THE MONTY HALL PROBLEM
1 of these 3 doors hides a car. The other 2 hide goats.
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You get to pick a door. You randomly pick A.
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You get to pick a door. You randomly pick A.

Then a door you didn’t pick is opened (say, B) revealing a goat.
The Monty Hall Problem

1 of these 3 doors hides a car. The other 2 hide goats.

You get to pick a door. You randomly pick A.

Then a door you didn’t pick is opened (say, B) revealing a goat.

You’re given the choice: keep your door or switch?
Roll 2 dice...

Conditional Probability
Roll 2 dice ...

$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$
Roll 2 dice ...

\[ P(A) = P(\text{sum} = 8) \]

\[ P(B) = P(\text{both are even}) \]

If we knew that both are even, then what is the probability that the sum is 8?
Conditional Probability

Roll 2 dice...

$P(A) = P(\text{sum} = 8)$

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"prob. A given B"

$P(A | B)$

If we knew that both are even, then what is the probability that the sum is 8?
CONDITIONAL PROBABILITY

Roll 2 dice ...

\[ P(A) = P(\text{sum} = 8) \]

\[ P(B) = P(\text{both are even}) \]

"prob. A given B"

\[ P(A \mid B) \]

If we knew that both are even, then what is the probability that the sum is 8?

\[ A = \{ (2,6), (3,5), (4,4), (5,3), (6,2) \} \]
Roll 2 dice ...

A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}

B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}

P(A) = P(sum = 8)

P(B) = P(both are even)

If we knew that both are even, then what is the probability that the sum is 8?

"prob. A given B"

P(A | B)
Conditional Probability

Roll 2 dice...

- \( P(A) = P(\text{sum} = 8) \)
- \( P(B) = P(\text{both are even}) \)

\( \text{“prob. A given B”} \)

\( P(A \mid B) \)

If we knew that both are even, then what is the probability that the sum is 8?

\[
A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}
\]

\[
B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}
\]
**Conditional Probability**

Roll 2 dice ... $P(A) = P(\text{sum} = 8)$

Roll 2 dice a b

$P(B) = P(\text{both are even})$

"prob. A given B"

$P(A \mid B)$

If we knew that both are even, then what is the probability that the sum is 8?

$A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$

$P(A \mid B) = \frac{3}{9}$
Conditional Probability

Roll 2 dice...

$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$

"prob. A given B"

$P(A | B)$

If we knew that both are even, then what is the probability that the sum is 8?

$A = \{ (2,6), (3,5), (4,4), (5,3), (6,2) \}$

$B = \{ (2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6) \}$

$P(A | B) = \frac{3}{9}$

$\neq P(A)$ in this example
$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$
\[ P(A) = P(\text{sum} = 8) \]
\[ P(B) = P(\text{both are even}) \]

(if you want the universe to have area = 1, divide all by 36.)
\[ P(A) = P(\text{sum} = 8) \]
\[ P(B) = P(\text{both are even}) \]
\[ P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36} \]
\[ P(A) = P(\text{sum} = 8) \]

\[ P(B) = P(\text{both are even}) \]

\[ P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36} \]

When we establish \( B \) then the universe shrinks.
\[ P(A) = P(\text{sum} = 8) \]
\[ P(B) = P(\text{both are even}) \]

\[ P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36} \]

When we establish \( B \) then the universe shrinks.

The probability that \( A \) holds is normalized: \( \frac{\text{remaining valid green area}}{\text{new universe (pink area)}} \)
\[ P(A) = P(\text{sum} = 8) \]
\[ P(B) = P(\text{both are even}) \]

\[ P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36} \]

When we establish \( B \) then the universe shrinks. The probability that \( A \) holds is normalized:

\[ \frac{\text{remaining valid green area}}{\text{new universe (pink area)}} \]

\[ P(A|B) = ? \]
\[ P(A) = P(\text{sum} = 8) \]
\[ P(B) = P(\text{both are even}) \]
\[ P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36} \]

When we establish \( B \) then the universe shrinks. The probability that \( A \) holds is normalized:

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{3+6}
\]
another example  Flip a coin 5 times.  \( P(1st \ flip = T) = \frac{1}{2} \)
another example  Flip a coin 5 times. \( P(1\text{st flip } = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?
another example  Flip a coin 5 times.  \[ P(1\text{st flip} = T) = \frac{1}{2} \]

But what if you know that 3 of the 5 flips were H?

\[ P(1\text{st flip} = T \mid 3 \cdot H) = ? \]
another example  Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(1\text{st flip} = T \mid 3\cdot H) = \frac{P[(1\text{st flip} = T) \cap (3\cdot H)]}{P(3\cdot H)}
\]
another example  Flip a coin 5 times.  \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(1\text{st flip} = T \mid 3 \cdot H) = \frac{P[(1\text{st flip} = T) \cap (3 \cdot H)]}{P(3 \cdot H)}
\]

\( P(3 \cdot H) = \) ?
another example

Flip a coin 5 times. \( P(\text{1st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(\text{1st flip} = T | 3 \cdot H) = \frac{P[(\text{1st flip} = T) \cap (3 \cdot H)]}{P(3 \cdot H)}
\]

\[
P(3 \cdot H) = \frac{\binom{5}{3}}{2^5} \quad \rightarrow \text{Ways to choose 3 positions for H.}
\]

\[
\quad \rightarrow \text{Sample space.}
\]
**another example**

Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(1\text{st flip} = T \mid 3\cdot H) = \frac{P[(1\text{st flip} = T) \cap (3\cdot H)]}{P(3\cdot H)}
\]

\[
P(3\cdot H) = \frac{\binom{5}{3}}{2^5} \rightarrow \text{Ways to choose 3 positions for H.}\]

\[
= \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}
\]

→ Sample space.
another example  
Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
\text{P}(1\text{st flip} = T \mid 3\cdot H) = \frac{\text{P}[(1\text{st flip} = T) \cap (3\cdot H)]}{\text{P}(3\cdot H)}
\]

\[
\text{P}(3\cdot H) = \frac{\binom{5}{3}}{2^5} \rightarrow \text{Ways to choose 3 positions for H.}
\]

\[
\rightarrow \text{Sample space.} = \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}
\]

\[
\text{P}[(1\text{st flip} = T) \cap (3\cdot H)] = ?
\]
another example  
Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[ P(1\text{st flip} = T \mid 3\cdot H) = \frac{P[(1\text{st flip} = T) \cap (3\cdot H)]}{P(3\cdot H)} \]

\[ P(3\cdot H) = \frac{\binom{5}{3}}{2^5} \quad \text{→ Ways to choose 3 positions for H.} \]
\[ = \frac{5!}{3!2!} = \frac{5}{32} = \frac{5}{16} \]

\[ P[(1\text{st flip} = T) \cap (3\cdot H)] : \begin{cases} T & HHHHT \\
T & HHHTH \\
T & HTHHH \\
T & TTHHH \end{cases} \]
\[ \frac{4}{32} \]
another example  Flip a coin 5 times.  \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(1\text{st flip} = T \mid 3 \cdot H) = \frac{P[(1\text{st flip} = T) \cap (3 \cdot H)]}{P(3 \cdot H)}
\]

\[
P(3 \cdot H) = \frac{\binom{5}{3}}{2^5} \quad \rightarrow \quad \text{Ways to choose 3 positions for H.}
\]

\[
\quad \rightarrow \quad \text{Sample space.} \quad = \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}
\]

\[
P[(1\text{st flip} = T) \cap (3 \cdot H)] = \frac{4}{32} \quad \text{OR} \quad \frac{1}{2} \cdot \frac{\binom{4}{3}}{4} = \frac{1}{8}
\]
another example

Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

\[
P(\text{1st flip} = T \mid 3 \cdot H) = \frac{P[(\text{1st flip} = T) \cap (3 \cdot H)]}{P(3 \cdot H)}
\]

\[
P(3 \cdot H) = \frac{\binom{5}{3}}{2^5} \quad \rightarrow \quad \text{Ways to choose 3 positions for H.}
\]

\[
= \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}
\]

\[
\frac{2/16}{5/16} = \frac{2}{5}
\]

\[
P[(\text{1st flip} = T) \cap (3 \cdot H)] = \frac{4}{32}
\]

OR \( \frac{1}{2} \cdot \frac{\binom{4}{3}}{2} = \frac{1}{8} \)
another example  Flip a coin 5 times. \( P(1\text{st flip} = T) = \frac{1}{2} \)

But what if you know that 3 of the 5 flips were H?

think of having a biased coin: 60-40 vs 50-50 then the first flip has 40% for T \( \rightarrow \frac{2}{5} \)
\[
P(\geq 2 \text{ people in a group of } k \text{ have same birthday})
\]
\[
= 1 - P(\text{all } k \text{ have distinct birthdays})
\]
\[
\downarrow P(\text{2nd person has different bday than 1st})
\]
\[
\cdot P(\text{3rd} \ldots \cdot \cdot \cdot 1\text{st & 2nd})
\]
\[
\Rightarrow \quad \frac{364}{365} = P(A)
\]
\[
\Rightarrow \quad \frac{363}{365} = P(B|A)
\]
assuming 1st & 2nd differ
\[ P(\geq 2 \text{ people in a group of } k \text{ have same birthday}) \]

\[ = 1 - P(\text{all } k \text{ have distinct birthdays}) \]

\[ \geq P(2\text{nd person has different bday than 1st}) \]

\[ \cdot P(3\text{rd \ldots \ldots \ldots 1st & 2nd}) \]

\[ \geq B \]

\[ \cdot P(4\text{th \ldots \ldots \ldots \ldots (1-3)}) \]

\[ \implies \frac{364}{365} = P(A) \]

\[ \rightarrow \frac{363}{365} = P(B|A) \]

assuming 1st & 2nd differ

\[ \implies \frac{362}{365} \]

\[ = P(C|A \cap B) \]

\[ \text{etc} \]

\[ = P(A) \cap P(B|A) \cap P(C|A \cap B) \ldots \]
Flip a coin x3 : \[ P(3rd = T \mid 1st = H) = ? \]
Flip a coin x3 : \[ P(3\text{rd} = T \mid 1\text{st} = H) = \]

\[\frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)}\]
Flip a coin x3 : \[ P(3\text{rd} = T \mid 1\text{st} = H) = \]
\[ = \frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)} \]
\[ = \frac{\frac{2}{8}}{\frac{1}{2}} \]
\[ = \frac{1}{2} \]
Flip a coin x3:  \[ P(3\text{rd} = T \mid 1\text{st} = H) = \]

\[
= \frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)} \quad \rightarrow \quad \frac{2}{8} \quad \text{# outcomes} \quad \rightarrow \quad \frac{1}{2} \quad \text{sample space} \quad = \frac{1}{2}
\]

Notice  \( P(3\text{rd} = T) = \frac{1}{2} \)
Flip a coin x3 : \[ P(3\text{rd} = T \mid 1\text{st} = H) = \]

\[
= \frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)}
\]

\[= \frac{\frac{2}{8}}{\frac{1}{2}} \rightarrow \text{sample space} = \frac{1}{2}
\]

Notice \[ P(3\text{rd}=T) = \frac{1}{2} \] so knowledge of \( (1\text{st}=H) \) was useless.
Flip a coin x3:  \( P(3\text{rd} = T \mid 1\text{st} = H) = \)

\[
\frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)} = \frac{2}{8} \rightarrow \text{sample space} = \frac{1}{2}
\]

Notice  \( P(3\text{rd} = T) = \frac{1}{2} \) so knowledge of \((1\text{st} = H)\) was useless.

A & B are independent if  \( P(A) = P(A \mid B) \)
INDEPENDENCE

Flip a coin $\times 3$: 

$$P(3\text{rd} = T \mid 1\text{st} = H) =$$

$$= \frac{P[(3\text{rd} = T) \land (1\text{st} = H)]}{P(1\text{st} = H)} = \frac{2}{8} \quad \text{# outcomes}$$

$$\quad \frac{1}{2} \quad \text{sample space} = \frac{1}{2}$$

Notice $P(3\text{rd} = T) = \frac{1}{2}$ so knowledge of $(1\text{st} = H)$ was useless.

$A \ & \ B$ are independent if $P(A) = P(A \mid B)$

if $P(B) = P(B \mid A)$ [equivalent]
\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

always
\[ P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A) \]

- always

- if \( A \) & \( B \) independent
\[ P(A | B) = \frac{P(A \cap B)}{P(B)} = P(A) \]

always

if A & B independent

alternate definition: A & B are independent if

\[ P(A \cap B) = P(A) \cdot P(B) \]
another example

4 boys & 4 girls

Select 2 people from this group
another example

4 boys & 4 girls
Select 2 people from this group

A: 1st person is a girl
B: 2nd person is a girl
Another example

4 boys & 4 girls
Select 2 people from this group

A: 1st person is a girl
B: 2nd person is a girl

P(A) =
another example

4 boys & 4 girls
Select 2 people from this group

A: 1st person is a girl
B: 2nd person is a girl

\[ P(A) = \frac{4}{8} \]
\[ P(B) = \]
Another example

A: 1st person is a girl
B: 2nd person is a girl

\[ P(A) = \frac{4}{8} \]

\[ P(B) = \frac{1}{2} \quad \text{(by symmetry)} \]

(or: sample space = 8, 7
& for each girl=2nd, #outcomes = 7)

4 boys & 4 girls

Select 2 people from this group
another example

4 boys & 4 girls
Select 2 people from this group

A: 1st person is a girl
B: 2nd person is a girl

\[ P(A) = \frac{4}{8} \]

\[ P(B) = \frac{1}{2} \] (by symmetry)

(or: sample space = 8, 7
& for each girl=2nd, #outcomes = 7)

\[ P(B|A) = \]
Another example

4 boys & 4 girls
Select 2 people from this group

A: 1st person is a girl
B: 2nd person is a girl

\[ P(A) = \frac{4}{8} \]

\[ P(B) = \frac{1}{2} \text{ (by symmetry)} \]

\[ P(B|A) = \frac{3}{7} \]

\[ \frac{4}{8} \neq \frac{1}{2} \]

(by sample space = 8 ± 7 & for each girl = 2nd, # outcomes = 7)
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]
Back to Monty Hall

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

Now we know the car is not at B
What is the probability it’s at A?

\[ P(A | \overline{B}) \]
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

Now we know the car is not at B
What is the probability it's at A?

\[ P(A | \neg B) = \frac{P(A \cap \neg B)}{P(\neg B)} \]
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

Now we know the car is not at B
What is the probability it's at A?

\[ P(A | \overline{B}) = \frac{P(A \cap \overline{B})}{P(B)} \]

...because \[ P(\overline{B} | A) = \frac{P(A \cap \overline{B})}{P(A)} \] ... \[ = \frac{P(\overline{B} | A) \cdot P(A)}{P(B)} \]
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

Now we know the car is not at B
What is the probability it's at A?

\[ P(A | \overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} \]

...because \[ P(\overline{B} | A) = \frac{P(A \cap \overline{B})}{P(A)} \]

\[ = \frac{P(\overline{B} | A) \cdot P(A)}{P(\overline{B})} = \frac{1 \cdot P(A)}{P(\overline{B})} \]
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

Now we know the car is not at B
What is the probability it's at A?

\[ P(A | \overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} \]

...because \[ P(\overline{B} | A) = \frac{P(A \cap \overline{B})}{P(A)} \] ...

\[ = \frac{P(\overline{B} | A) \cdot P(A)}{P(\overline{B})} = \frac{1 \cdot P(A)}{\frac{2}{3}} = \frac{1}{2} \]
BACK TO MONTY HALL

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

\[ P(A | \overline{B}) \leftarrow \text{back up} \]
BACK TO MONTY HALL

A  B  C

\[ P(\text{car} @ A) = P(A) = \frac{1}{3} = P(B) = P(C) \]

A  B  C

\[ P(A | (\text{door B was opened } \land \text{ we chose A})) \]

\[ \land \text{ not } = \overline{B} \quad \text{extra info} \]
BACK TO MONTY HALL: intuition

$P(\text{car} @ A) = P(A) = \frac{1}{3} = P(B) = P(C)$

w.l.o.g. guess A
& B is shown.
BACK TO MONTY HALL: intuition

w.l.o.g. guess A & B is shown.

$\text{P(car @ A)} = \text{P}(A) = \frac{1}{3} = \text{P}(B) = \text{P}(C)$
BACK TO MONTY HALL: intuition

w.l.o.g. guess A & B is shown.

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

A (car @ A)

\[ \frac{1}{3} \]

B

\[ \frac{1}{3} \]

if C, then B revealed \(\rightarrow\) switch to win
BACK TO MONTY HALL: intuition

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

w.l.o.g. guess A & B is shown.

- A (car @ A)
- if B, then C revealed → switch to win
- if C, then B revealed → switch to win
BACK TO MONTY HALL: intuition

w.l.o.g. guess A & B is shown.

P(car @ A) = P(A) = \frac{1}{3} = P(B) = P(C)

- if A (car @ A) then
  - \frac{1}{2} \rightarrow B \text{ revealed}
  - \frac{1}{2} \rightarrow C \text{ revealed}

- if B, then C revealed \rightarrow switch to win
- if C, then B revealed \rightarrow switch to win
BACK TO MONTY HALL: intuition

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

w.l.o.g. guess A & B is shown.

4 events:

- if A (car @ A) then
  \[ \frac{1}{3} \rightarrow B \text{ revealed} \]
  \[ \frac{1}{3} \rightarrow C \text{ revealed} \]

- if B, then C revealed \rightarrow switch to win

- if C, then B revealed \rightarrow switch to win

\{ switch \rightarrow lose \}
BACK TO MONTY HALL: intuition

\[ P(\text{car @ A}) = P(A) = \frac{1}{3} = P(B) = P(C) \]

w.l.o.g. guess A & B is shown.

4 events:

- if A (car @ A) then \( \frac{1}{2} \) switch
- if B, then C revealed → switch to win
- if C, then B revealed → switch to win \( \frac{1}{3} \)
\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[
\begin{array}{c}
A \\
B \\
C
\end{array}
\]

\[ \frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[
\begin{array}{c}
A \& \ B \text{ shown} \\
A \& \ C \text{ shown} \\
B \& \ C \text{ shown} \\
B \& \ B \text{ shown}
\end{array}
\]
When we establish that B is shown, the universe shrinks.
When we establish that \( B \) is shown, the universe shrinks.

\[
P(A \cap B \text{ shown}) = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3}
\]
When we establish that $B$ is shown, the universe shrinks.  

\[ P(A \cap \text{B shown}) = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3} \]

\[ P(C \cap \text{B shown}) = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3} \]
$P[A | (\text{we chose A} \land \text{door B was opened})]$
\[ P[A \mid (\text{we chose } A \land \text{ door } B \text{ was opened})] \]

apply: \[ P(x \mid y) = \frac{P(x \cap y)}{P(y)} \]

\[ = \frac{P[A \cap (\text{we chose } A \land \text{ door } B \text{ was opened})]}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]
\[ P[A | (\text{we chose } A \land \text{ door } B \text{ was opened})] \]

apply \( P(x|y) = \frac{P(x \land y)}{P(y)} \)

just moving parentheses

\[ P[A \land (\text{we chose } A \land \text{ door } B \text{ was opened})] \]

\[ \frac{P(\text{we chose } A \land \text{ door } B \text{ was opened})}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]

\[ P[(A \land \text{ we chose } A) \land \text{ door } B \text{ was opened}] \]

\[ \frac{P(\text{we chose } A \land \text{ door } B \text{ was opened})}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]
\[
P[\text{we chose } A \land \text{ door B was opened}] = \frac{P[A \land \text{ we chose } A \land \text{ door B was opened}]}{P(\text{we chose } A \land \text{ door B was opened})} = \frac{P[(A \land \text{ we chose } A) \land \text{ door B was opened}]}{P(\text{we chose } A \land \text{ door B was opened})} = \frac{P[\text{door B was opened } | (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A)}{P(\text{we chose } A \land \text{ door B was opened})}
\]
So far,

\[ P[A \mid (\text{we chose } A \land \text{ door } B \text{ was opened})] \]

\[ = \frac{P[\text{door } B \text{ was opened } \mid (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A)}{P(\text{we chose } A \land \text{ door } B \text{ was opened})} \]
So far, \[
P[A \mid (\text{we chose } A \land \text{ door } B \text{ was opened})]
\]

\[
= \frac{P[\text{door } B \text{ was opened} \mid (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A)}{P(\text{we chose } A \land \text{ door } B \text{ was opened})}
\]

\[
= \frac{\frac{1}{2} \cdot P(A) \cdot P(\text{we chose } A)}{P(\text{we chose } A \land \text{ door } B \text{ was opened})}
\]

(host picks randomly independent)

\[
= \frac{1}{2} \cdot P(A) \cdot P(\text{we chose } A)
\]

\[
= \frac{P(\text{we chose } A \land \text{ door } B \text{ was opened})}{P(\text{we chose } A \land \text{ door } B \text{ was opened})}
\]
So far, \[ P[A \mid (\text{we chose } A \land \text{door } B \text{ was opened})] \]

\[= \frac{P[\text{door } B \text{ was opened} \mid (A \land \text{we chose } A)] \cdot P(A \land \text{we chose } A)}{P(\text{we chose } A \land \text{door } B \text{ was opened})} \]

\[= \frac{1/2 \cdot P(A) \cdot P(\text{we chose } A)}{P(\text{we chose } A \land \text{door } B \text{ was opened})} \]

\[= \frac{1/2 \cdot (1/3 \cdot 1/3)}{P(\text{we chose } A \land \text{door } B \text{ was opened})} \]
So far,

\[ P[A \mid (\text{we chose A} \land \text{door B was opened})] = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose A} \land \text{door B was opened})} \]
So far, \[ P[A \mid (\text{we chose } A \land \text{ door B was opened})] = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \]

\[ P(\text{we chose } A \land \text{ door B was opened}) \]

\[ = \frac{1}{18} \]

The 2 terms in the denominator are "mutually exclusive", which means that they have no intersection. We can see that because one asks for A to happen, but the other asks for C to happen (among other things). In fact we should also include the term \[ +P(\text{we chose A AND door B was opened AND B}) \]

but this term is equal to zero.

Here's what is going on. There is an event, X. In our case, \[ X = (\text{we chose } A \land \text{ door B was opened}) \]. We are interested in \[ P(X) \], as shown in orange above. We can say that \[ P(X) = P(X \text{ and } A) + P(X \text{ and } B) + P(X \text{ and } C) \], if A, B, C are mutually exclusive and cover all possibilities. Each of those 3 terms can have different values, but they will sum to \[ P(X) \]. Think of each term as a subset of X. (it makes sense that they are subsets of X, because they are more restrictive, and maintain all requirements of X). In fact not only are they subsets, but the 3 terms partition X. If X happens, then either \[ (X \text{ and } A) \] happens, or \[ (X \text{ and } B) \] happens, or \[ (X \text{ and } C) \] happens. In our case we have some additional information; \[ X \text{ and } B \] cannot happen.
So far,

\[ P[A \mid (\text{we chose A} \land \text{door B was opened})] \]

\[
= \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose A} \land \text{door B was opened})}
\]

\[ P(\text{open B} \mid (C \land \text{choose A})) = 1 \]

Could not have

door B opened
and
car at B

\[
= \frac{\frac{1}{8}}{\left[ P(\text{we chose A} \land \text{door B was opened} \land A) \right] + P(\text{we chose A} \land \text{door B was opened} \land C)}
\]
So far,

\[ P[A | (\text{we chose A} \land \text{door B was opened})] \]

\[ = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose A} \land \text{door B was opened})} \]

\[ = \frac{1}{18} \left[ P(\text{we chose A} \land \text{door B was opened} \land A) \right]^{+} \]

\[ P(\text{open B} | (C \land \text{choose A})) = 1 \]

\[ = \frac{P(C \land \text{open B} \land \text{choose A})}{P(C \land \text{choose A})} \]
So far,

\[
P[A | (\text{we chose A } \land \text{ door B was opened})] = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(\text{we chose A } \land \text{ door B was opened})}
\]

\[
= \frac{\frac{1}{18}}{\frac{1}{18}}
\]

\[
P(\text{open B} | (C \land \text{ choose A})) = \frac{1}{3}
\]

\[
P(C \land \text{ open B } \land \text{ choose A}) = \frac{P(C \land \text{ open B } \land \text{ choose A})}{P(C \land \text{ choose A})}
\]

\[
= \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3}}
\]

\[
P(\text{we chose A } \land \text{ door B was opened } \land \text{ A}) + \frac{1}{3} \cdot \frac{1}{3}
\]

\[
= \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3}}
\]
So far,

\[ P[A \mid (\text{we chose } A \land \text{ door B was opened})] \]

\[ = \frac{\frac{1}{18}}{P(\text{we chose } A \land \text{ door B was opened } \land A) + \frac{1}{3} \cdot \frac{1}{3}} \]
So far,

\[ P[A \ | \ (\text{we chose } A \land \text{ door } B \text{ was opened})] = \frac{\frac{1}{18}}{P(\text{we chose } A \land \text{ door } B \text{ was opened} \land A) + \frac{1}{3} \cdot \frac{1}{3}} = \frac{\frac{1}{18}}{P[(A \land \text{ we chose } A) \land \text{ door } B \text{ was opened}] + \frac{1}{9}} \]
So far, 

\[ P \left[ A \mid (\text{we chose } A \land \text{door } B \text{ was opened}) \right] \]

\[ = \frac{\frac{1}{18}}{P(\text{we chose } A \land \text{door } B \text{ was opened} \land A) + \frac{1}{3} \cdot \frac{1}{3}} \]

\[ = \frac{\frac{1}{18}}{P[(A \land \text{we chose } A) \land \text{door } B \text{ was opened}] + \frac{1}{9}} \]

\[ = \frac{\frac{1}{18}}{P[\text{door } B \text{ was opened} \mid (A \land \text{we chose } A)] \cdot P(A \land \text{we chose } A) + \frac{1}{9}} \]
So far,

\[ P[A \mid (\text{we chose } A \land \text{ door } B \text{ was opened})] \]

\[ = \frac{1}{18} \frac{P(\text{we chose } A \land \text{ door } B \text{ was opened} \land A)}{P[(A \land \text{ we chose } A) \land \text{ door } B \text{ was opened}] + \frac{1}{3} \cdot \frac{1}{3}} \]

\[ = \frac{1}{18} \frac{P[(A \land \text{ we chose } A) \land \text{ door } B \text{ was opened}]}{P[\text{door } B \text{ was opened} \mid (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A) + \frac{1}{9}} \]

\[ = \frac{1}{18} \text{ (host's random choice independent)} \]
So far, 

\[
P[A \mid (\text{we chose A} \land \text{door B was opened})]
\]

\[
= \frac{\frac{1}{18}}{P(\text{we chose A} \land \text{door B was opened} \land A) + \frac{1}{3} \cdot \frac{1}{3}}
\]

\[
= \frac{\frac{1}{18}}{P[(A \land \text{we chose A}) \land \text{door B was opened}]} + \frac{1}{9}
\]

\[
= \frac{\frac{1}{18}}{P[\text{door B was opened} \mid (A \land \text{we chose A})] \cdot P(A \land \text{we chose A}) + \frac{1}{9}}
\]

host's random choice \hspace{1cm} \text{independent}

\[
= \frac{\frac{1}{18}}{\frac{1}{2} \cdot (\frac{1}{3} \cdot \frac{1}{3}) + \frac{1}{9}}
\]
So far, \[ P[A | (\text{we chose } A \land \text{ door } B \text{ was opened})] \]

\[ = \frac{1/18}{P(\text{we chose } A \land \text{ door } B \text{ was opened} \land A) + \frac{1}{3} \cdot \frac{1}{3}} \]

\[ = \frac{1/18}{P[(A \land \text{ we chose } A) \land \text{ door } B \text{ was opened}] + \frac{1}{9}} \]

\[ = \frac{1/18}{P[\text{door } B \text{ was opened} | (A \land \text{ we chose } A)] \cdot P(A \land \text{ we chose } A) + \frac{1}{9}} \]

\[ = \frac{1/18}{\frac{1}{2} \cdot (\frac{1}{3} \cdot \frac{1}{3}) + \frac{1}{9}} = \frac{1/18}{\frac{1}{18} + \frac{2}{18}} = \frac{1}{3} \]
TESTING FOR A DISEASE
TESTING FOR A DISEASE

• Suppose 1% of the population has a disease
TESTING FOR A DISEASE

- Suppose 1% of the population has a disease
- There is a diagnostic test, that finds it 80% of the time (assuming the subject has it)
TESTING FOR A DISEASE

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- There is a diagnostic test, that finds it 80% of the time (assuming the subject has it)
- The test also produces false positives, at a rate of 9.6% (you're fine, but the test says you're not)
TESTING FOR A DISEASE

- Suppose 1% of the population has a disease
- there is a diagnostic test, that finds it 80% of the time (assuming the subject has it)
- the test also produces false positives, at a rate of 9.6% (you're fine, but the test says you're not)

If someone tests positive, what are the odds that they have the disease?
<table>
<thead>
<tr>
<th></th>
<th>Have disease</th>
<th>Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test VERTISE</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test CLEAN</td>
<td>20%</td>
<td>90.4%</td>
</tr>
<tr>
<td></td>
<td>1% Have disease</td>
<td>99% Don't have disease</td>
</tr>
<tr>
<td>----------------</td>
<td>-----------------</td>
<td>------------------------</td>
</tr>
<tr>
<td><strong>Test ₁</strong></td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td><strong>Test ₂</strong></td>
<td>20%</td>
<td>90.4%</td>
</tr>
<tr>
<td>1% Have disease</td>
<td>99% Don't have disease</td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>------------------------</td>
<td></td>
</tr>
<tr>
<td>Test ✅</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test ✅</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease} | \text{test ✅}) = \ ?
\]
<table>
<thead>
<tr>
<th>Test</th>
<th>1% Have Disease</th>
<th>99% Don't Have Disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test $\not\in$</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test $\in$</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease}| \text{test } \not\in) = \frac{P(\text{disease } \cap \text{ test } \not\in)}{P(\text{test } \not\in)}
\]
<table>
<thead>
<tr>
<th></th>
<th>1% Have disease</th>
<th>99% Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test ☐</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test ☐</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease} | \text{test ☐}) = \frac{P(\text{disease} \cap \text{test ☐})}{P(\text{test ☐})} = \frac{P(\text{test ☐} | \text{disease}) \cdot P(\text{disease})}{P(\text{test ☐})}
\]
<table>
<thead>
<tr>
<th></th>
<th>1% Have disease</th>
<th>99% Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Test ☐</strong></td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td><strong>Test ☐</strong></td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease}|\text{test ☐}) = \frac{P(\text{disease} \cap \text{test ☐})}{P(\text{test ☐})} = \frac{P(\text{test ☐}|\text{disease}) \cdot P(\text{disease})}{P(\text{test ☐})} \]

\[
P(\text{test ☐}|\text{disease}) \cdot P(\text{disease}) = ?
\]
## 4 events

<table>
<thead>
<tr>
<th></th>
<th>1% Have disease</th>
<th>99% Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test ∅</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test ☐</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease} | \text{test ∅}) = \frac{P(\text{disease} \cap \text{test ∅})}{P(\text{test ∅})} = \frac{P(\text{test ∅} | \text{disease}) \cdot P(\text{disease})}{P(\text{test ∅})}
\]

\[
P(\text{test ∅} | \text{disease}) \cdot P(\text{disease}) = 0.8 \cdot 0.01 = 0.008 \ (0.8%)
\]

\[
P(\text{test ∅}) = ?
\]
<table>
<thead>
<tr>
<th></th>
<th>1% Have disease</th>
<th>99% Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test ✗</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test ☑</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease} | \text{test ✗}) = \frac{P(\text{disease} \cap \text{test ✗})}{P(\text{test ✗})} = \frac{P(\text{test ✗} | \text{disease}) \cdot P(\text{disease})}{P(\text{test ✗})}
\]

\[
P(\text{test ✗}) = \left\{ \begin{array}{l}
P(\text{test ✗} | \text{disease}) \cdot P(\text{disease}) = 0.8 \cdot 0.01 = 0.008 (0.8%) \\
\end{array} \right.
\]
<table>
<thead>
<tr>
<th>Test</th>
<th>Have disease</th>
<th>Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease}|\text{test } \overline{\cdot}) = \frac{P(\text{disease} \cap \text{test } \overline{\cdot})}{P(\text{test } \overline{\cdot})} = \frac{P(\text{test } \overline{\cdot} | \text{disease}) \cdot P(\text{disease})}{P(\text{test } \overline{\cdot})}
\]

\[
P(\text{test } \overline{\cdot}) = \left\{ \begin{array}{l}
P(\text{test } \overline{\cdot} | \text{disease}) \cdot P(\text{disease}) = 0.8 \cdot 0.01 = 0.008 (0.8\%) \\
+ P(\text{test } \overline{\cdot} | \text{no disease}) \cdot P(\text{no disease})
\end{array} \right.
\]
<table>
<thead>
<tr>
<th>Test</th>
<th>1% Have Disease</th>
<th>99% Don't Have Disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test $\checkmark$</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test $\not\checkmark$</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease} | \text{test } \checkmark) = \frac{P(\text{disease} \cap \text{test } \checkmark)}{P(\text{test } \checkmark)} = \frac{P(\text{test } \checkmark | \text{disease}) \cdot P(\text{disease})}{P(\text{test } \checkmark)}
\]

\[
P(\text{test } \checkmark) = \begin{cases} 
P(\text{test } \checkmark | \text{disease}) \cdot P(\text{disease}) & = 0.8 \cdot 0.01 = 0.008 (0.8\%) \\
+ & \\
P(\text{test } \not\checkmark | \text{no disease}) \cdot P(\text{no disease}) & = 0.096 \cdot 0.99 \approx 0.095 (9.5\%)
\end{cases}
\]
<table>
<thead>
<tr>
<th></th>
<th>1% Have disease</th>
<th>99% Don't have disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test ☑</td>
<td>80%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Test ☒</td>
<td>20%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>

\[
P(\text{disease}|\text{test ☑}) = \frac{P(\text{disease} \cap \text{test ☑})}{P(\text{test ☑})} = \frac{P(\text{test ☑}|\text{disease}) \cdot P(\text{disease})}{P(\text{test ☑})}
\]

\[
P(\text{test ☑}) = P(\text{test ☑}|\text{disease}) \cdot P(\text{disease}) + P(\text{test ☑}|\text{no disease}) \cdot P(\text{no disease}) = 0.8 \cdot 0.01 + 0.096 \cdot 0.99 \approx 0.095 (9.5%)
\]

\[
P(\text{disease}|\text{test ☑}) = \frac{0.008}{0.008 + 0.095} \approx 7.8%
\]
Bayes theorem
\[ P(A \cap B) = P(A|B) \cdot P(B) \]

\[ = P(B|A) \cdot P(A) \]
\[ P(A \cap B) = P(A|B) \cdot P(B) \]
\[ = P(B|A) \cdot P(A) \]

\[ \leftarrow P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)} \]
Bayes' theorem

\[ P(A \cap B) = P(A|B) \cdot P(B) \]

\[ = P(B|A) \cdot P(A) \]

\[ P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)} \]

\[ P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\overline{A}) \cdot P(\overline{A})} \]
\[ P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})} \]
\[ P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\overline{A}) \cdot P(\overline{A})} \]

A: have disease
B: test ✗
\[
P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid \overline{A}) \cdot P(\overline{A})}
\]

A: have disease
B: test ☐

<table>
<thead>
<tr>
<th></th>
<th>1% (A)</th>
<th>99% (\overline{A})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B \mid A)</td>
<td>80%</td>
<td>(B \mid \overline{A})</td>
</tr>
</tbody>
</table>
\[ P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\overline{A}) \cdot P(\overline{A})} \]

\( A: \) have disease

\( B: \) test

\[ \begin{array}{c|c|c}
1\% \ A & 99\% \ \overline{A} \\
B|A = 80\% & B|\overline{A} = 9.6\% \\
\end{array} \]

- \( P(A) = 0.01 \)
- \( P(\overline{A}) = 0.99 \)
- \( P(B|A) = 0.8 \)
- \( P(B|\overline{A}) = 0.096 \)
\[ P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})} \]

A: have disease
B: test

\[ \begin{array}{c|c|c}
1\% & 1\%A & 99\% \bar{A} \\
\hline
B|A = 80\% & B|\bar{A} = 9.6\% \\
\end{array} \]

\[ P(A) = 0.01 \quad P(\bar{A}) = 0.99 \]
\[ P(B|A) = 0.8 \quad P(B|\bar{A}) = 0.096 \]

\[ P(A|B) = \frac{0.8 \cdot 0.01}{0.8 \cdot 0.01 + 0.096 \cdot 0.99} \]