What is common about these functions?

\[ 5n + \frac{1}{2} n^2 + \frac{1}{100} \cdot n^3 \quad 18n^3 - 12 \quad 40n^3 + \log n^{10} \]

- The dominant term contains \( n^3 \)

What does this mean? For \( n=5, \ \frac{1}{100} \cdot n^3 \) doesn't dominate.

\[ n^3 \text{ dominates for all } n \text{ larger than some integer.} \]

- All three functions have \( 50n^3 \) as an upper bound, for \( n \geq 1 \)
- All three functions have \( \frac{1}{100} n^3 \) as a lower bound, for \( n \geq 1 \)
There exist constants $c > 0$, $n_0 > 0$ such that for all $n \geq n_0$:

- All three functions have $cn^3$ as an upper bound
  \[(all \leq cn^3)\]

There exist constants $d > 0$, $n_1 > 0$ such that for all $n \geq n_1$:

- All three functions have $dn^3$ as a lower bound
  \[(all \geq dn^3)\]

- All three functions have $50n^3$ as an upper bound, for $n \geq 1$
- All three functions have $\frac{1}{100}n^3$ as a lower bound, for $n \geq 1$
If there exist constants $c > 0$, $n_0 > 0$ such that for all $n \geq n_0$:

$$f(n) \leq cn^3$$

then we say $f(n) = O(n^3)$

If there exist constants $d > 0$, $n_1 > 0$ such that for all $n \geq n_1$:

$$f(n) \geq dn^3$$

then we say $f(n) = \Omega(n^3)$
If there exist constants $c > 0$, $n_0 > 0$ such that for all $n \geq n_0$:

$$f(n) \leq c \cdot g(n)$$

then we say $f(n) = O(g(n))$  \hspace{1cm} \text{Big-O}

If there exist constants $d > 0$, $n_1 > 0$ such that for all $n \geq n_1$:

$$f(n) \geq d \cdot g(n)$$

then we say $f(n) = \Omega(g(n))$  \hspace{1cm} \text{Omega}
If $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ then $f(n) = \Theta(g(n))$
\[ 5n, \quad \frac{1}{2}n^2, \quad \frac{1}{100} \cdot n^3 \]
$5n + \frac{1}{2}n^2 + \frac{1}{100} \cdot n^3 < 6n^3 = \mathcal{O}(n^3)$

(exaggerate terms)
For large $n$, these are within a constant multiplicative factor. Anything with a larger dominating term will eventually surpass and exceed by a lot.
Polynomials: \( a + bn + cn^2 + dn^3 \ldots + zn^k = O(n^k) \)

\( a, b, c, d, \ldots, z : \text{constants} \)

Also assuming one of each term (compare to: \( a_1n + a_2n + \ldots + a_nn \))

Logarithms: \( 50 \cdot \log n^3 + \log n^2 + n^{0.1} = O(n^{0.1}) \)

"weaker" than polynomial

Exponential: \( 100 \cdot n^{50} + 3^n + 40 \cdot 2^n = O(3^n) \)

"stronger" than polynomial
Ordering some common functions

\[ n^n \]
\[ n! \]

exponential: \[ k^n \quad (k>1) \]
\[ 1.1^n, \ 2^n, \ 3^n \ \text{etc} \]

polynomial: \[ n^k \]
\[ n^{0.1}, \sqrt{n}, \ n, \ n^{0.1}, \ n^2, \ n^3, \ etc \]

powers of logs: \[ \log n = (\log n)^k \]
\[ \log n, \ \log^2 n, \ \log^3 n, \ etc \]

constants: \[ 1, \ 50, \ 2^{100} = O(1) \]
\[(50 \cdot \log n + 10 \log^5 n + n^{0.1}) \cdot (100 \cdot n^{50} + 3^n + 40 \cdot 2^n) = O(n^{0.1}) \cdot O(3^n) = O(n^{0.1} \cdot 3^n)\]

- is \(5n^2 = O(n^3)\)? Yes. but a better answer is \(O(n^2)\)
- is \(n^3 = O(5n^2)\)? No.

There is no \(c\) such that \(n^3 \leq c \cdot 5n^2\) (for all large \(n\))

Also, the 5 doesn't belong in \(O(5n^2)\)
Typically, a "better" bound will be valid only for higher values of $n$. $f(n) = O(g(n))$
Prove $\frac{1}{2} n^2 + 3n - 10 = \Theta(n^2)$

\[ \text{prove } = O(n^2) \quad \rightarrow \quad \text{find } c_1 \text{ & } n_1 \text{ s.t. } \frac{1}{2} n^2 + 3n - 10 \leq c_1 \cdot n^2 \text{ for } n > n_1. \]

\[ \therefore \frac{1}{2} n^2 + 3n - 10 < 3.5n^2 \Rightarrow c_1 = 3.5 \text{ & } n_1 = 1 \text{ work (exaggerate & simplify)} \]

\[ \text{prove } = \Omega(n^2) \quad \rightarrow \quad \text{find } c_2 \text{ & } n_2 \text{ s.t. } \frac{1}{2} n^2 + 3n - 10 \geq c_2 \cdot n^2 \text{ for } n > n_2. \]

\[ \therefore \frac{1}{2} n^2 + 3n - 10 > \frac{1}{2} n^2 - 10 = \frac{4}{10} n^2 + \left(\frac{1}{10} n^2 - 10\right) \text{ (underestimate & simplify)} \]

\[ > \frac{4}{10} n^2 \quad \text{for } n > 10 \]

\[ c_2 = 0.4 \text{ & } n_2 = 10 \text{ work} \]
How NOT to prove $f(n) = \frac{1}{2}n^2 + 3n - 10 = O(n^2)$

- Obviously the dominant term is $\frac{1}{2}n^2$, so it's $O(n^2)$

- As $n \to \infty$, the function approaches $\frac{1}{2}n^2$, so $f(n) \leq cn^2$, for $c = \frac{1}{2}$

- We need to show $\frac{1}{2}n^2 + 3n - 10 \leq cn^2$...

\[
\frac{\frac{1}{2}n^2}{n^2} + \frac{3n}{n^2} - \frac{10}{n^2} \leq \frac{cn^2}{n^2} \quad \text{so as} \quad n \to \infty, \quad \frac{1}{2} \leq c \quad \text{[NOT OK]}
\]
Prove $6n^3 \neq \Theta(n^2)$

if it were true, then $c_1 n^2 \leq 6n^3 \leq c_2 n^2$ for $n > n_0$

$6n^3 \gg c_1 \cdot n \rightarrow$ trivially true for $n > 1$ & $c_1 = 6$

is $6n^3 = O(n^2)$ ? $\rightarrow$ $6n^3 \leq c_2 n^2$ ?

\[ \fbox{NO} \]

$6n \leq c_2$ ?

$n \leq \frac{c_2}{6}$ \{ NO. \}

Whatever constant $c_2$ we choose, $n$ will eventually surpass it.
Notes:

- $f(n) = O(g(n))$ can be expressed as $f(n) \in O(g(n))$

- Further reading: little-o & little omega ($\omega$)

  \[
  f(n) = o(g(n)) \iff f(n) = O(g(n)) \text{ but } f(n) \neq \Theta(g(n))
  \]

  e.g., $n^2 = o(n^3)$ but $0.5n^2 \neq o(n^2)$

  \[
  f(n) = \omega(g(n)) \iff f(n) = \Omega(g(n)) \text{ but } f(n) \neq \Theta(g(n))
  \]

  e.g., $n^3 = \omega(n^2)$ but $5n^2 \neq \omega(n^2)$
Recap of rules and model of computation used in this course. (unless mentioned otherwise)

- Any number occupies $O(1)$ storage ... and can be read in $O(1)$ time
  Even irrationals.

- We can do simple arithmetic in $O(1)$ time. (on $O(1)$ elements)
  $\left(+,-,\times,\div, \text{but also } \sqrt{}, \ln, \text{powers, etc}\right)$

- We care only about what happens for unimaginably large input size $n$

- We focus on worst-case time/space complexity