 \)

Given \( G = (V,E) \) w/ \( F > 1 \) faces, remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).
Either \( f_1 \) or \( f_2 \) is a bounded face,
so \( e \) is on a cycle (\( e \) is not a cut edge)
\( G - e \) is connected & \( f_1, f_2 \) merge:

Given \( G = (V,E) \) w/ \( F > 1 \) faces, remove an edge \( e \) between 2 faces, \( f_1 \) & \( f_2 \).
Either \( f_1 \) or \( f_2 \) is a bounded face,
so \( e \) is on a cycle (\( e \) is not a cut edge)
\( G - e \) is connected & \( f_1, f_2 \) merge:

\[ V - (E-1) + (F-1) = 2 \]
\[ \Rightarrow V - E + F = 2 \]

Note that this also holds for multigraphs.
Euler formula \( V - E + F = 2 \)

\( V - E + F = 2 \) applies to any connected planar graph (in fact, to convex polyhedra) by projection.

Induction on faces:

- \( F = 1 \) : tree. \( E = V - 1 \)

- \( F > 1 \):
  Remove an edge between 2 faces.
  Remains connected.

\( F \to F - 1 \quad E \to E - 1 \)
Euler formula: \( V - E + F = 2 \)

The formula applies to any connected planar graph (in fact, to convex polyhedra) by projection.

**Induction on faces:**
- \( F = 1 \):
  - Tree: \( F = 1 \), \( E = V - 1 \)
- \( F > 1 \):
  - Remove an edge between 2 faces.
  - Remains connected.
  - \( F \rightarrow F - 1 \), \( E \rightarrow E - 1 \)

**Induction on vertices:**
- \( V = 1 \):
  - Only loops: \( F = E + 1 \)
Euler formula \( V - E + F = 2 \)

Euler’s formula applies to any connected planar graph (in fact, to convex polyhedra) by projection.

**Induction on faces:**
- \( F = 1 \): tree, \( V = 2 \), \( E = V - 1 \)
- \( F > 1 \):
  - Remove an edge between 2 faces.
  - Remains connected.
  - \( F \rightarrow F - 1 \), \( E \rightarrow E - 1 \)

**Induction on vertices**
- \( V = 1 \): only loops, \( F = E + 1 \)
- \( V > 1 \):
  - Contract edge \( x \neq y \)
  - \( V \rightarrow V - 1 \), \( E \rightarrow E - 1 \)
Euler formula \( V - E + F = 2 \)

\[ \begin{align*}
V - E + F &= 2 \\
\text{applies to any connected planar graph (in fact, to convex polyhedra)} \\
\text{by projection}
\end{align*} \]

**Induction on faces:**
- \( F = 1 \) : tree
  - \( E = V - 1 \)
- \( F > 1 \):
  - Remove an edge between 2 faces.
  - Remains connected.
  - \( F \to F - 1 \), \( E \to E - 1 \)

**Induction on vertices:**
- \( V = 1 \) : only loops
  - \( F = E + 1 \)
- \( V > 1 \):
  - Contract edge \( x \neq y \)
  - \( V \to V - 1 \), \( E \to E - 1 \)

**Induction on edges:**
- \( E = 0 \) : one vertex, one face
Euler formula \( V - E + F = 2 \)

\( V - E + F = 2 \) applies to any connected planar graph (in fact, to convex polyhedra) by projection.

### Induction on faces:
- \( F = 1 \): tree, \( E = V - 1 \)
- \( F > 1 \):
  - Remove an edge between 2 faces.
  - Remains connected.
  - \( F \to F - 1 \), \( E \to E - 1 \)

### Induction on vertices:
- \( V = 1 \):
  - Only loops, \( F = E + 1 \)
- \( V > 1 \):
  - Contract edge \( x \neq y \)
  - \( V \to V - 1 \), \( E \to E - 1 \)

### Induction on edges:
- \( E = 0 \):
  - One vertex, one face
- \( E > 1 \):
  - If \( x \neq y \) contract as before
  - Else \( \bullet \) remove as before
  - \( E \to E - 1 \) & \( F \to F - 1 \)

\( V \to V - 1 \)
use the Euler formula $V - E + F = 2$

to show that a connected plane graph has $E \leq 3V - 6$ for $V \geq 3$

Not allowed: ☐
use the Euler formula \( V - E + F = 2 \)

to show that a connected plane graph has \( E \leq 3V - 6 \)

Every edge belongs to 1 or 2 faces \( \sum_e \leq 2E \) for all faces
use the Euler formula \( V - E + F = 2 \)

to show that a connected plane graph has \( E \leq 3V - 6 \)

Every edge belongs to 1 or 2 faces \( \sum_{\text{all faces}} e \leq 2E \)

Every face has \( \geq 3 \) edges (for \( V > 3 \)) \( \sum_{\text{all faces}} e \geq 3F \)
use the Euler formula \( V - E + F = 2 \) to show that a connected plane graph has \( E \leq 3V - 6 \)

Every edge belongs to 1 or 2 faces \[ \sum_{\text{all faces}} e \leq 2E \]

Every face has \( \geq 3 \) edges (for \( V > 3 \)) \[ \sum_{\text{all faces}} e \geq 3F \]
Use the Euler formula \( V - E + F = 2 \)

to show that a connected plane graph has \( E \leq 3V - 6 \)

Every edge belongs to 1 or 2 faces \[ \sum_{\text{all faces}} e \leq 2E \]

Every face has \( \geq 3 \) edges (for \( V > 3 \)) \[ \sum_{\text{all faces}} e \geq 3F \]

\[ E - F = V - 2 \]
use the Euler formula $V - E + F = 2$

to show that a connected plane graph has $E \leq 3V - 6$

Every edge belongs to 1 or 2 faces

Every face has $\geq 3$ edges (for $V \geq 3$)

$E - F = V - 2$

$E - \frac{2E}{3} \leq V - 2$

$E \leq 3V - 6$
use the Euler formula \[ V - E + F = 2 \]

to show that a connected plane graph has \[ E \leq 3V - 6 \]

Every edge belongs to 1 or 2 faces

Every face has \( \geq 3 \) edges (for \( V > 3 \))

\[
\begin{align*}
E - F &= V - 2 \\
E - \frac{2E}{3} &\leq V - 2 \\
E &\leq 3V - 6 \\
\sum e &\leq 2E \\
\sum e &\geq 3F \\
2E &\geq 3F
\end{align*}
\]
$E \leq 3v - 6$
$E \leq 3v - 6$

$K_5$
$E \leq 3V - 6$

10 \leq 15 - 6

Not planar
$E \leq 3v - 6$

$10 \leq 15 - 6$

$K_5$

$E \leq 3v - 6$

$K_{3,3}$
\[ E \leq 3V - 6 \]
\[ 10 \leq 15 - 6 \]

!!!

\[ K_5 \]

\[ E \leq 3V - 6 \]
\[ 9 \leq 18 - 6 \quad \text{ok!} \]

\[ K_{3,3} \]
$E \leq 3V - 6$

$10 \leq 15 - 6$

$9 \leq 18 - 6$  OK!

$E \leq 3V - 6$

$K_5$

not if

All planar graphs have $E \leq 3V - 6$

Some non-planar graphs can too
V - E + F = 2

What if G has no triangles?
\[ V - E + F = 2 \]

What if \( G \) has no triangles?

Every edge belongs to 1 or 2 faces

\[ \sum e \leq 2E \quad \text{all faces} \]

\[ E \geq 2F \]

Every face has \( \geq 4 \) edges (for \( V > 4 \))

\[ \sum e \geq 4F \quad \text{all faces} \]
$V - E + F = 2$

What if $G$ has no triangles?

Every edge belongs to 1 or 2 faces

Every face has $\geq 4$ edges (for $v > 4$)

$E - F = V - 2$

$E - \frac{E}{2} \leq V - 2$

$E \leq 2V - 4$

Instead of $\leq 3V - 6$

$\sum_{\text{faces}} e \leq 2E$

$\sum_{\text{faces}} e \geq 4F$

$E \geq 2F$
triangle free
for triangle free:

\[ E \leq 2V - 4 \]

\[ 9 \leq 2 \cdot 6 - 4 \]

\[ \text{!!!} \]

\[ K_{3,3} \]

\[ v = 6, \quad E = 9 \]

\[ \text{NOT PLANAR} \]
It turns out that every non-planar graph "contains" one of these two shapes.

\[ K_5 \quad \text{non-planar} \quad \Rightarrow \quad K_{3,3} \]
TRIANGULATIONS
triangulation

add edges while possible
E = 3V - 6

Why?

{ Assume outer face is a triangle }
\[ E = 3V - 6 \]

Why?

Assume outer face is a triangle

Every edge belongs to 1 or 2 faces

\[ \sum e \leq 2E \]

\[ \sum e \geq 3F \]

Every face has \( \geq 3 \) edges (for \( v > 3 \))

\( 2E \geq 3F \)
\[ E = 3V - 6 \]

Why?

\{ \text{Assume outer face is a triangle} \}

Every edge belongs to 1 or 2 faces \[ \sum e \leq 2E \] all faces

Every face has 3 edges (for \( V > 3 \)) \[ \sum e \geq 3F \] all faces

\[ 2E \times 3F \]
E = 3V - 6

Why?

{ Assume outer face is a triangle

Every edge belongs to \( \times 2 \) faces

Every face has \( \times 3 \) edges (for \( V > 3 \))

\[
E - F = V - 2
\]

\[
\sum e = 2E \quad \sum e = 3F \quad \begin{cases} \sum e = 2E \\ \sum e = 3F \end{cases} \]

\[
V - E + F = 2
\]
\[ E = 3V - 6 \]

\[ \text{Why?} \]

\[ \begin{array}{c}
\{ \\
\text{Assume outer face is a triangle} \\
\end{array} \]

\[ \begin{align*}
\text{Every edge belongs to} & \quad 1 \times \text{or} \quad 2 \times \text{faces} \\
\text{Every face has} & \quad \not= \times 3 \text{ edges} \ (\text{for} \ V > 3) \\
\end{align*} \]

\[ \sum e \not= \times 2E \quad \text{all faces} \]

\[ \sum e \not= \times 3F \quad \text{all faces} \]

\[ 2E \times 3F \]

\[ E - F = V - 2 \]

\[ E - \frac{2E}{3} = V - 2 \]

\[ E = 3V - 6 \]

\[ V - E + F = 2 \]
What is the average degree of a triangulation?
What is the average degree of a triangulation?

\[ \frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) \]
What is the average degree of a triangulation?

\[
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E
\]
What is the average degree of a triangulation?

\[
\frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} \leq 6
\]
What is the average degree of a triangulation?

\[ \frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} \leq 6 \]

Every triangulation has a vertex w/ degree \( \leq 5 \)
What is the average degree of a triangulation?

\[ \frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6 \]

\[ \Rightarrow \text{Every triangulation has a vertex w/ degree} \leq 5 \]

\[ \Rightarrow \text{Immediately applies to any planar graph (fewer edges)} \]
What is the average degree of a triangulation?

\[
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6
\]

\[\Rightarrow\text{Every planar graph has a vertex with degree} \leq 5\]
What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \frac{V}{i=1} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} \leq 6$$

Every planar graph has a vertex with degree $$\leq 5$$

Can we find many low-degree vertices?
$E = 3V - 6$

What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} < 6$$

$\Rightarrow$ Every planar graph has a vertex w/ degree $\leq 5$

Can we find many low-degree vertices? $\rightarrow$ not if "low" $= 5$. What if "low" $= 8$?
What is the average degree of a triangulation?

\[
\frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6
\]

\[\implies\] Every planar graph has a vertex w/ degree \leq 5

Can we find many low-degree vertices? \[\rightarrow \text{not if "low" = 5. what if "low" = 8?}\]

\[\rightarrow \frac{V}{2}\] degree-8

Prove by contradiction
What is the average degree of a triangulation?

\[
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6
\]

\[\Rightarrow\] Every planar graph has a vertex w/ degree \( \leq 5 \)

Can we find many low-degree vertices? \( \rightarrow \) not if "low" = 5.
what if "low" = 8?

Say you had \( \frac{V}{2} \) vertices w/ degree \( \geq 9 \)
What is the average degree of a triangulation?

\[ \frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6 \]

\[ \Rightarrow \text{Every planar graph has a vertex w/ degree } \leq 5 \]

Can we find many low-degree vertices? \[ \Rightarrow \text{not if } "low" = 5, \text{ what if } "low" = 8? \]

Say you had \( \frac{V}{2} \) vertices w/ degree \( \gg 9 \)

\[ \sum_{i=9}^{\gg} d(v_i) \gg 9 \cdot \frac{V}{2} \]

sum degrees of \( \frac{V}{2} \) of them
What is the average degree of a triangulation?

\[ \frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6 \]

\( \iff \) Every planar graph has a vertex w/ degree \( \leq 5 \)

Can we find many low-degree vertices? → not if "low" = 5. What if "low" = 8?

Say you had \( \gg \frac{V}{2} \) vertices w/ degree \( \gg 9 \)

\[ \Sigma d(v_i) \gg 9 \cdot \frac{V}{2} \]

\sum degrees of \( \frac{V}{2} \) of them \( \iff \) all other vertices have degree \( \gg 3 \)

\[ \Sigma d(v_i) \gg 3 \cdot \frac{V}{2} \]
What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6$$

$\Rightarrow$ Every planar graph has a vertex w/ degree $\leq 5$

Can we find many low-degree vertices? $\rightarrow$ not if "low" = 5.
what if "low" = 8?

Say you had $\frac{V}{2}$ vertices w/ degree $\geq 9$

$\Rightarrow \sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$

sum degrees of $\frac{V}{2}$ of them $\Rightarrow \sum d(v_i) \geq 3 \cdot \frac{V}{2}$

all other vertices have degree $\geq 3$

$\Rightarrow \Sigma + \Sigma \geq 6V$

contradiction

always have $\frac{V}{2}$ w/ $\deg \leq 8$