Game for 2 players:

Draw a complete graph, each player can draw edges using one color, take turns coloring one edge at a time.

Whoever completes a triangle first wins.

Is there always a winner?
Cliques

How many cliques?
What is the largest clique?

Given $G$, a subset $S$ of $V(G)$ is a clique if every $s_i, s_j \in S$ share an edge in $G$.

The induced subgraph obtained by removing all but $S$ from $V(G)$ is a complete graph ($K_S$).
Independent Sets

Given $G$, a subset $S$ of $V(G)$ is an independent set if $\forall s_i, s_j \in S$ share an edge in $G$.

The induced subgraph obtained by removing all but $S$ from $V(G)$ is an edgeless graph.

i.e. its complement is a complete graph.
cliques vs. independent sets

max clique

complement

max ind. set

max ind. set

max clique
cliques vs. independent sets

max clique

max ind. set

complement

ind. set

max clique
Claim: Every graph with $|V| \geq 6$ contains a triangle \textit{(clique of size 3)} \textit{OR} an independent set of size 3.

Rephrase: \textit{(for $|V| \geq 6$)} a graph contains a triangle \textit{OR} its complement does.

Proof: pick any vertex $v$.

If $d(v) \geq 3$, we have $v \rightarrow x \rightarrow y \rightarrow z$. If $xy$ or $xz$ or $yz$: we find a clique $\Delta$.

Otherwise, $x, y, z$ are an independent set.

If $d(v) \leq 2$, there are $\geq 3$ vertices not neighboring $v$. \rightarrow v \rightarrow a \rightarrow b \rightarrow c$.

If $ab, bc, ac$ are edges, they are a clique $\Delta$.

Otherwise one edge is missing (w.l.o.g. $ab$) \ldots so $vab$ is an ind. set. \quad $\blacksquare$
(for $n \geq 6$) a graph contains a triangle or its complement does.

Equivalent statement
If we color each edge of $K_n$ red or black, then we must get a red triangle or a black triangle.
Recap: if you want a clique or an independent set of size 3 then you'll be happy as long as $|V| \geq 6$

$\iff R(3,3) = 6$
Recap: if you want a clique or an independent set of size 3
then you'll be happy as long as $|V| \geq 6$

$\Rightarrow R(3,3) = 6$

$R(4,4)$

$\downarrow$

18

(one direction to be shown)

$R(5,5)$

we don't know!

[43...49]

$R(n,n)$

~ exponential

even for small values we will probably never know the exact answer

For more, see Ramsey's Thm.
$R(x,y)$: smallest number $N$ such that any graph with $\geq N$ vertices has a clique of size $x$ or an independent set of size $y$.

$R(4,2) = 4$
Suppose $|V| \geq 10$

Pick any vertex, A.  $\geq 9$ vertices remain.

Form 2 groups:

S & T

If $|S| \geq 6$, use $R(3,3) = 6$ : S has 3 independent vertices (done), or S has a 3-clique, so with A we get a 4-clique.

If $|T| \geq 4$, if T is a clique: done.

Otherwise $\exists a,b$ in T w/ no edge. Combine w/ A.  $\Box$
$R(4,3) \leq 10$

$R(4,3) > 8$

... turns out $R(4,3) = 9$

$\Leftrightarrow$ not terribly hard

$\Leftrightarrow$ notice $R(x,y) = R(y,x)$

no 4-clique

complement(G)

no 3-clique = no 3-independent in G
Suppose $|V| \geq 18$

Pick any vertex, $A$. $\geq 17$ vertices remain.

Form 2 groups: $S$ & $T$

edges to $A$

no edges to $A$

If $|S| \geq 9$, use $R(3,4) = 9$ : $S$ has 4 independent vertices (done)

or $S$ has a 3-clique, so with $A$ we get a 4-clique.

If $|T| \geq 9$, use $R(4,3) = 9$ : $T$ has a 4-clique, (done) or $T$ has 3 independent vertices, so with $A$ we have 4.
$R(4,4) \leq 18$

Suppose $|V| \geq 18$

Pick any vertex, $A$. $\geq 17$ vertices remain.

Form 2 groups: $S$ & $T$

If $|S| \geq 9$, use $R(3,4) = 9$; $S$ has 4 independent vertices (done) or $S$ has a 3-clique, so with $A$ we get a 4-clique.

If $|T| \geq 9$, use $R(3,4) = 9$ on the complement graph. (Symmetric)
Notes:

(1) if we only knew that $R(4,3) \leq 10$ (instead of $= 9$)
we could have used $|V| \geq 20$ for $R(4,4)$

As you bound smaller $R(\cdot)$ values, you can get (loose) bounds for larger ones

(2) there is a graph w/ 17 vertices with no
clique or independent set of size 4

$\Rightarrow R(4,4) = 18$