We want to insert n elements into a BST so that they will be stored in sorted order.

InOrder walk: 1 2 3 5 6 7 8

Insertion: nothing fancy. Just read elements and insert into current tree.
Given array of elements: 3 1 8 2 6 7 5

This is a sorting algorithm.
Given the very simple BST-sort/construction algorithm:

- how long can it take to build the BST?
- what shape will it have? How balanced? What depth?

Depends on input sequence

1 2 3 4 5 → 1

Can be bad:
- \(O(n)\) depth
- \(O(n^2)\) time
- \(\Omega(n\log n)\) worst-case time: sorting lower bound
Even for a balanced tree, 
$$\Theta(n) \approx \frac{n}{2}$$ nodes have height = $$\Theta(\log n)$$
so it must take $$\Omega(n\log n)$$ time to build.

Any algorithm producing any tree shape : $$\Omega(n\log n)$$ time

So ... if lucky , $$\Theta(n\log n)$$ time
if unlucky, $$O(n^2)$$ time

Sounds like... quicksort
Stable quicksort

- Use first elt to partition
- Repeat on each side

Quicksort round 1: compare all els to 3

BST sort: 3 = root; eventually all els pass through.

Quicksort: partitions into 2 groups

\(<3\) & \(>3\)

Each is independent

BST sort: same

Exactly same comparisons

But in different order

Same tree as BST
We've seen \( \text{QUICKSORT} \preceq \text{BST-SORT} \) \( \downarrow \) (stable)

\[ E[\text{BST build}] = \Theta(n \log n) \text{ time} \]

\[ E[\text{insert node}] = \Theta(\log n) \text{ time} \]

Randomized versions have same analysis.
Intuition: $E[\text{depth}] = \Theta(\log n)$ so it should be \textit{not} balanced? No

\[
\log(n - \sqrt{n}) < \log n \quad \Rightarrow \quad \sqrt{n} \quad \Rightarrow \quad \sqrt{n}
\]

average depth $\leq \frac{1}{n} \cdot (n \log n + \sqrt{n} \cdot (\sqrt{n} + \log n))$

$= \log n + 1 + \frac{\log n}{\sqrt{n}}$

$= O(\log n)$ so $E[\text{depth}] \nRightarrow \text{balance}$
\[ H(n) = 1 + \max \left\{ H(k), H(n-k-1) \right\} \]

for some random \( k \) \((0 \leq k \leq n-1)\)

- If \( \frac{n}{4} < k < \frac{3n}{4} \)
  \[ \iff H(n) \leq 1 + H(\frac{3n}{4}) \]
- Else \( H(n) \leq 1 + H(n-1) \)
  \[ < 1 + H(n) \]

\[
\begin{align*}
E[H(n)] &\leq 1 + \frac{1}{2} E[H(\frac{3n}{4})] + \frac{1}{2} E[H(n)] \\
\frac{1}{2} E[H(n)] &\leq 1 + \frac{1}{2} E[H(\frac{3n}{4})] \\
E[H(n)] &\leq 2 + E[H(\frac{3n}{4})] = O(\log n)
\end{align*}
\]

\[ 2 \log_{4/3} n \approx 2 \times 2.4 \log_2 n < 5 \log n \]